Evolutionary game theory

Concept & Application & Theory

Equilibrium selection problem

- NE exists for any game with a finite number of players and strategies

- NE is guaranteed to be unique

- NE does not always seem reasonable

Static concept of Evolutionary stable strategy (ESS)

1. $\pi(\sigma \mid \sigma) \geq \pi(\mu \mid \sigma)$ (NE)

2. If $\pi(\sigma \mid \sigma) = \pi(\mu \mid \sigma)$, then $\pi(\sigma \mid \mu) > \pi(\mu \mid \mu)$ $\mu \neq \sigma$

Example

	Hunter A		
Hunter B		Hunt Stag	Hunt Hare
	Hunt Stag	4, 4	0, 4
	Hunt Hare	4, 0	3, 3

Analysis

- Hunt Stag is NE, but not ESS.
- Hunt Hare is both NE and ESS.
- Hare hunters are better than stag hunter no matter the choice of the other.

	Hunter A		
Hunter B		Hunt Stag	Hunt Hare
	Hunt Stag	4, 4	0, 4
	Hunt Hare	4, 0	3, 3

Uniform invasion boundary

- For strategy σ, there exists some $\epsilon' > 0$ such that for all $0 < \epsilon < \epsilon'$

$$\pi(\sigma \mid \epsilon\mu + (1 - \epsilon)\sigma) > \pi(\mu \mid \epsilon\mu + (1 - \epsilon)\sigma)$$

Local superiority

- σ is locally superior if there exists a neighborhood U around σ such that for all strategies $\mu \in U$, $\mu \neq \sigma$, $\pi(\sigma \mid \mu) > \pi(\mu \mid \mu)$

Theorem Hofbauer et al., 1979

- The following are equivalent:

1. σ is an evolutionarily stable strategy
2. σ has a uniform invasion barrier
3. σ is locally superior

Dynamic Concept of ESS (Replicator Dynamics)

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- For each strategy S_i, there is a proportion p_i of the population adopting it.
- Each agent selects someone else to play at random and compare the payoffs, then adopts the strategy from the agent who had a higher payoff.
- In summary, an equation can be written as follows

$$\frac{dp_i}{dt} = (\text{ Rate at which people start using } S_i) - (\text{ Rate at which people stop using } S_i)$$

Application: Cake Cutting (Skyrms, 1996, pp. 3–4)

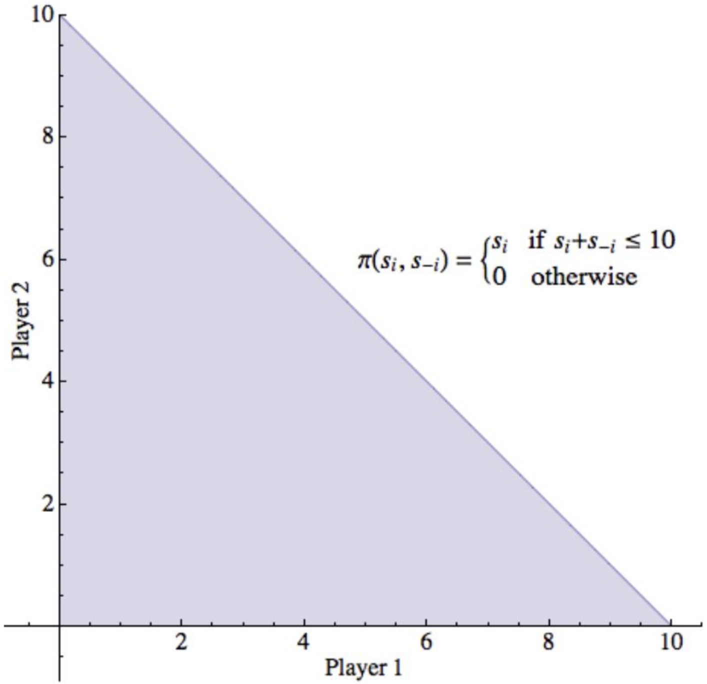
- A chocolate cake of 10 portions
- Neither of us has any special claim as against the other
- If we cannot agree how to share it, the cake will spoil and we will get nothing

