Applications of Prospect Theory

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Prospect Theory

- Analysis of decisional risk
- First introduced by Daniel Kahneman and Amos Tversky in 1979.
- Human’s utility functions are dependent on how they value risk.

Which of the following would you prefer?

A: 50% chance to win 1,000, B: 450 for sure.
50% chance to win nothing;

- Aims to better explain how agents make choices, specifically compared to expected utility theory:
  - Reference dependence, Loss aversion, Risk aversion / risk seeking, and Probability weights

Reference Dependence

- Utility in Expected Utility Theory:
  - Utility(playing game) = Pr(winning) * **U(W + x)**
  - Agents derive utility from **their final total wealth**
- Utility in Prospect Theory:
  - Utility(playing game) = Pr(winning) * **U(x)**
  - Agents derive utility from **their change in wealth**
  - Temperature, Brightness, Loudness

Loss Aversion

- In Expected Utility Theory:
  - Winning $x is an equal opposite to losing $x, so v(x) = -v(-x)
- In Prospect Theory:
  - Agents are more sensitive to losses than they are to gains.
  - An agent’s value function v(x) is steeper when x is negative

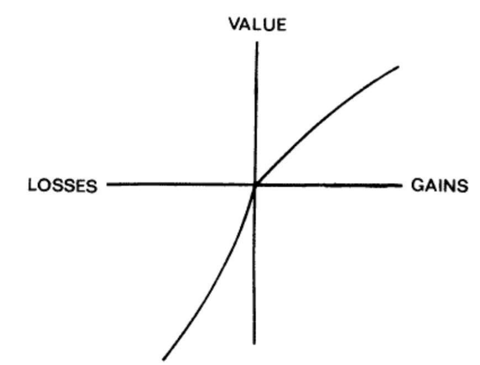


FIGURE 3.—A hypothetical value function.

Risk Aversion + Risk Seeking

- In Prospect Theory:
  - An agent's value function v(x) is
    - Concave when positive
    - Convex when negative
  - Agents are risk seeking with losses and risk averse with gains

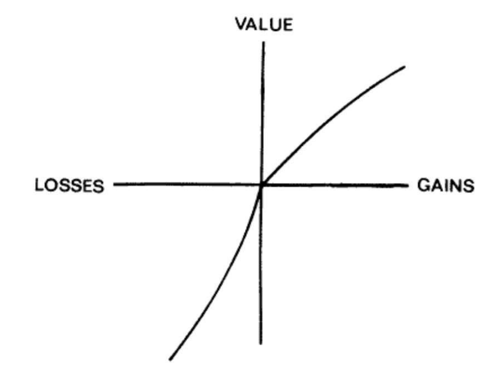


FIGURE 3.—A hypothetical value function.

PROBLEM 11: In addition to whatever you own, you have been given 1,000.
You are now asked to choose between

A: (1,000, .50), and B: (500).

$N = 70$ [16] [84]*

PROBLEM 12: In addition to whatever you own, you have been given 2,000.
You are now asked to choose between

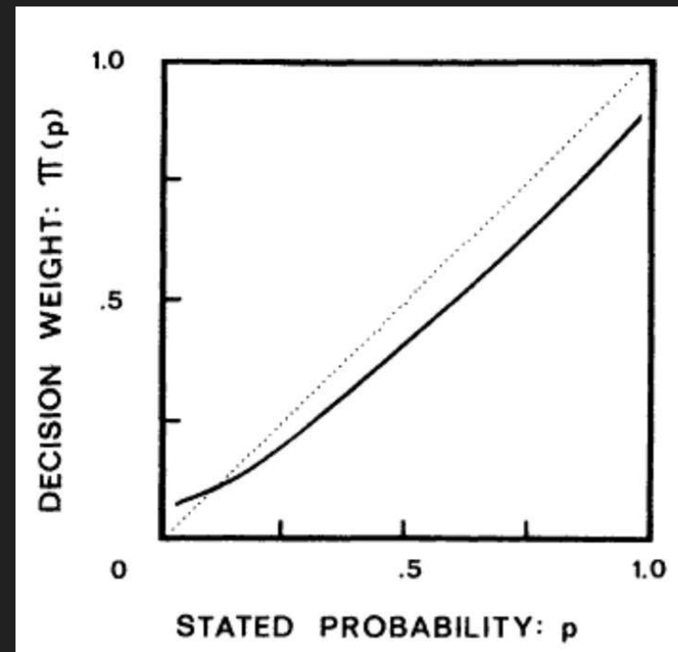
C: (−1,000, .50), and D: (−500).

