Lecture 10
Other Solution Concepts and Game Models

CS580

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Some slides are borrowed from V. Conitzer's presentations.

So far

- Normal-form games
  - Multiple rational players, single shot, simultaneous move
- Nash equilibrium
  - Existence
  - Computation in two-player games.

Today:

- Issues with NE
  - Multiplicity
  - Selection: How players decide/reach any particular NE
- Possible Solutions
  - Dominance: Dominant Strategy equilibria
  - Arbitrator/Mediator: Correlated equilibria, Coarse-correlated equilibria
  - Communication/Contract: Stackelberg equilibria, Nash bargaining
- Other Games
  - Extensive-form Games, Bayesian Games

Formally: Games and Nash Equilibrium

- N: Set of players/agents
- $i \in N$, S_i: Set of strategies/moves of player i
- $s = (s_1, \ldots, s_n) \in S_1 \times S_2 \times \cdots \times S_n$,
  $u_i(s)$: payoff/utility of player i

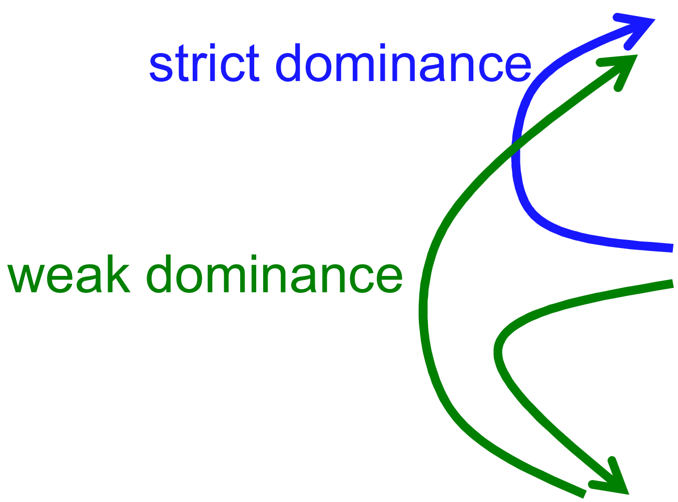
  $(\sigma_i(1), \ldots, \sigma_i(s_i)) \geq 0$, $\sigma_i(1) + \cdots + \sigma_i(s_i) = 1$

- $\sigma_i \in \Delta(S_i)$ randomized strategy of i
  - Probability distribution over the moves in S_i
- *Nash equilibrium:* $\sigma = (\sigma_1, \ldots, \sigma_n)$ *s.t.*
  $\forall i \in N, \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \quad \forall \tau_i \in \Delta(S_i)$

Dominance

- Strict dominance: For a player, move s strictly dominates t if no matter what others play, s gives her better payoff than t
  - for all s_{-i}, $u_i(s, s_{-i}) > u_i(t, s_{-i})$
- s weakly dominates t if
  - for all s_{-i}, $u_i(s, s_{-i}) \geq u_i(t, s_{-i})$; and
  - for some s_{-i}, $u_i(s, s_{-i}) > u_i(t, s_{-i})$

-i = "the player(s) other than i"



	L	**M**	**R**
U	0, 0	1, -1	1, -1
G	-1, 1	0, 0	-1, 1
B	-1, 1	1, -1	0, 0

Dominant Strategy Equilibrium

Playing move s is best for me, no matter what others play.

- $s = (s_1, \ldots, s_n)$ is DSE if for each player i, there is a (strategy) move s_i that (weakly) dominates all other moves.
  - for all i, s_i', s_{-i}, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$;

Example?

Prisoner's Dilemma

- Pair of criminals has been caught
- They have two choices: {confess, don't confess}

	confess	don't confess
confess	-5, -5	0, -6
don't confess	-6, 0	-1, -1

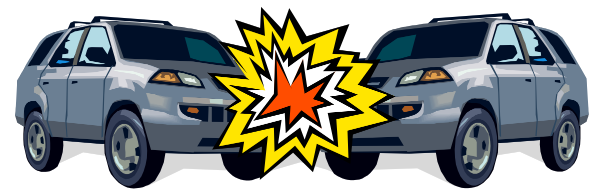
“Should I buy an SUV?”