Analysis

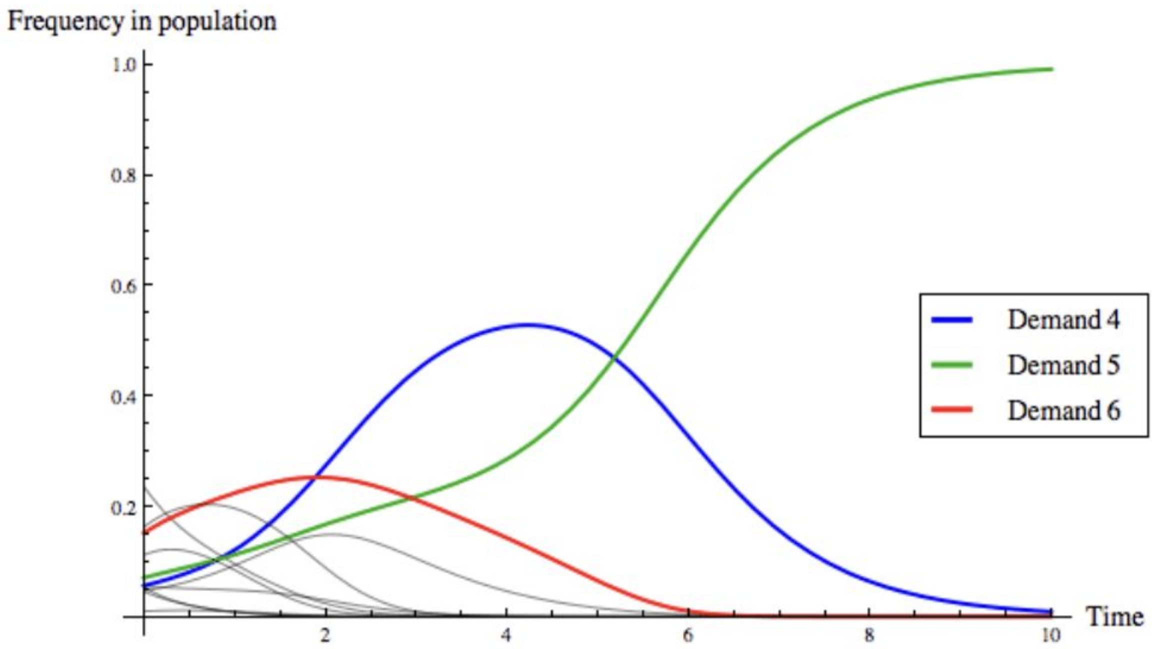
- The set of all feasible choices:

- All strategies on the diagonal is NE

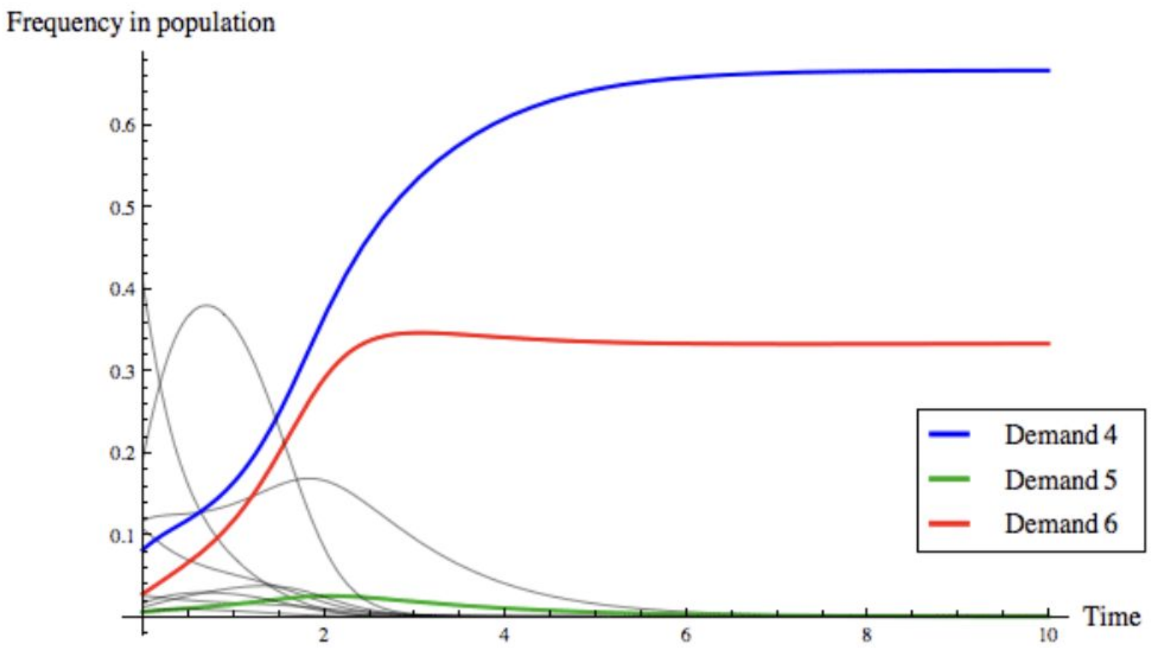
- Rational choice “seems” to be half and half?