$N = 68$ [69*] [31]

Probability Weights

- In Expected Utility Theory:
  - Utility = Pr[outcome] * value(outcome)
- In Prospect Theory:
  - We use $\pi = w(p)$ instead of p
  - $w(p) > p$ when p is low, and $w(p) < p$ when p is high
  - $w(p)$ is further from p by 0 and 1

Examples: The Lottery, Dividend Stocks



Model Comparison

Expected Utility:

Given a set of outcome probability pairs, $\{(x_i, p_i)\}$:

- x_i is the gain/loss
- p_i is the probability of event i happening

An agent's utility under Prospect Theory:

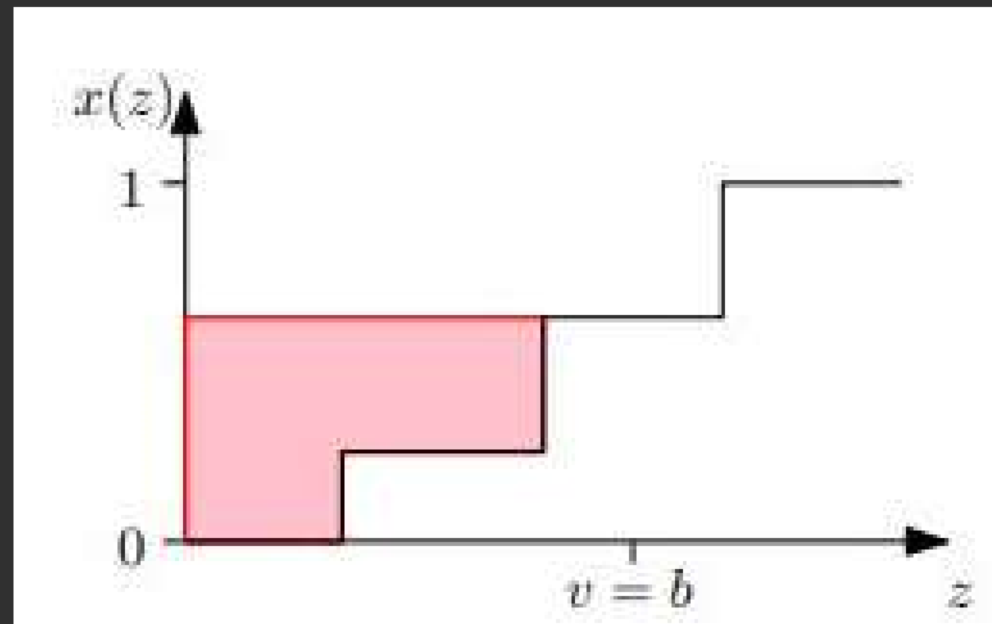
$$\Sigma\ \pi_i * v(x_i)$$

Auction Design

Myerson’s Revenue Maximizing Auction Review

Myerson’s Lemma

- For the single-parameter environment
  - Sealed-bid auction
- We must use the following payment rule:



Picture from [1]

Maximizing Revenue

$$\max_{x} \mathop{\mathbb{E}}_{v\sim D}\left[Rev\right] = \max_{x} \mathop{\mathbb{E}}_{v\sim D}\left[\sum_{i=1}^{n} p_i(v)\right]$$

$$= \max_{x} \mathop{\mathbb{E}}_{v\sim D}\left[\underbrace{\sum_{i=1}^{n} \phi_i(v_i)\, x_i(v_i, v_{-i})}_{\text{virtual social welfare.}}\right]$$

Where agent i's virtual value = $\left[v_i - \left(\frac{1-F_i(v_i)}{f_i(v_i)}\right)\right].$

The assumption

We continue to use a quasilinear utility model, so, in an auction with allocation and payment rules $\mathbf{x}$ and $\mathbf{p}$, respectively, bidder i has utility

$$u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$$

on the bid profile (i.e., bid vector) $\mathbf{b}$.

[1]

What if this assumption is not true???

Maximizing Auction Revenue when agents act under Prospect Theory

Auctions under the Prospect Theory model

In class we learned about Myerson’s Revenue Maximizing Auction = MYE.

