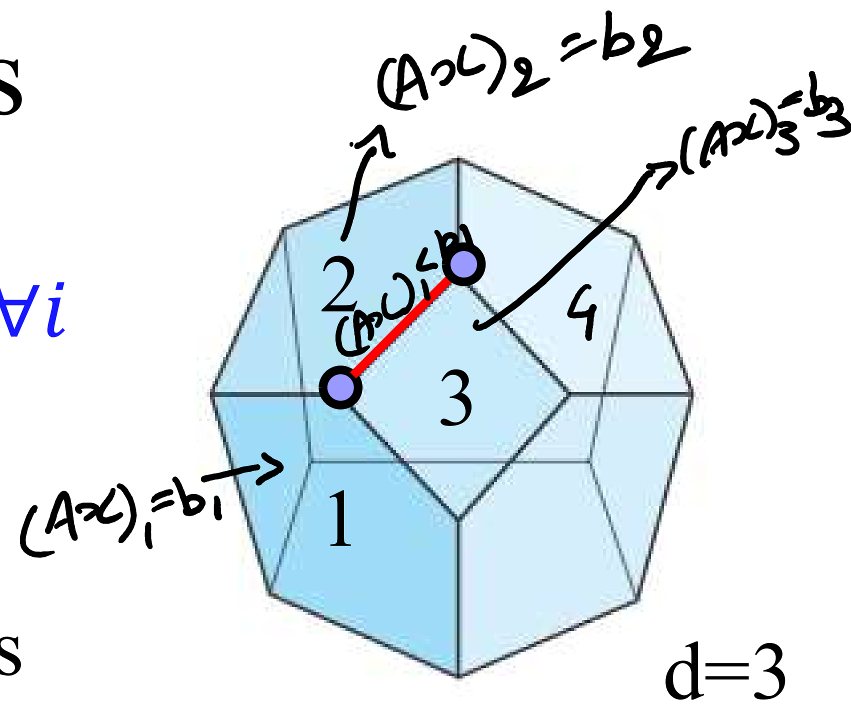
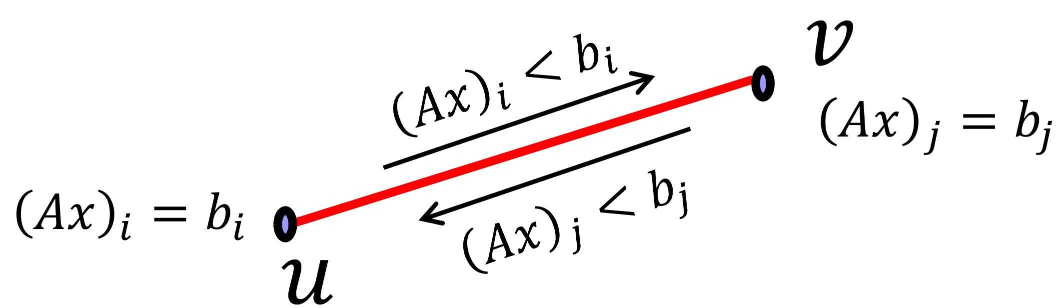
Basic Polytope Properties

- Given $A_{m \times d}, b_{m \times 1}$: $(Ax)_i \le b_i, \forall i$
  - In d dimension
- At a vertex (0-dim), d equalities
- On an edge (1-dim), d-1 equalities
- 1-skeleton → vertices + edges → graph





u, v share d-1 equalities.
These also hold on connecting edge

Finding NE in game (A, B)

$m \times n$ $m \times n$

moves of Alice

moves of Bob

$$d = m + n$$

Given $M_{d\times d} > 0$, find $x \in R^d$, $x \neq \mathbf{0}$ s.t.

$$\forall i \leq d,\ \ x_i \geq 0, \qquad (Mx)_i \leq 1$$

$$x_i > 0 \ \Rightarrow (Mx)_i = 1$$

$$\equiv x_i = 0 \ \text{ OR } (Mx)_i = 1$$

Find $x \neq \mathbf{0}$ s.t.

$\forall i \leq d, \quad x_i \geq 0, \qquad (Mx)_i \leq 1$ → d-dim polytope P

$x_i = 0 \text{ or } (Mx)_i = 1$ → Label/color i is present

$x_1 = 0 \text{ or } (Mx)_1 = 1$ $\quad x_4 = 0 \text{ or } (Mx)_4 = 1$

$x_2 = 0 \text{ or } (Mx)_2 = 1$ $\quad \ddots$

$x_3 = 0 \text{ or } (Mx)_3 = 1$ $\quad x_d = 0 \text{ or } (Mx)_d = 1$

- Define $L(x) = \{i \mid \text{label/color } i \text{ is present at } x\}$
- *Fully-labeled/panchromatic* set of points

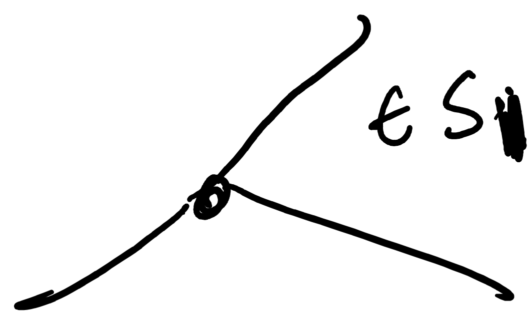
  $S = \{x \mid L(x) = \{1, \ldots, d\}\}$ (all colors are present)

