EFX Allocations for Additive Valuations

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Problem Setting

We are given n agents and items $\{1, 2, \ldots, k\}$. We must allocate these items into n bundles such that $\bigcup B_i = \{1, 2, \ldots, k\}$ and for $i \neq j$, $B_i \cap B_j = \emptyset$.

Agent i has *additive* valuation function v_i, meaning for any item x, $v_i(x) \geq 0$, and for some bundle of items B, we have $v_i(B) = \sum_{x \in B} v_i(x)$.

Definition (Envy)

Agent i *envies* a bundle of items X if $v_i(X) > v_i(B_i)$

Definition (Envy-Free Allocation [Fol66])

An allocation is i *envy-free* if no agent envies the bundle of any other agent.

Weaker Fairness Notions

Recall that an envy-free allocation does not always exist given some agents and valuation functions. In particular, if there are 2 agents and 1 item that the agents both positively value, there is no envy-free allocation.

This leads us to weaker fairness notions

Definition (Envy-Free Up to 1 Item – *EF1* [Bud11])

An allocation is *EF1* if $\forall i, j$, there exists $x \in B_j$ such that agent i doesn't envy $B_j \setminus \{x\}$

Definition (Envy-Free Up to Any Item – *EFX* [Car+19])

An allocation is *EFX* if $\forall i, j$, for **all** $x \in B_j$, agent i doesn't envy $B_j \setminus \{x\}$

Background

Here are some results for additive valuations:

- EF1 allocations always exist (agents draft the items) [Lip+04]
- EFX allocations always exist for two agents (you split I choose) [PR20]
- EFX allocations always exist for three agents. [CGM20]
- EFX allocations always exist for n agents where all agents have identical valuation functions. [PR20]
- EFX allocations always exist for n agents when there are only two unique valuation functions among the agents. [Mah20]
- EFX allocations always exist for n agents where $n-2$ agents have identical valuation functions. [GNV+23]
- EFX allocations always exist for n agents where for any item x, $v_i(x) = 0$ or $v_i(x) = 1$. [BSY23]

Our Approach

Given some non-EFX allocation $B = (B_1, B_2, \ldots, B_n)$, we can find agents i, j and item x such that $v_i(B_j \setminus \{x\}) > v_i(B_i)$. Let $B'_i = B_i \cup \{x\}$. Let $B'_j = B_j \setminus \{x\}$.

Let $B''_j = \begin{cases} B'_i & v_j(B'_i) > v_j(B'_j) \\ B'_j & \text{otherwise} \end{cases}$

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Then, we define the function f that takes an input of a non-EFX allocation, and outputs a new allocation such that for any k, we have

$$f(B)_k := \begin{cases} B''_i & k = i \\ B''_j & k = j \\ B_k & \text{otherwise} \end{cases}$$

Properties Of f

- Agent j does not envy agent i in allocation $f(B)$.

Proof

Note that $f(B)_j = B''_j$ and $f(B)_i = B''_i$. If $B''_j = B'_i$ then $v_j(B''_j) = v_j(B'_i) > v_j(B'_j) = v_j(B''_i)$. On the other hand, if $B''_j = B'_j$ then $v_j(B''_j) = v_j(B'_j) \geq v_j(B'_j) = v_j(B''_i)$.

Properties Of f (Continued)

- $v_i(f(B)_i) > v_i(B_i)$.

Proof

Consider the following two cases:

1. $B_i'' = B_i'$. Then, $v_i(f(B)_i) = v_i(B_i') = v_i(B_i \cup \{x\}) > v_i(B_i)$.
2. $B_i'' = B_j'$. Then, $v_i(f(B)_i) = v_i(B_j') = v_i(B_j \setminus \{x\}) > v_i(B_i)$

Properties Of f (Continued) (Continued)

- Agent i envies agent j less in allocation $f(B)$ than in allocation B. In other words, $v_i(f(B)_j) - v_i(f(B)_i) < v_i(B_j) - v_i(B_i)$.