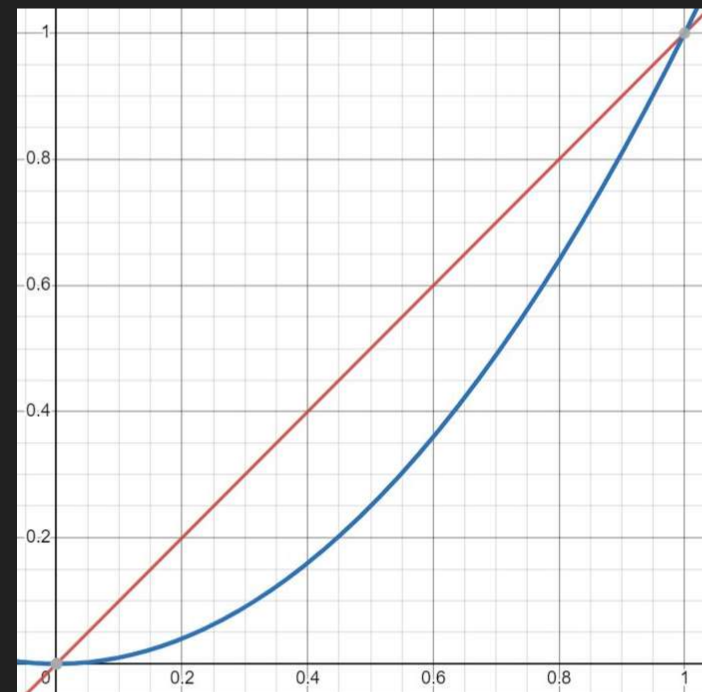
This auction relied on each agent being expected utility maximizing.

If this assumption is changed, we can design auctions that generate more revenue than MYE.

The following auction is based off the paper “Revenue Maximization with an Uncertainty-Averse Buyer” [2]

The Risk Model

- We assume that agents are risk averse
- Let x = the probability of an event
- $0 < w(x) < x$ when $0 < x < 1$
- $w(x)$ is convex
- $w(x)$ is weakly increasing
- $w(0) = 0$ and $w(1) = 1$



The Setting

We also simplify the setting to the following:

- We sell 1 item to 1 buyer
- $v \in$ known distribution F

We know that in the MYE setting a posted price mechanism is optimal, in particular:

p^* when the agent is risk averse = p^* when the agent isn’t

A Motivating Example

Let v be uniformly distributed over $[0, 1]$

We know from class $p^* = \frac{1}{2}$ and $E[\text{revenue}] = \frac{1}{4}$, no matter what $w(x)$ is

Can we do better?

A Motivating Example - Binary Lottery

Yes!

We can add the following option #2: pay a slightly lower price p_2 with probability ½

Now how the agent assesses risk affects the outcome.

For this example, assume $w(x) = x^2$ (the agent is risk averse)

And $p_2 = 3/8$

A Motivating Example - Binary Lottery

Let v_1 = the lowest value where the agent will choose option #1

Let v_2 = the lowest value where the agent will choose option #2

Solving for v_1 and v_2, $v_1 = 13/24$ and $v_2 = 3/8$

$E[revenue] = Pr[v > v_1] * Revenue_{\#1} + Pr[v_2 < v < v_1] * Revenue_{\#2}$

$= 25/96 > 1/4$

So if the agent is risk averse enough, we can make more revenue using this binary lottery option.

The New Auction

Call the new auction A

We present the buyer with a menu of binary lotteries M

Each option = (x_i, p_i)

In each lottery you either get the item or you don’t and pay nothing

Revenue Maximization

Does A maximize revenue?

Lemma: For any w(x) and lottery (x, p), the agent’s utility function is concave wrt v

So the agent’s utility from M = the maximum of concave functions

=> can convert to a convex and non decreasing utility function

The paper uses this information to derive the following theorem:

There exists an optimal ex-post IR mechanism that is described by a menu of binary lotteries.

Assumptions and Limitations

- This auction does not give a nice expression for the revenue it achieves
- Nor does it lead to a clean theorem like in MYE
- We assume a known distribution F
- We assume we know $w(x)$

Revenue Maximization on Variations

F is unknown

The agent is extremely risk averse

THEOREM 3.2. *For every $\varepsilon > 0$ and $H > 1$, if the buyer's weighting function satisfies $y(1-\varepsilon) \leq 2^{-H/\varepsilon}$, there exists a mechanism that for any value distribution F supported over $[1, H]$ obtains revenue at least $1-O(\varepsilon)$ times the buyer's expected value $\mathbf{E}_{v\sim F}[v]$.*

Note: the paper uses y(x) instead of w(x)

y(x) is unknown, but is picked from a monotone non-crossing family of weighting functions

F is supported on [0, H]