  - Vertices.
  - $\mathbf{0} \in S$. $x \in S \setminus \{\mathbf{0}\}$ iff x is a solution → **new goal!**

$x_i = 0$ or $(Mx)_i = 1$ ⟶ Label/color i present

- Define $L(x) = \{i \mid \text{label/color } i \text{ is present at } x\}$
- *Fully-labeled* set $S = \{x \mid L(x) = \{1, \dots, d\}\}$.
  - Vertices.
  - $\mathbf{0} \in S$. $x \in S \setminus \{\mathbf{0}\}$ iff x is a solution → **new goal!**

- *1-almost* fully-labeled set, $S_1 = \{x \mid L(x) \supseteq \{2, \dots, d\}\}$.
  - Only color 1 is missing.
  - $S \subset S_1$. Vertices + edge.



Lemke-Howson follows a path in S_1

Structure of S_1
(Paths and Cycles)

$x_i = 0$ or $(Mx)_i = 1 \longrightarrow$ Label/color i

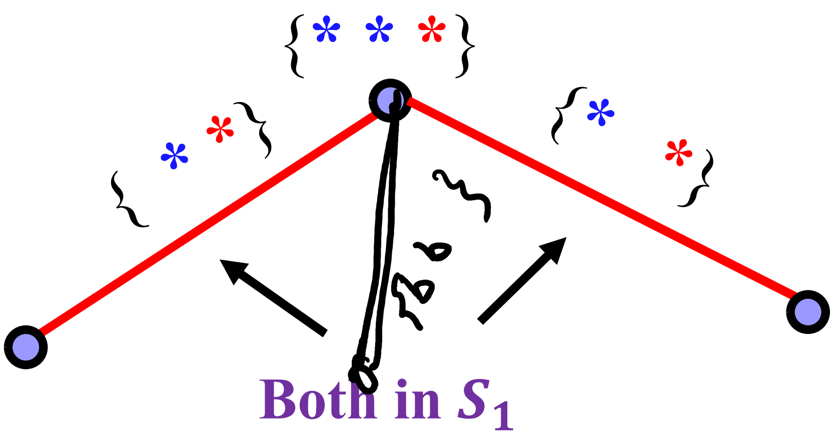
Panchromatic *set*, $S = \{x \mid L(x) = \{1,2,\dots,d\}\}$.

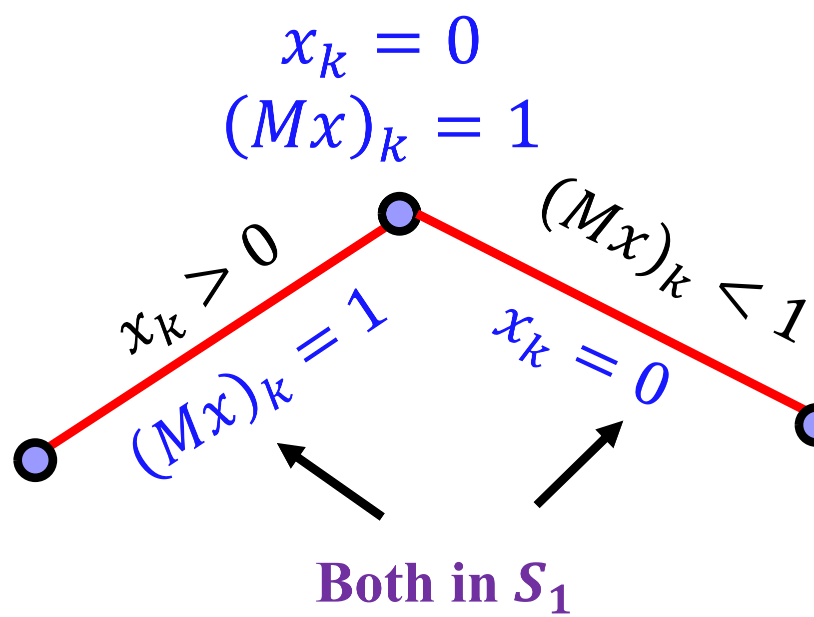
1-almost panchromatic set, $S_1 = \{x \mid L(x) \supseteq \{2,\dots,d\}\}$.

- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2,\dots,d\}$
  - For each $i \in \{2,\dots,d\}$, $x_i = 0$ *or* $(Mx)_i = 1$
  - Unique k $\in \{2,\dots,d\}$ s.t. $x_k = 0$ **and** $(Mx)_k = 1$

$d = 3$. Colors: {* * *}

S_1: Points with colors {* *}





$x_i = 0$ or $(Mx)_i = 1 \longrightarrow$ Label/color i

Panchromatic *set*, $S = \{x \mid L(x) = \{1,2,\ldots,d\}\}$.