Replicator Simulation



Replicator Simulation 2



Rescue

- Evolution rate of Demand 5 is approximately 62%

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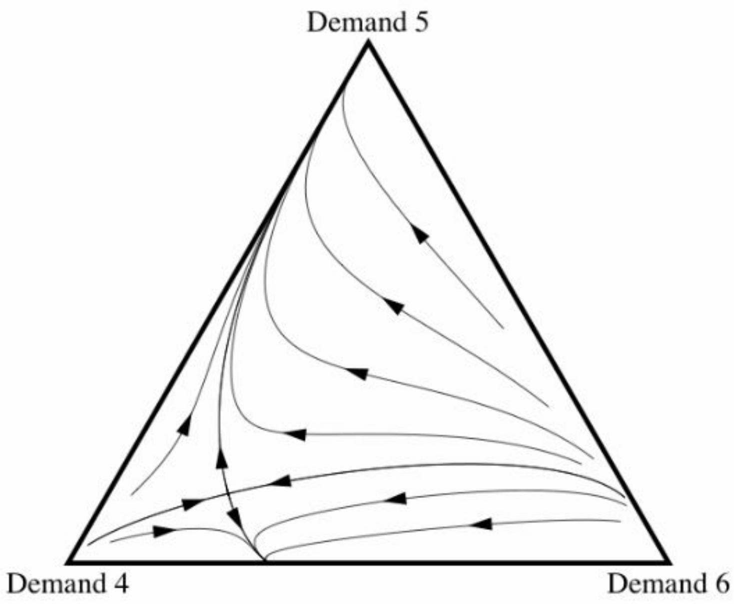
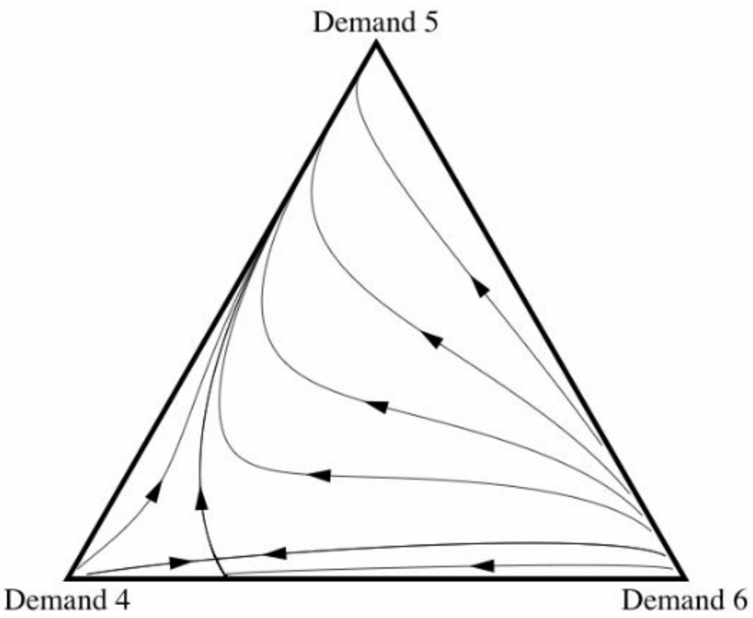
- Evolution rate of Demand 5 is approximately 62%
- What about adding correlation coefficient?
  - A correlation coefficient determines the percentage of time a player is played against its own kind