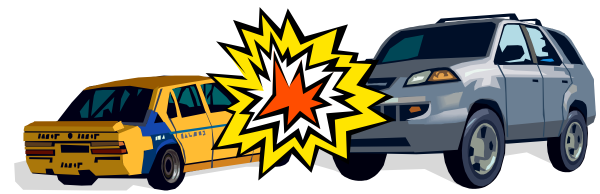
purchasing cost


cost: 5


cost: 3

accident cost

cost: 5  cost: 5

cost: 8  cost: 2

cost: 5  cost: 5

		
	-10, -10	-7, -11
	-11, -7	-8, -8

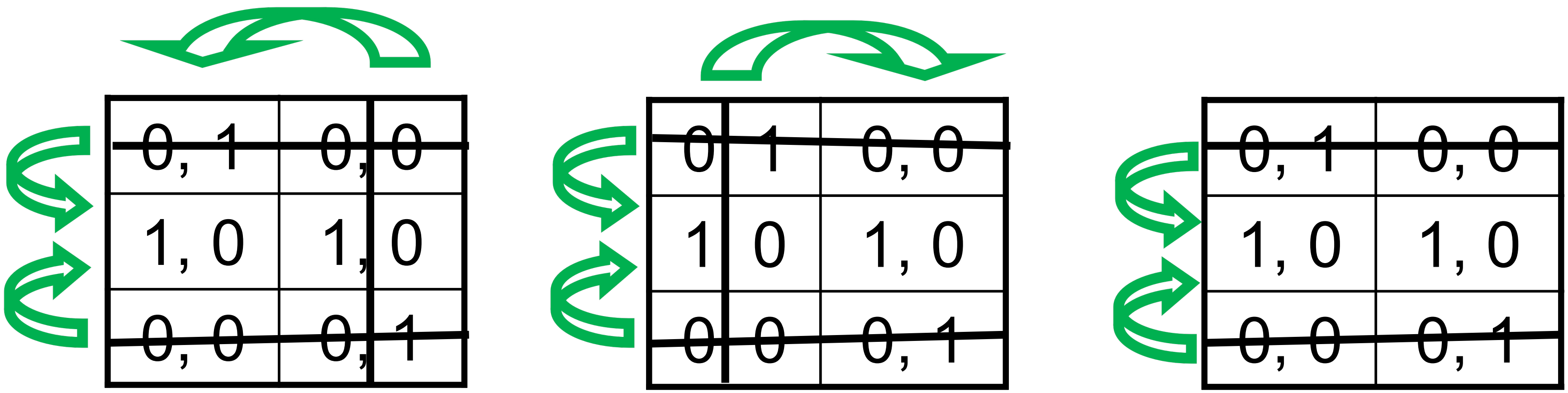
Dominance by Mixed strategies

- Example of dominance by a mixed strategy:

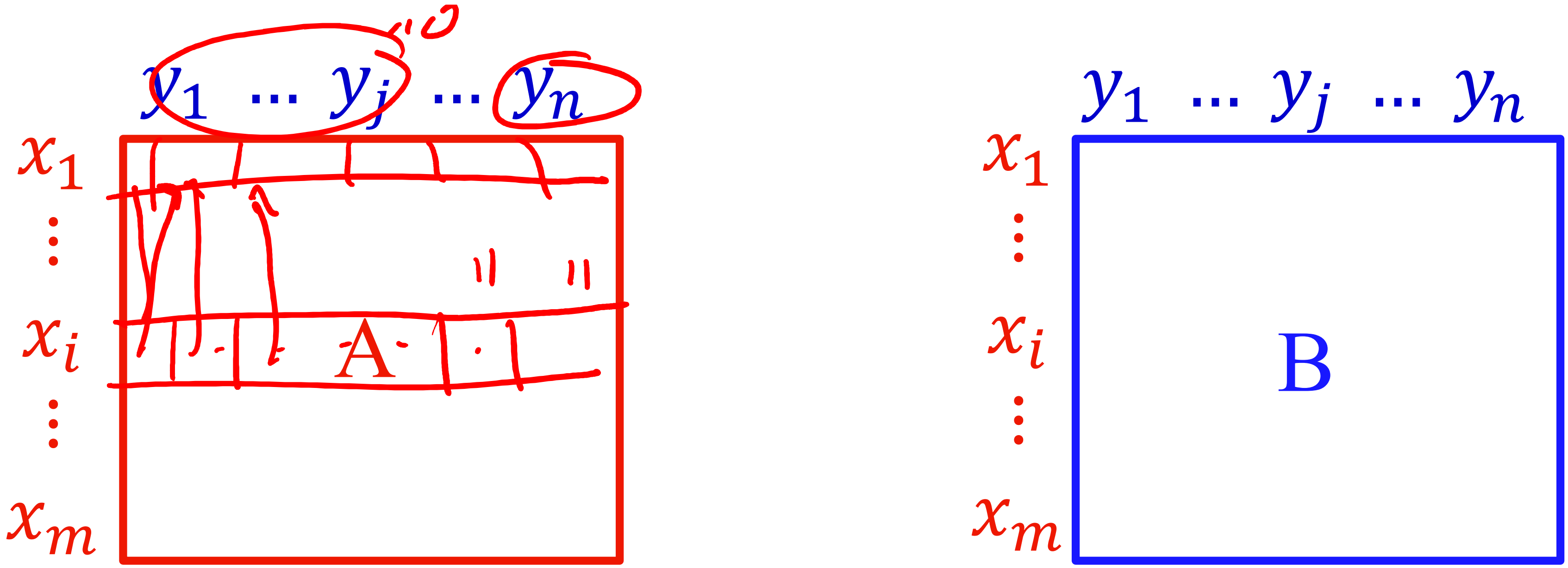
1/2	3, 1	0, 0
1/2	0, 0	3, 2
	1, 0	1, 1

Iterated dominance: path (in)dependence

Iterated weak dominance is path-dependent: sequence of eliminations may determine which solution we get (if any)
(whether or not dominance by mixed strategies allowed)



Iterated strict dominance is path-independent: elimination process will always terminate at the same point
(whether or not dominance by mixed strategies allowed)



NE: $x^T A y \geq x'^T A y,\ \forall x'$ $\quad x^T B y \geq x^T B y',\ \forall y'$

No one plays dominated strategies.