Proof

Note that $B_i \cup B_j = f(B_i) \cup f(B_j)$. Thus,
$v_i(B_i) + v_i(B_j) = v_i(B_i \cup B_j) = v_i(f(B_i) \cup f(B_j)) = v_i(f(B_i)) + v_i(f(B_j))$.
Then recall that $v_i(f(B_i)) > v_i(B_i)$ so
$v_i(f(B_i)) + v_i(B_i) + v_i(B_j) > v_i(B_i) + v_i(f(B_i)) + v_i(f(B_j))$ so
$v_i(f(B_j)) < v_i(B_j)$. Thus, $v_i(f(B)_j) - v_i(f(B)_i) < v_i(B_j) - v_i(B_i)$.

Main Question

Define $f^k(B) = \underbrace{f(f(\ldots(f(f(}_{k \text{ times}} B))))).$ For any number of agents, number of items, valuation functions, and initial allocation B, is there some k such that $f^k(B)$ is an EFX allocation?

Short answer: no.

Degenerate Cases

A *degenerate* valuation function v is a valuation function such that for some sets of items A, B with $A \neq B$, we have $v(A) = v(B)$.

Theorem ([CGM20])

If an EFX allocation always exists for n agents with non-degenerate additive valuation functions, then an EFX allocation always exists for n agents with any additive valuation functions.

[Akr+23] extends this result to more general monotone valuation functions.

Two Agent Case

Consider the case where there are two agents. We know from [PR20] that "you split I choose" always gives an EFX allocation. However, we have another proof that there is always an EFX allocation for two agents.

Claim

Given two agents with additive valuation functions and any initial allocation B, there is some k such that $f^k(B)$ is an EFX allocation.

Proof

For two agents and for any allocation B with EFX envy, $f(B)$ has strictly less total envy than B. Thus, $f^k(B)$ has less total envy than B, so $f^k(B) \neq B$. Assume that there is no k such that $f^k(B)$ is an EFX allocation. Then, the set $\{f^k(B) : \forall k \geq 0\}$ is a set of infinite distinct allocations, which is a contradiction.

n Agents With Identical Valuation Functions

Consider the case where there are n agents that all have the same valuation function v. Plaut and Roughgarden [PR20] give a somewhat complicated way of constructing an EFX allocation. We have another proof that there is always an EFX allocation for n agents with the same additive valuation function.

Claim

Let $g(B) = f(B)$ except that agent i is specifically $\text{argmin}_i(v(B_i))$. Given n agents with additive valuation function v and any initial allocation B, there is some k such that $g^k(B)$ is an EFX allocation.

Continued

Proof

For n agents with valuation function v and for allocation B with EFX envy, $\min_i v(g(B)_i) > \min_i v(B_i)$. Thus, $f^k(B) \neq B$. Assume that there is no k such that $g^k(B)$ is an EFX allocation. Then, the set $\{g^k(B) : \forall k \geq 0\}$ is a set of infinite distinct allocations, which is a contradiction.

Counterexample for Non-Identical Valuations

	item 1	item 2	item 3	item 4
agent *a*	0.6	0.3	0.1	ϵ
agent *b*	0.1	0.6	0.3	ϵ
agent *c*	0.3	0.1	0.6	ϵ

Here, agent *a* values item 1 the most, agent *b* values item 2 the most, and agent *c* values item 3 the most. Additionally, we have a fourth item which all the agents value minimally (ϵ). Let's allocate item 2 to *a*, item 3 to *b*, and item 1 to *c*. Let's also allocate item 4 to *a*.

Conjecture 1

Definition

The *Nash welfare* of an allocation is defined as the geometric mean of agents' valuations for their own bundles [Nas50]:

$$\text{Nash Welfare} = \left(\prod_{i=1}^{n} v_i(B_i)\right)^{1/n}$$

Conjecture

If M is the maximum Nash welfare allocation, there is some k such that $f^k(M)$ is an EFX allocation.

Conjecture 2

Definition

A *cyclic* allocation is an allocation such that for some $k > 0$ we have $f^k(B) = B$

Definition

An allocation B' dominates allocation B if B' is better for all agents, meaning for all i, $v_i(B'_i) \geq v_i(B_i)$.

Definition

A *Pareto optimal* allocation is an allocation B such that there is no allocation B' that dominates B.

Conjecture

There are no Pareto optimal allocations that are cyclic.

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