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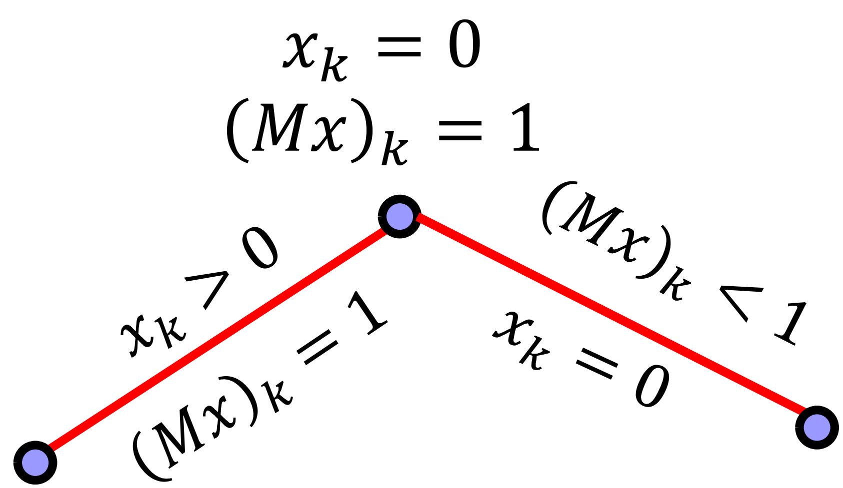
- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2,\ldots,d\}$
  - For each $i \in \{2,\ldots,d\}$, $x_i = 0$ *or* $(Mx)_i = 1$
  - Unique k $\in \{2,\ldots,d\}$ s.t. $x_k = 0$ **and** $(Mx)_k = 1$
    - k: Duplicate label

Both edges are in S_1

Any other? No!



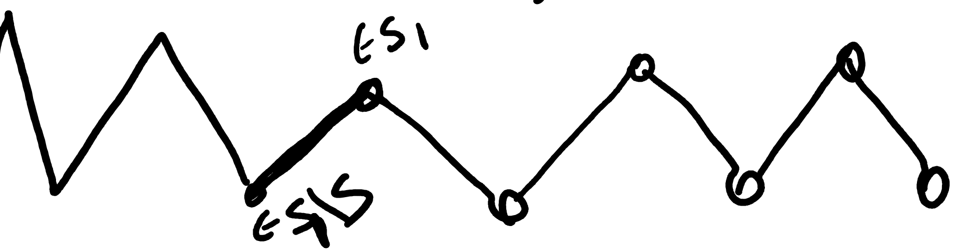
Claim 1. $\deg(v) = 2$ if $v \in S_1 \setminus S$ in S_1

Starting vertex

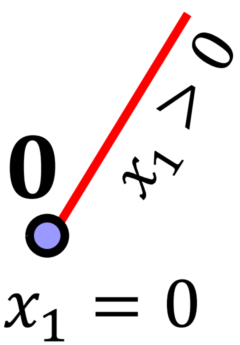
$x_i = 0$ or $(Mx)_i = 1$ $\longrightarrow$ Label/color i

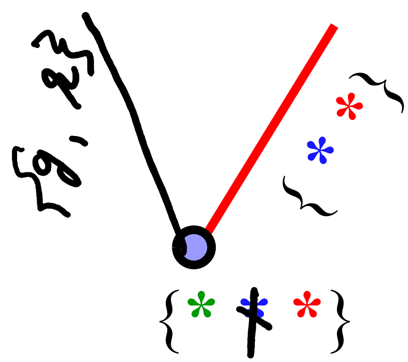
Panchromatic *set*, $S = \{x \mid L(x) = \{1,2, \dots, d\}\}$.
1-almost panchromatic set, $S_1 = \{x \mid L(x) \supseteq \{2, \dots, d\}\}$.

- Vertex $v \in S(\subset S_1)$. Then $L(v) = \{1, \dots, d\}$
  - No duplicate label.
    - Can only leave label 1 to remain in S_1



$d = 3$. Colors: {* * *}
S_1: Points with colors {* *}

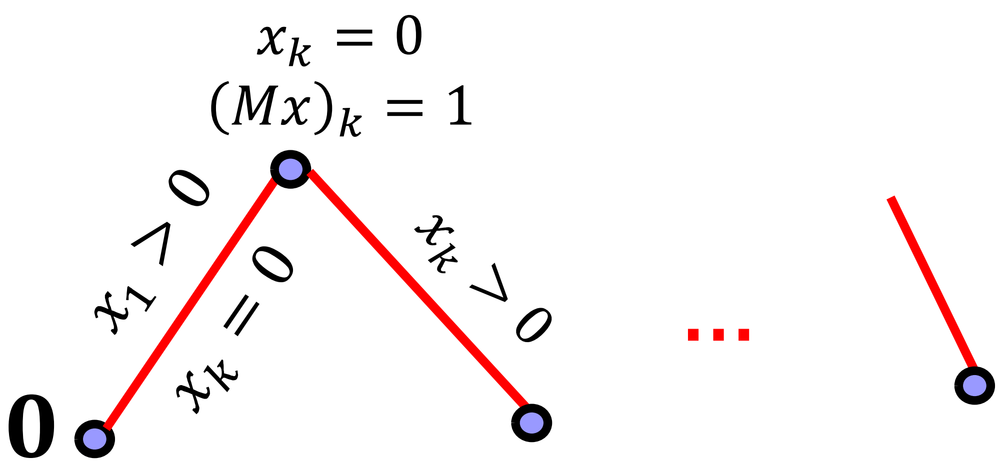




Claim 2. $\deg(v) = 1$ if $v \in S$, within S_1

Lemke-Howson: Follow path starting at **0**

- Vertex $v \in S(\subset S_1)$. Then $L(v) = \{1, \dots, d\}$
  - No duplicate label



Thumb rule: Relax the one that is tight on the previous edge.