THEOREM 3.3. *Let $\mathcal{Y}$ be a monotone non-crossing family of weighting functions and let F be a value distribution supported on $[0, H]$. Then there exists a mechanism $\mathcal{M}$ that for any weighting function in $\mathcal{Y}$ achieves an $O(\log\log H)$ approximation to revenue. Formally, for all $y \in \mathcal{Y}$,*

$$\mathrm{REV}_{y,F}(\mathcal{M}) \geq \Omega\left(\frac{1}{\log\log H}\right) \mathrm{OPT}(y, F).$$

Bounding MYE's performance

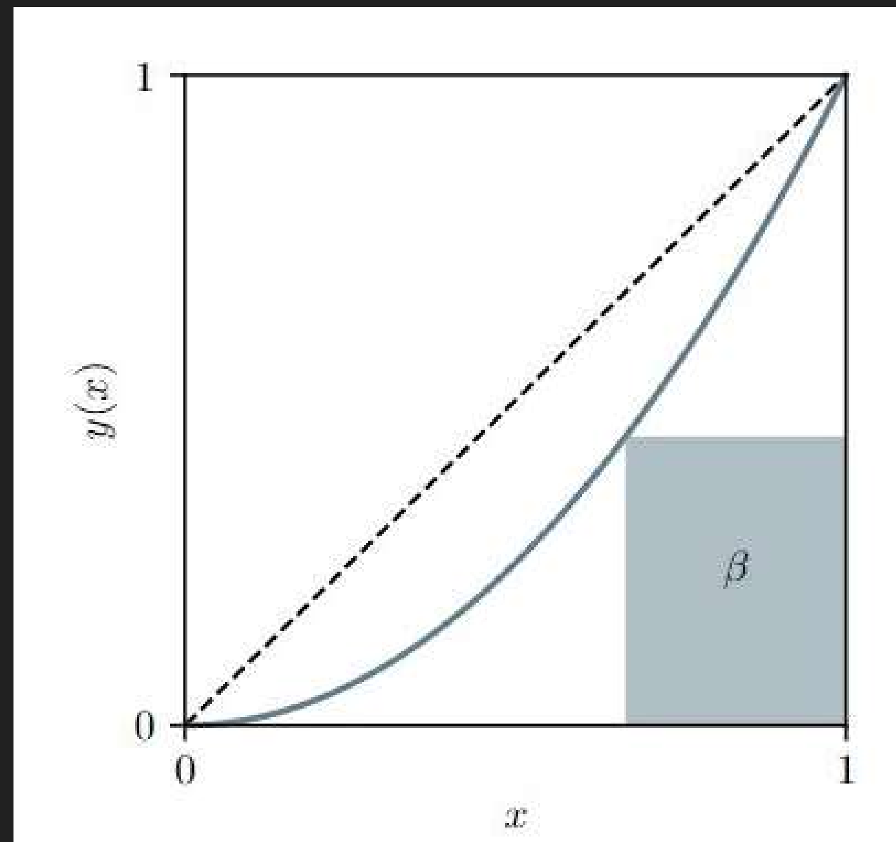
Finding the actual Menu can be challenging

β = maximum area rectangle under w(x)

MYE >= β * OPT

If the agent is more risk averse, β is smaller

So less risk averse = MYE performs better



Results Summarized

- We can extract more revenue from risk averse buyers using binary lotteries
- When agents are less risk averse, MYE provides a decent approximation and is much simpler

Auctions for Risk Seekers

- This paper does not use Prospect Theory
- Instead $u(x) = \beta\ (e^{ax} - 1)$
- The optimal auction:
  - If 1 buyer: Randomized take it or leave it
  - If n >= 2 buyers: Loser pay auction where winner gets a refund

[3]

Future Work

- Auctions for general prospect theory agents
- Auctions for risk seeking prospect theory agents
- Simpler auctions for risk averse agents

Sources

[1] CS364A: Algorithmic Game Theory Lecture #3: Myerson's Lemma. Tim Roughgarden

http://theory.stanford.edu/~tim/f13/l/l3.pdf

[2] Shuchi Chawla, Kira Goldner, J. Benjamin Miller, and Emmanouil Pountourakis: Revenue Maximization with an Uncertainty-Averse Buyer. In: Proceedings of the 2018 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). 2018, 2050-2068

https://epubs.siam.org/doi/epdf/10.1137/1.9781611975031.134

[3] Nikolova, E., Pountourakis, E., Yang, G.: Optimal mechanism design with risk-loving agents. In: International Conference on Web and Internet Economics, pp. 375–392. Springer (2018)

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