How to eat 4/9 of a pizza

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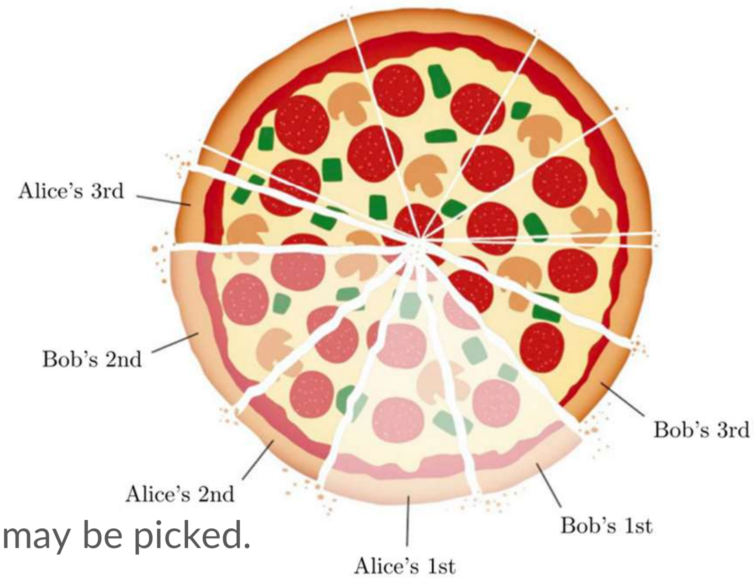
The problem

Alice and Bob share a pizza.

(i) They pick pieces in an alternating fashion;

(ii) Alice starts by eating any piece of the pizza;

(iii) Afterward only pieces adjacent to already eaten pieces may be picked.



Observation 1: Even count of pieces

Claim: Alice can get at least ½ of the pizza (For even pieces).

Proof idea: color the pizza alternatively by red and green, then Alice can decide to pick all the green (or red) pieces.

Some definitions/ notations:

Def: ‘interval’ of a pizza = set of consecutive pieces. Odd intervals= intervals with odd pieces(Same for even) Two cuts, say C_1 and C_2, encloses an interval.

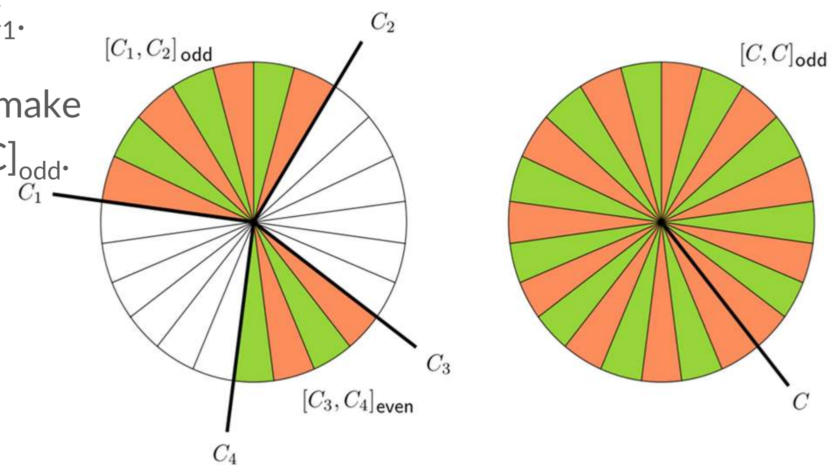
Def: ‘$[C_1, C_2]_{odd}$’ the odd interval between C_1 and C_2

‘$[C_1, C_2]_{even}$’ the even interval between C_1 and C_2

Coloring the intervals

Just like what we did in Observation 1, we want to color each interval by red and green. For an interval '[C_1, C_2]', we make sure that red is always adjacent to C_1.

Since now we are only focusing on odd pizzas, we can make a random cut, calling it C, then color the pizza w.r.t $[C,C]_{odd}$.



Follow Bob strategy (fB)

After the first pick, if Alice decides to pick the piece that was just revealed by Bob, we call this strategy 'Follow Bob'(fB).