1. Leave label 1
2. If Label 1 found
   - Then done.
3. Else leave duplicate label.
4. Go to 2.

Recall

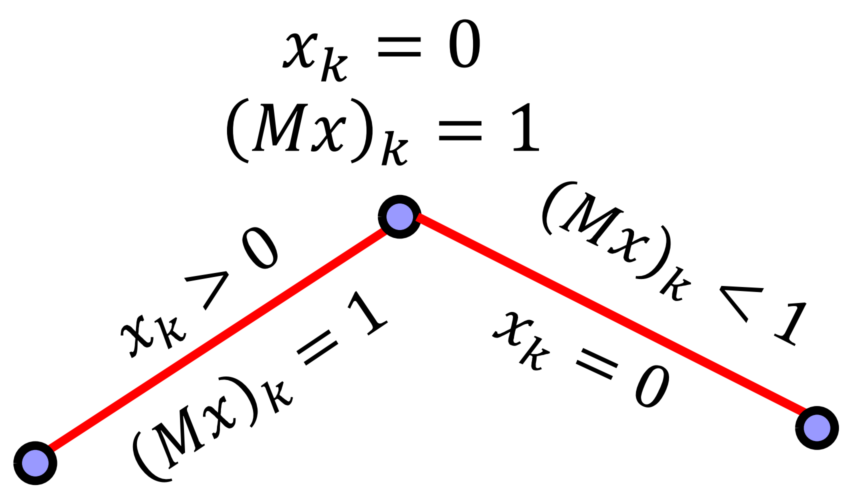
$\forall i, x_i \geq 0, \quad (Mx)_i \leq 1$ → d-dim P

$x_i = 0$ or $(Mx)_i = 1$ → Label/color i

- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2, \ldots, d\}$
  - For each $i \in \{2, \ldots, d\}$, $x_i = 0$ *or* $(Mx)_i = 1$
  - Unique k $\in \{2, \ldots, d\}$ s.t. $x_k = 0$ **and** $(Mx)_k = 1$
  - k is duplicate

Both edges are in S_1

Any other? No!



Claim 1. $\deg(v) = 2$ if $v \in S_1 \setminus S$

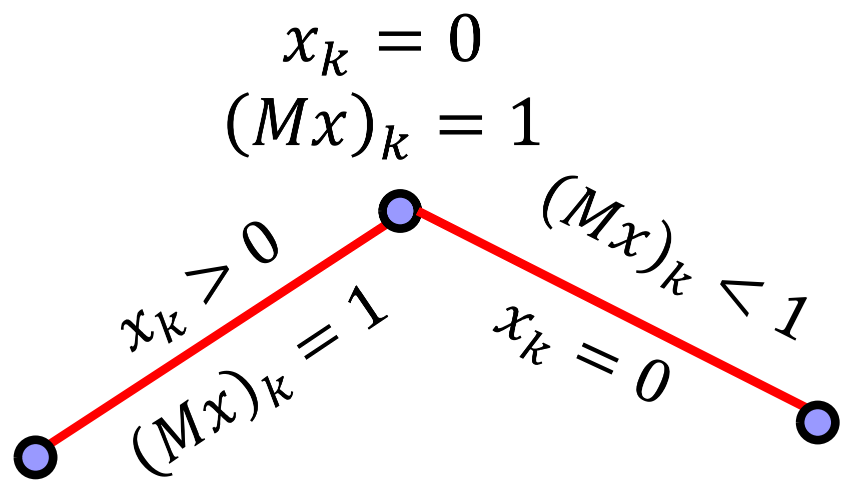
Recall

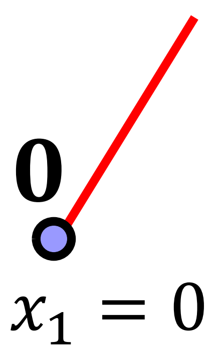
$x_i = 0$ or $(Mx)_i = 1$ ⟶ Label/color i

Panchromatic *set*, $S = \{x \mid L(x) = \{1,2,\dots,d\}\}$.
1-almost panchromatic set, $S_1 = \{x \mid L(x) \supseteq \{2,\dots,d\}\}$.

- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2,\dots,d\}$
  - k: duplicate label

Claim 1. $\deg(v) = 2$ if $v \in S_1 \setminus S$





- Vertex $v \in S (\subset S_1)$. Then $L(v) = \{1,\dots,d\}$
  - No duplicate label.

