Higher Dimensional Stable Matching Problem

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Applications

- Hospital / resident assignments (National Resident Matching Program [1])
- User / server assignments [4]
    - Due to sparsity, only partial preference lists may be available and used, e.g. a user in Sydney may not even have a preference evaluation on a server in New York

Recap: Two-Dimensional Case

Input: two sets of vertices A and B, each with n elements. Every element $a \in A$ has a preference list on elements in B, and so does everyone in B. Let $\mathscr{M}$ be a matching, which can be regarded as a bijection from A to B, i.e. $\mathscr{M}$ contains n pairs from $A \times B$ with no repetitive vertices.

Definition

A pair $(a, b) \in A \times B$ is a **blocking pair** with respect to $\mathscr{M}$ if all three conditions hold: 1) $(a, b) \notin \mathscr{M}$; 2) a prefers b over $\mathscr{M}(a)$, its partner in the matching $\mathscr{M}$; 3) b prefers a over $\mathscr{M}(b)$.

A matching is **stable** if no pair from $A \times B$ blocks it.

Preliminary: Three-Dimensional Case

Now we have three sets M, W, C (refered as men, women, and cats) with an equal cardinality.

A set of triples (families) $\mathscr{M}$ is a matching if every triple in $M \times W \times C$ belongs to exactly one family in $\mathscr{M}$.

Preliminary: Three-Dimensional Case (Cont.)

Every man (woman, cat) from every group has a **complete** preference order $\mathscr{P}_m$ ($\mathscr{P}_w$, $\mathscr{P}_c$) over $W \times C$ ($M \times C$, $M \times W$), meaning that nobody is "equally" liked by anyone else.

Definition

Like the Two-Dimensional case, a triple (m, w, c) is a **blocking** pair with respect to a matching $\mathscr{M}$ if every member prefers his companies in the triple than those in the family assigned by $\mathscr{M}$. A matching is **stable** if no triple blocks it.

Non-existence in Three Dimensions [2]

Let $n = 2$, and write $M = \{m_1, m_2\}$, $W = (w_1, w_2)$, $C = \{c_1, c_2\}$. So we only have $2^3 = 8$ possible families and four possible matchings:

$$\begin{aligned}
\mathscr{M}_1 &= \{F_{1,a} = (m_1, w_1, c_1), F_{1,b} = (m_2, w_2, c_2)\}, \\
\mathscr{M}_2 &= \{F_{2,a} = (m_1, w_2, c_2), F_{2,b} = (m_2, w_1, c_1)\}, \\
\mathscr{M}_3 &= \{F_{3,a} = (m_1, w_2, c_1), F_{3,b} = (m_2, w_1, c_2)\}, \\
\mathscr{M}_4 &= \{F_{4,a} = (m_1, w_1, c_2), F_{4,b} = (m_2, w_2, c_1)\}.
\end{aligned}$$

Non-existence in Three Dimensions (Cont.)

We then assign the following preference orders:

$$\begin{aligned}
\mathscr{P}_{m_1} &: (w_2, c_1) >_{m_1} (w_1, c_2) >_{m_1} (w_2, c_2) >_{m_1} (w_1, c_1),\\
\mathscr{P}_{m_2} &: (w_2, c_1) >_{m_2} (w_1, c_1) >_{m_2} (w_2, c_2) >_{m_2} (w_1, c_2),\\
\mathscr{P}_{w_1} &: (c_2, m_2) >_{w_1} (c_1, m_2) >_{w_1} (c_1, m_1) >_{w_1} (c_2, m_1),\\
\mathscr{P}_{w_2} &: (c_2, m_2) >_{w_1} (c_1, m_1) >_{w_1} (c_1, m_2) >_{w_1} (c_2, m_1),\\
\mathscr{P}_{c_1} &: (m_1, w_2) >_{c_1} (m_2, w_1) >_{c_1} (m_1, w_1) >_{c_1} (m_2, w_2),\\
\mathscr{P}_{c_2} &: (m_1, w_1) >_{c_1} (m_1, w_2) >_{c_1} (m_2, w_2) >_{c_1} (m_2, w_1).
\end{aligned}$$

Question: which matching(s) are blocked?

Purely Cyclic Preferences

A system with s groups is said to have **purely cyclic preferences** if everyone in A_i (modulo s) has preferences only on members in A_{i+1}.

Proposition

[3] Assume that, under cyclic preferences, a stable matching exists whenever $n \leq k$, then a stable matching also exists whenevere $n = k + 1$ and there is a triple $(m, w, c) \in M \times W \times C$ such that w (m, c) is the most preferred agent for m (c, w).

Purely Cyclic Preferences

For three sides, the state of the art is the existence when all groups has three [2], four [3], or five [5] members.

If $n \leq s$, then a stronger result holds but needs more machineries.

Purely Cyclic Preferences, $n \leq s$

For each member x in A_i, we will denote it as (x, i) to emphasize its membership in A_i. Consider a directed graph $G = (V, E)$, where $V = \bigcup_{r=0}^{s-1} A_r$, and $E = \{(x, j) \to (y, j+1) : j \in \mathbb{Z}_s\}$.

Definition

[2] For every edge $e = [(x, j) \to (y, j+1)]$, its **rank**, $r(e)$, is defined as the position of the vertex $(y, j+1)$ in the preference list of the previous vertex (x, i).

Purely Cyclic Preferences, $n \leq s$ (Cont.)

Based upon the rank, we characterize an "optimal" path on a vertex subset $W \subseteq V$ of G, where the subgraph induced by W is written as $G[W]$.

Definition

[2] A directed path $P = \{(x_j, j) \to (x_{j+1}, j+1) \to \cdots \to (x_{j+q}, j+q)\}$ is said to be a **best choice path** in $G[W]$ if every vertex in the path is in W and every edge in P has the lowest possible rank in $G[W]$, i.e. for every $r \in [q]$, $(x_{j+r}, j+r)$ is the favorable vertex of $(x_{j+r-1}, j+r-1)$ in $G[W]$.

Purely Cyclic Preferences, $n \leq s$ (Cont.)

Based upon the best choice paths, we now characterize the **best choice matching**. The algorithm starts with $W = V$, and we randomly pick a point $a_0^1 \in A_0$. We then write the unique best choice path from a_0^1 as

$$F^1 = \{(a_0^1, 0), (a_1^1, 1), \cdots, (a_{s-1}^1, s-1)\}. \quad (1)$$

Now, delete all vertices in F^1, and randomly pick another vertex $a_0^2 \in A_0 \cap W = A_0 \cap (V \setminus F^1)$, and write the unique best choice path from a_0^2 as

$$F^2 = \{(a_0^2, 0), (a_1^2, 1), \cdots, (a_{s-1}^2, s-1)\}. \quad (2)$$

Let π as a permutation of vertices in A_0, and one choice of π is $\pi = (a_0^1, a_0^2, \cdots, a_0^n)$, and define $\mathscr{M}_\pi = (F^1, \cdots, F^n)$.

Definition

[2] A matching $\mathscr{M}$ for the graph G defined above is a **best choice matching** if $\mathscr{M} = \mathscr{M}_\pi$ for some permutation π on A_0.

Purely Cyclic Preferences, $n \leq s$ (Cont.)

Theorem

[2] Suppose the system is under purely cyclic preferences and contains s groups, where each group has $n \leq s$ members, then a stable best choice matching always exists.

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Thank you!