Prop 2: Easy threshold for odd cases

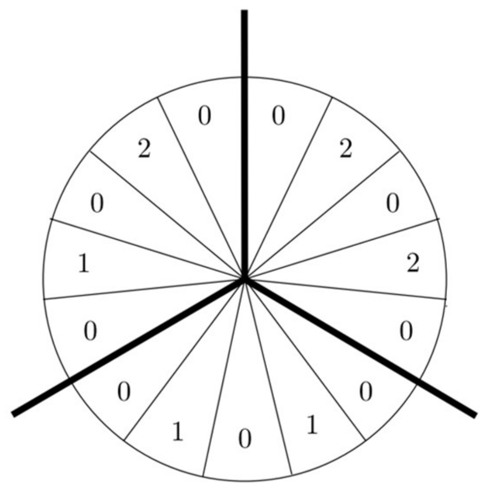
Claim: Alice can get at least ⅓ of the pizza (For odd pieces).

Proof idea: Randomly pick a cut C of the pizza, then color $[C,C]_{odd}$ with reds and greens. We observe that Alice can always pick all the red pieces, then we assume that $R([C,C]_{odd}) < ⅓$. Then the pizza has more greens. Starting with the 'mid-point green piece' opposing C, one can make sure that Alice gets at least half of the green areas, which is greater than ⅓.

Observation 3:

Claim: There are some pizzas where Alice can only get 4/

Proof: Picture

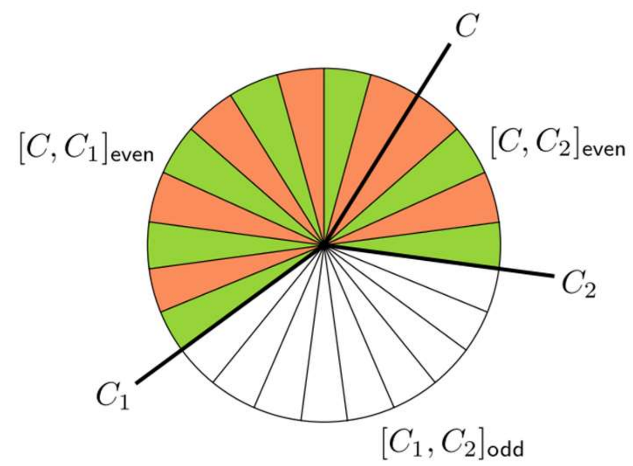


Now it’s time to start the proof!

Assume that $[C_1, C_2]_{odd}$' has already been eaten. Now it's Bob's turn, and Alice is using fB-strategy.

We analyze Bob's behavior: Then Bob is trying to find a cut C such that $[C, C_1]_{even}$ and $[C, C_2]_{even}$ would have maximized green area.

We call C 'the (Bob's) *best answer to* $[C_1, C_2]_{odd}$'



Definition: Heavy Greens Property

For an even interval $[C_1, C_2]_{even}$, if for every $[C_1, C]_{even}$ in $[C_1, C_2]_{even}$, the green pieces have larger(or equal) size.

One can prove that a cut C with $[C, C_1]_{even} \subseteq [C_1, C_2]_{even}$ is a best answer to $[C_1, C_2]_{odd}$ if and only if $[C, C_1]_{even}$ and $[C, C_2]_{even}$ have the heavy greens property.

Proof idea: We find C' that's also best answer, then [C,C'] would have equal reds and greens. Otherwise contradiction.

C best answer $\Leftrightarrow$ both partitions have heavy greens property.

Definition: Easy pizzas and hard pizzas

A pizza is *easy* if fB-strategy for Alice would yield at least ½ of the pizza.

Pizza is *hard otherwise.*

Theorem 4:

A hard pizza can be partitioned into three odd intervals each with heavy greens property.

Proof: Very long, will discuss if time allows.