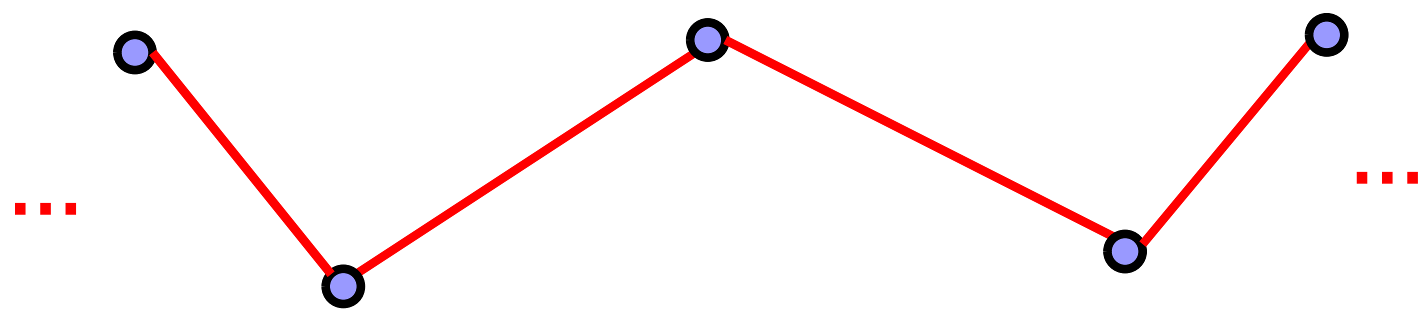
Claim 2. $\deg(v) = 1$ if $v \in S$

S_1: Structure

$\forall i, x_i \geq 0, \quad (Mx)_i \leq 1$ → d-dim P

$x_i = 0$ or $(Mx)_i = 1$ → Label/color i

- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2, \ldots, d\}$
  - Unique duplicate label



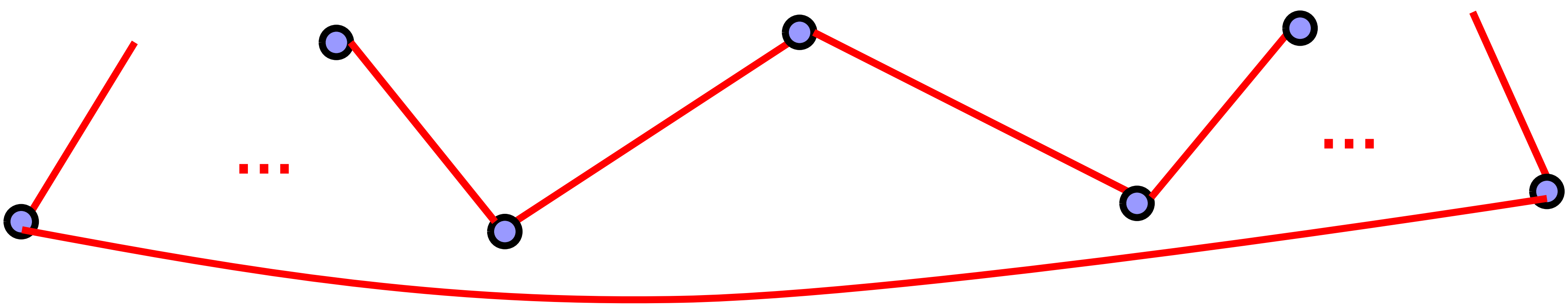
Both edges are in S_1

S_1: Structure

$\forall i, x_i \geq 0, \qquad (Mx)_i \leq 1$ → d-dim P

$x_i = 0$ or $(Mx)_i = 1$ → Label/color i

- Vertex $v \in S_1 \setminus S$. Then $L(v) = \{2, \dots, d\}$
  - Unique duplicate label