Why?

What if they can discuss beforehand?

Players: {Alice, Bob}

Two options: {Football, Tennis}

	2/3 F	1/3 T
1/3 F	1 2 (0.5)	0 0
2/3 T	0 0	2 1 (0.5)

At Mixed NE both get 2/3 < 1

Instead they agree on ½(F, T), ½(T, F)

Payoffs are (1.5, 1.5) Fair!

Needs a common coin toss!

Correlated Equilibrium – (CE)
(Aumann'74)

- **Mediator** declares a joint distribution P over $S = \times_i S_i$
- Tosses a coin, chooses $s = (s_1, \ldots, s_n) \sim P$.
- Suggests s_i to player i **in private**

- P is at equilibrium if each player wants to follow the suggestion when others do.
  - $U_i(s_i, P_{(s_i,.)}) \geq U_i(s_i', P_{(s_i,.)}), \forall s_i' \in S_1$

CE for 2-Player Case

- **Mediator** declares a joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$
- Tosses a coin, chooses $(i, j) \sim P$.
- Suggests i to Alice, j to Bob, in private.

- P is a CE if each player wants to follow the suggestion, when the other does.

Given Alice is suggested i, she knows Bob is suggested $j \sim P(i, .)$

$$\langle A(i, .), P(i, .)\rangle \geq \langle A(i', .), P(i, .)\rangle \quad : \forall i' \in S_1$$

$$\langle B(., j), P(., j)\rangle \geq \langle B(., j'), P(., j)\rangle \quad : \forall j' \in S_2$$

Players: {Alice, Bob}

Two options: {Football, Shopping}

	F	S
F	1 2 (0.5)	0 0 (0)
S	0 0 (0)	2 1 (0.5)

Instead they agree on ½(F, S), ½(S, F)

Payoffs are (1.5, 1.5) Fair!

CE!

Prisoner's Dilemma

	C	NC
C	-5, -5 (1)	0, -6 (0)
NC	-6, 0 (0)	-1, -1 (0)

C strictly dominates NC

Rock-Paper-Scissors (Aumann)

	R	P	S
R	0, 0 (0)	0, 1 (1/6)	1, 0 (1/6)
P	1, 0 (1/6)	0, 0 (0)	0, 1 (1/6)
S	0, 1 (1/6)	1, 0 (1/6)	0, 0 (0)

When Alice is suggested R

Bob must be following $P_{(R,.)} \sim (0,1/6,1/6)$

Following the suggestion gives her 1/6

While P gives 0, and S gives 1/6.

Computation: Linear Feasibility Problem

Game (A, B). Find, joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

s.t.

$$\sum_j A_{ij} p_{ij} \geq \sum_j A_{i'j} p_{ij} \quad \forall i, i' \in S_1$$

$$\sum_i B_{ij} p_{ij} \geq \sum_i B_{ij'} p_{ij} \quad \forall j, j' \in S_2$$

$$\sum_{ij} p_{ij} = 1; \quad p_{ij} \geq 0, \ \forall (i,j)$$

N-player game: Find distribution P over $S = \times_{i=1}^{N} S_i$

s.t. $U_i\left(s_i, P_{(s_i,\cdot)}\right) \geq U_i\left(s_i', P_{(s_i,\cdot)}\right), \ \forall s_i, s_i' \in S_i$

$\sum_{s \in S} P(s) = 1$

↑

$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i})$ Linear in P variables!

Computation: Linear Feasibility Problem

N-player game: Find distribution P over $S = \times_{i=1}^{N} S_i$

s.t. $U_i\left(s_i, P_{(i,.)}\right) \geq U_i\left(s_i', P_{(s_i,.)}\right), \forall s_i, s_i' \in S_i$

$\uparrow$ $\sum_{s \in S} P(s) = 1$

$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i})$ Linear in P variables!

Can optimize any convex function as well!

Coarse-Correlated Equilibrium

- After mediator declares P, each player opts in or out.
- Mediator tosses a coin, and chooses s ~ P.
- If player i opted in, then the mediator suggests her s_i in private, and she has to obey.
- If she opted out, then (knowing nothing about s) plays a fixed strategy $t \in S_i$
- At equilibrium, each player wants to opt in, if others are.

$$U_i(P) \geq U_i(t, P_{-i}),\ \forall t \in S_i$$

Where P_{-i} is joint distribution of all players except i.

Importance of (Coarse) CE

- Natural dynamics quickly arrive at approximation of such equilibria.
  - No-regret, Multiplicative Weight Update (MWU)

- Poly-time computable in the size of the game.
  - Can optimize a convex function too.

Show the following