Remarks: There's a stronger version of the theorem,

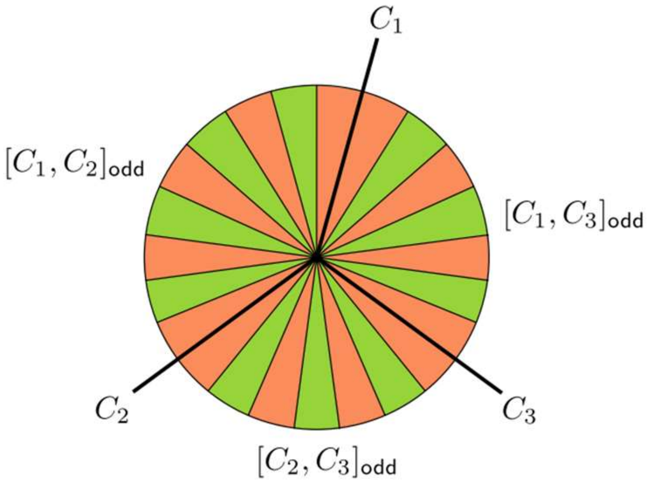
but for now we can just assume that there are some

properties for the chosen cuts:

i) g1 + r2 + r3 ⩽ r1 + r2 + g3 ⩽ r1 + g2 + r3

where r_i means the sum of red areas in the interval opposing C_i. Similar for g_i

ii) Each cut is the best answer to some piece p. (best answer w.r.t one piece).



‘Partial pizza’

We partition the pizza into 3 odd intervals, where the three cuts are named C_1, C_2 and C_3.

Consider $[C_2 , C_3]_{odd}$, we can think of it as a self-contained pizza, which glues together C_2 and C_3. We call this $C_{2,3}$, and the resulting pizza ‘partial pizza’.

Note. The reason why 2 and 3 were picked are because of the stronger version of Theorem 4.

strategies on the partial pizza

A strategy is said to be *plugged into the whole pizza(call it Pstrat) if:*

1) If none of C_2 and C_3 is revealed, Alice follow the good strategy.
2) If Bob reveals C_2 or C_3, Alice does not follow Bob. Instead, pick the other option.
3) Alice follows Bob if Bob picks something outside the partial pizza.
4) Inside the partial pizza, Alice has some specific strategy.

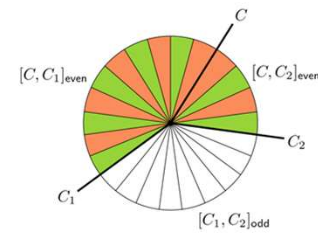
A Pstrat ensures the outcome of Alice to be at least the strategy inside the partial pizza, plus r_2+ g_3.

Proof on next page

Lemma 5: Best answer on the edge.

Claim: Let C_i, C_j be two cuts in our 3 cuts. Then if $[C, C_j]_{odd}$ is in $[C_i, C_j]_{odd}$, then either C_i or C_j is a best answer to it.

Reminder: best answer means that it maximizes the the partitioned green area.

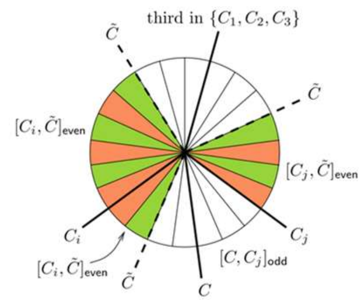


Proof: Consider ~C, a best answer to $[C, C_j]_{odd}$, then

clearly it's not in $[C_i, C_j]_{odd}$, then at least one of $[\sim C, C_j]_{ev}$

and $[\sim C, C_j]_{odd}$ is a subset of an interval with the heavy

green property. WLOG suppose that's *i*.