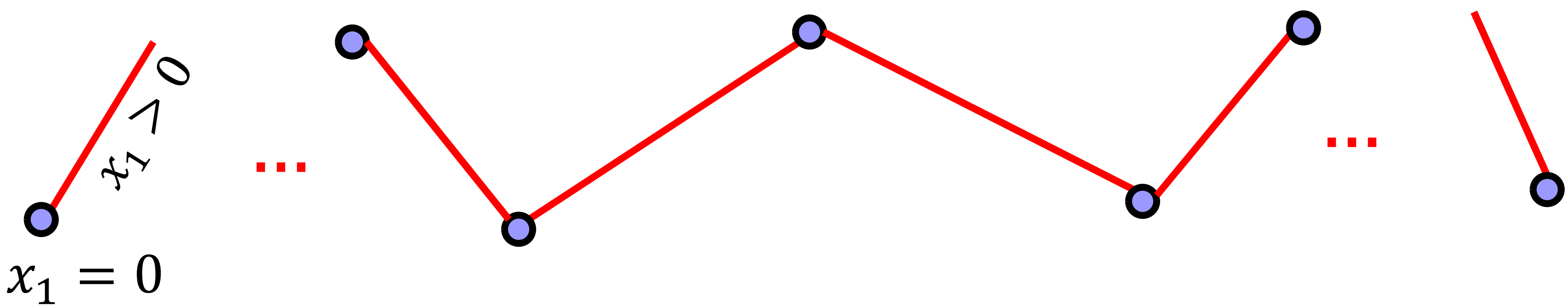
Cycle

S_1: Set of paths and cycles

$\forall i, x_i \geq 0, \quad (Mx)_i \leq 1$ → d-dim P

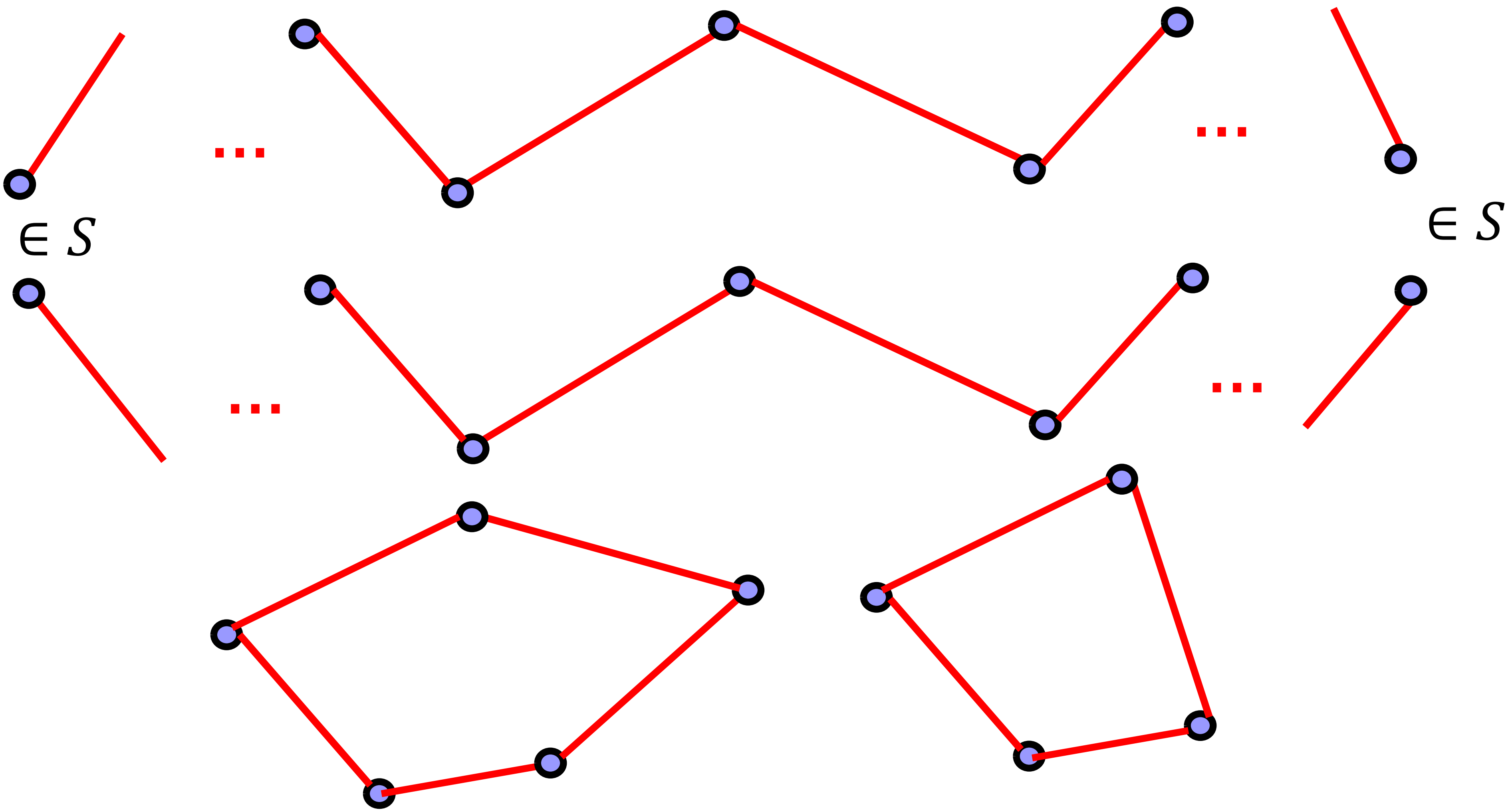
$x_i = 0$ or $(Mx)_i = 1$ → Label/color i

- Vertex $v \in S(\subset S_1)$. Then $L(v) = \{1, \ldots, d\}$
  - No duplicate label



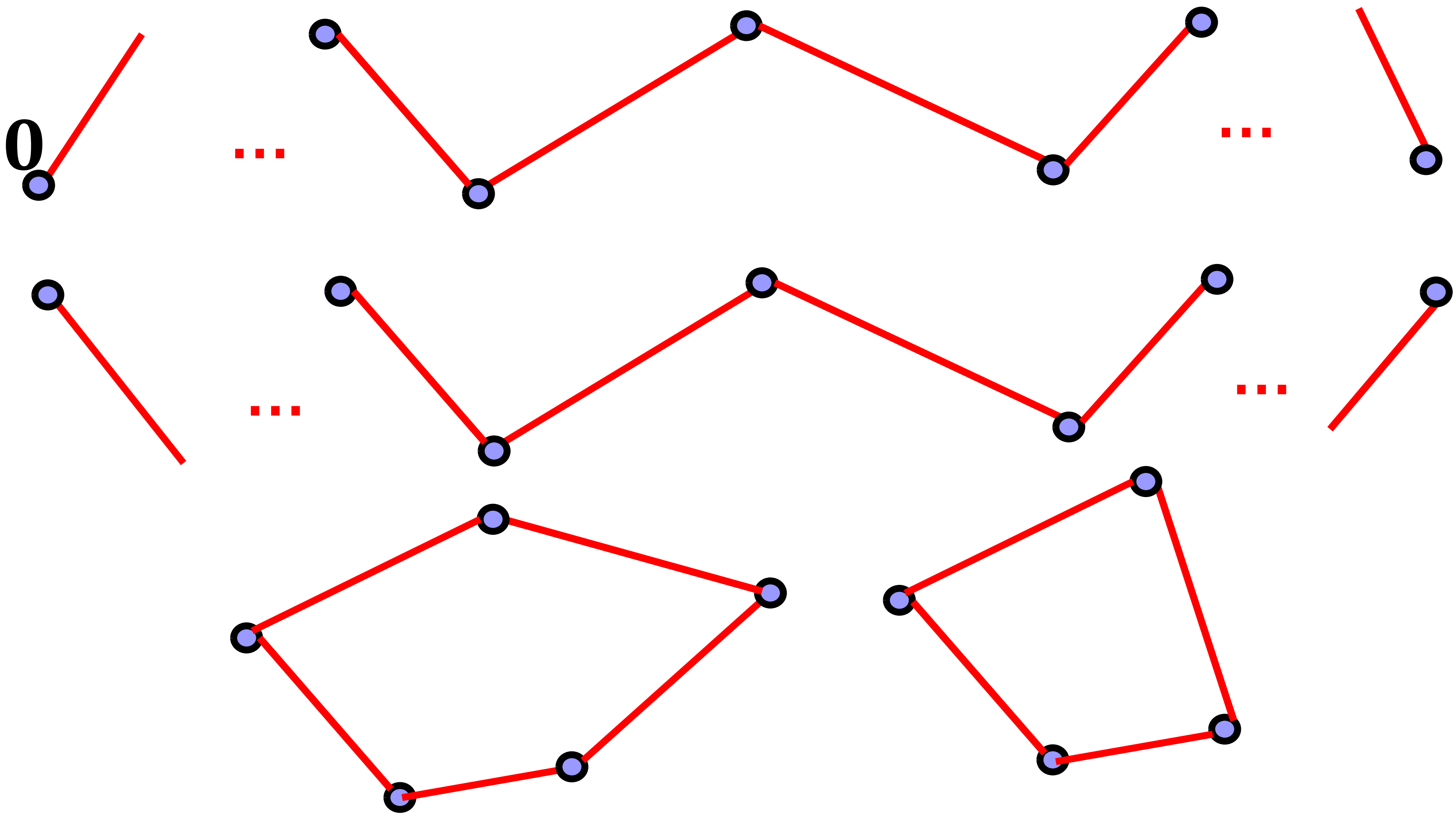
End in S

S_1: Set of paths and cycles



$S = \text{Solutions} \cup \{\mathbf{0}\}$

S_1: Set of paths and cycles

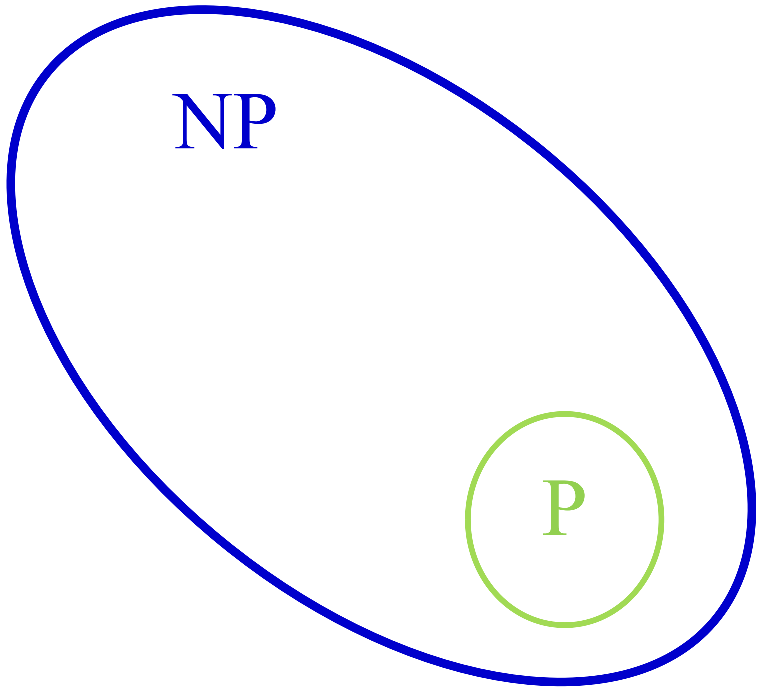


Goal: Find any other end-point   Defn of PPAD!

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

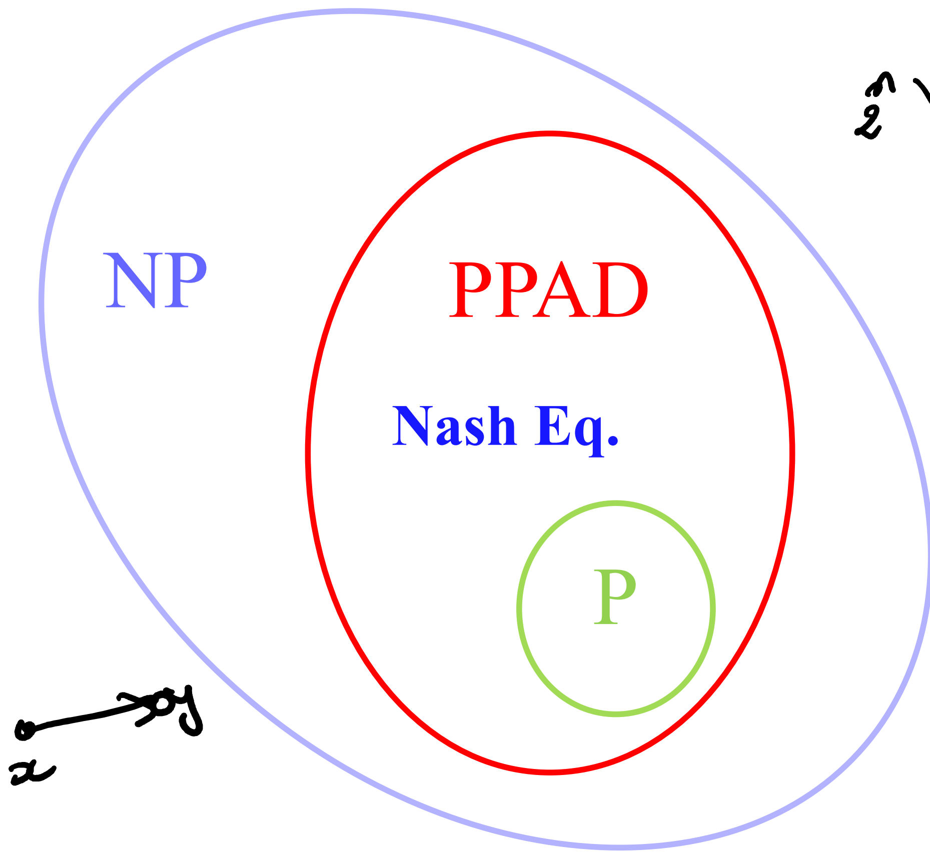
NP-hardness is ruled out!