-continue: We know ~C is best answer, so $[\sim C, C_i]_{even}$ is heavy green. We just mentioned that $[C_i, \sim C]_{even}$ is also heavy green. So it has equal reds and greens. Therefore C_i is also a best answer. □

After proving this we prove

Theorem 6: **A Pstrat ensures the outcome of Alice to be at least the strategy inside the partial pizza, plus $r_2 + g_3$.**

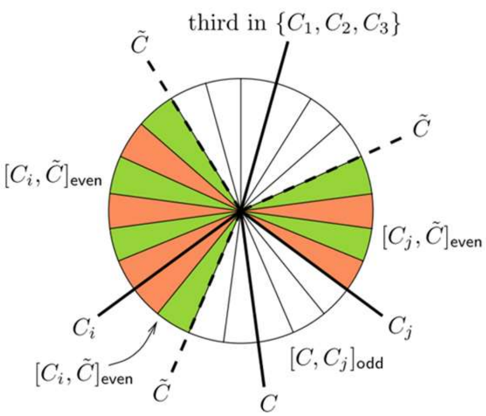
Note that Alice follow Bob outside the partial pizza when some

$[C, C_2]_{odd} \subseteq [C_2, C_3]_{odd}$ or $[C, C_3]_{odd} \subseteq [C_2, C_3]_{odd}$ is eaten.

By lemma 5 we know that the best answer of it is given by C_2 or C_3.

In the worst case Alice gets $r_2 + g_3$ or $r_3 + g_2$, by our Observation 1.

$r_2 + g_3 \leqslant r_3 + g_2$ by our assumption in Theorem 4 proof.



Last before final: calculating some strategies outcome.

Our final result is the combination of a list of strategies, taking the optimal one.

-fB

-mfB

-Pstrat combined with fB or mfB.

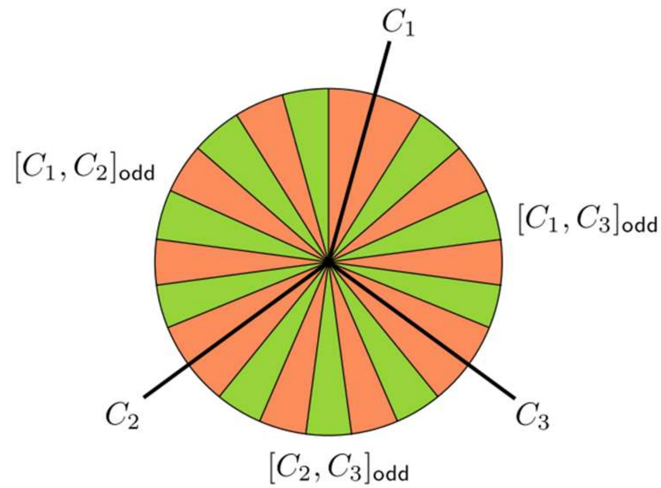
fB-strategy associated with cut C

Just like what we assumed in the proof of Theorem 4,

$g1 + r2 + r3 \leqslant r1 + r2 + g3 \leqslant r1 + g2 + r3$

These values are actually the least outcomes of the fB-strategy:

Let C be the cut we pick among the three, and p the piece which has the best answer of C, then if Alice starts with p and follows Bob afterwards, she gets all the red pieces of [C, C], which corresponds to to the values above.



Cut	Alice's outcome	Bob's outcome
C_1	$g_1 + r_2 + r_3$	$r_1 + g_2 + g_3$
C_2	$r_1 + g_2 + r_3$	$g_1 + r_2 + g_3$
C_3	$r_1 + r_2 + g_3$	$g_1 + g_2 + r_3$

modified follow-Bob strategy(mfB) associated with cut C

Idea: eat more green pieces.

(i) Alice starts with eating $p_i \in G_i$.

(ii) As long as Bob's moves reveal pieces in G_i Alice picks them, i.e., follows Bob.

(iii) At the moment Bob's move reveals the first red piece from another of the three odd intervals, Alice makes a single move that does not follow Bob. This means she picks a piece from R_i .

(iv) Alice follows Bob from then on.

If the starting piece is the middle-piece of the green pieces in G_i, then Alice gets at least half of the size of the green pieces in G_i. One can show that the following table is t

mfB-strategy	Alice's outcome
mfB_1	$\frac{g_1}{2} + g_2 + r_3$
mfB_2	$g_1 + \frac{g_2}{2} + r_3$
mfB_3	$g_1 + r_2 + \frac{g_3}{2}$

Final proof: 4/9.

Consider two cases, the partial pizza is easy or hard.