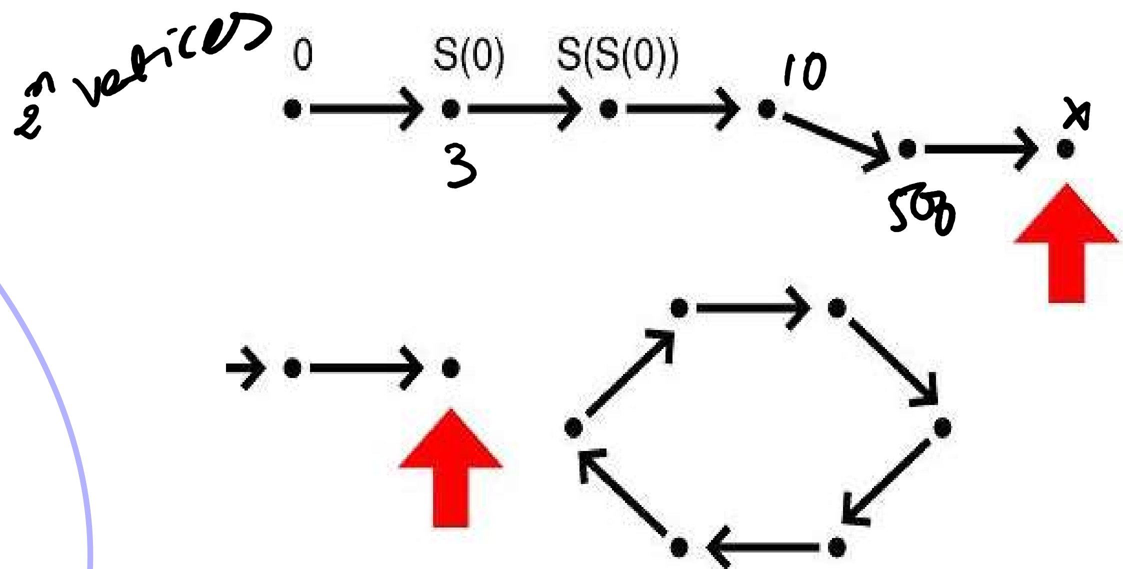
Complexity Classes

2-Nash is PPAD-complete!
[DGP'06, CDT'06]

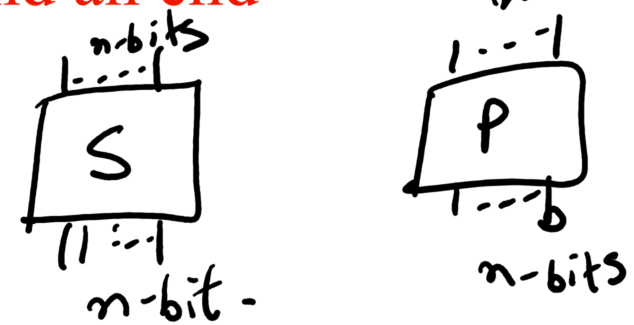
Papadimitriou'94

PPAD Polynomial Parity Argument for Directed graph





Find an end



$P(y) = x$ & $y = S(x)$

Brute-force Algorithm?

P

$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

Let (x, y) be a NE. Suppose we know supp(x) and supp(y).

Now can we find a NE?

Can we do better than "brute-force"?

Not so far. And may be never!

It is one of the hardest problems in PPAD.

What about special cases/approximation?

- Rank(A) or rank(B) is constant
- O(1)-approximate NE: quasi-polynomial time algorithm $n^{\log n}$
- Constant rank games: rank(A+B) is a constant
  - FPTAS

P $\begin{array}{c}\forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n\end{array}$

Q $\begin{array}{c}\forall j, \left(x^T B\right)_j \leq \pi_B \\ x \in \Delta_m\end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

$$P \quad \boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

Rank of a game: rank(A+B)

Poly-time approximation for constant rank games [KT'03].

Poly-time exact for rank-1 games [AGMS'11].

Exact for rank > 2 is PPAD-hard [M'13].

Open Problems

- Status of PPAD.
  - Is constant factor approximation of 2-Nash PPAD-hard?

- Not risk neutral? → Prospect Theory
  - Expected utility ≡ risk neutral