First case: partial is easy

Use the following strategies:

(i) The fB-strategy associated to C2, which yields at least $r_1 + g_2 + r_3$;

(ii) The mfB-strategy mfB2, which yields at least $g_1 + g_2/2 + r_3$;

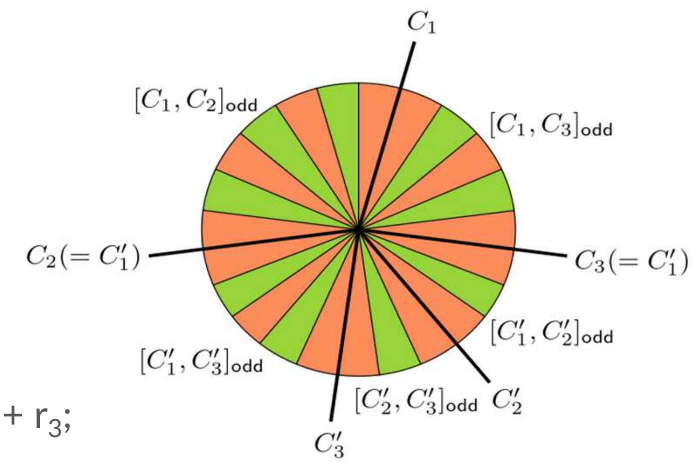
(iii) The fB-strategy plugged into the whole pizza, which yields at least $(g_1+r_1)/2 + r_2 + g_3$

Summing them up with weights we have:

$3/2\ (r_1 + g_2 + r_3) + 2\ \ (g_1 + g_2/2 + r_3) + ((g_1 + r_1)/2 + r_2 + g_3) \geqslant 2(r_1 + g_1 + r_2 + g_2 + r_3 + g_3) = 2.$

Since we are having total weight = 1.5+2+1=4.5, this mixed strategy has outcome = 2/4.5 = 4/9

Second case: partial is hard.



We partition the partial pizza into 3 intervals again.

(i) The fB-strategy associated to C2, which yields at least $r_1 + g_2 + r_3$;

(ii) The mfB-strategy mfB2, which yields at least $g_1 + g_2 /2 + r_3$;

(iii) The fB-strategy associated to C'_2 plugged into the whole pizza, which yields at least $r'_1 + g'_2 + r'_3 + r_2 + g_3$;

(iv) The strategy mfB1 for the partial pizza plugged into the whole pizza, which yields at least $g'_1 / 2 + g'_2 + r'_3 + r_2 + g_3$

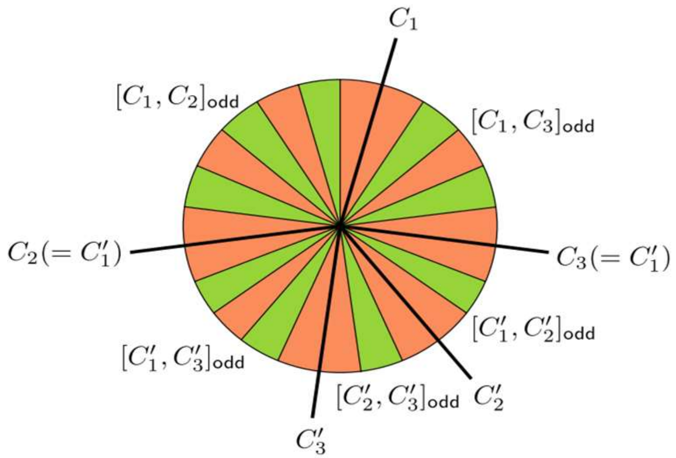
Similarly we have:

3/2 ($r_1 + g_2 + r_3$) + $g_1 + g_2$ / 2 + r_3 + ($r'_1 + g'_2 + r'_3 + r_2 + g_3$) + g'_1 / 2 + $g'_2 + r'_3 + r_2 + g_3$.

Plug in $r_1 = g'_1 + r'_2 + r'_3$ and $g_1 = r'_1 + g'_2 + g'_3$

The above is greater or equal to 2, which again, would

Imply that this mixed strategy gives 4/9



Done! Thanks for watching!

Citations:

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