Result

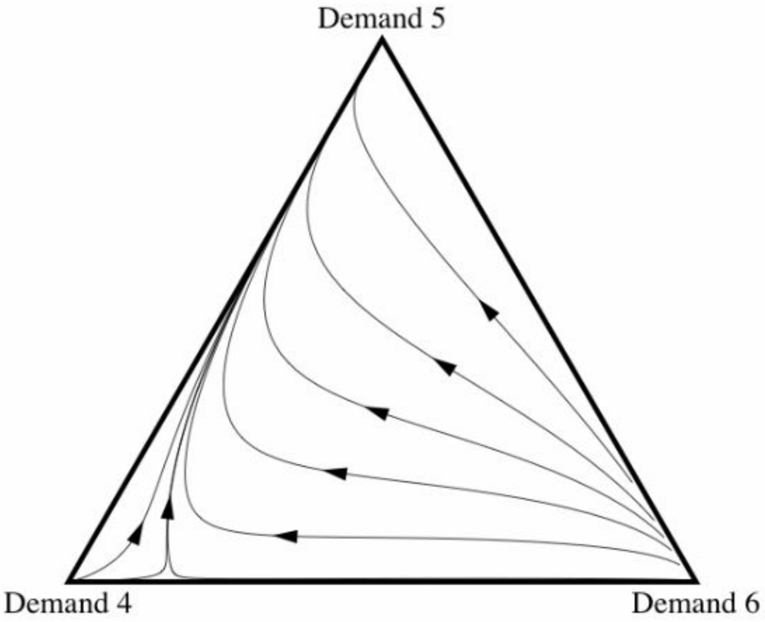


(a) $\varepsilon = 0$



(b) $\varepsilon = 0.1$

Result cont.



(c) $\varepsilon = 0.2$

Other applications

- Emergence of Language (*Why We Talk: The Evolutionary Origins of Language)*
- Sociology
- Psychology

ESS is NP-hard (Etessami, 2008)

- Reduction from SAT

Definitions

- Φ: Boolean formula in CNF
- V = {x1, . . . , xn} the set of Φ's variables
- L = {x1, ¬x1, x2, ¬x2, . . . , xn, ¬xn} the set of literals over V
- C = {c1, . . . , cr} ⊆ 2^L − {∅} the set of clauses of Φ (the empty clause is not allowed).
- Define the function $\chi : C \times L \mapsto \left\{ \frac{n-1}{n}, -1 \right\}$ as follows:

$$\chi(c, l) = \begin{cases} \frac{n-1}{n} & \text{if } l \notin c \\ -1 & \text{if } l \in c \end{cases}$$

Payoff matrix for player 1 (symmetric for player 2)

		L: x_1 $\neg x_1$ $\cdots$ x_n $\neg x_n$	C: c_1 $\cdots$ c_r
L	x_1, $\neg x_1$, $\vdots$, x_n, $\neg x_n$	B	-1
C	c_1, $\vdots$, c_r	$\chi(c, l)$	-1

$$B = \begin{pmatrix} 0 & -2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -2 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & -2 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & & & & \ddots & & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -2 & 0 \end{pmatrix}$$

Intuition

- Only strategies correspond to truth assignments to the variables in V are potentially an ESS.

Proof

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    - If $s(c) = 1$, then choosing any literal of c for player 2 is a best response (payoff -1), then player 1 choosing the same literal will yield a better payoff (0), violating NE

Proof

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- Claim 1: s(c) = 0 for all c ∈ C.
    - If s(c) = 1, then choosing any literal of c for player 2 is a best response (payoff -1), then player 1 choosing the same literal will yield a better payoff (0), violating NE
    - If 0 < s(c) < 1, then some strategy s' equal s except for choosing a different c' such that s'(c) = s(c) will result in U(s | s) = U(s' | s) and U(s | s') = U(s' | s'), violating ESS

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- Claim 2: for each variable at least one corresponding literal is played.
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- Then let l is such that s(l) = s(¬l) = 0 U(l | s) = 1 > U(s | s), so s is not NE

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- It can be shown that U(s, t) <= U(t, t), contradicting s being ESS

Proof

- Claim 4: Suppose the n literals in (l1, …, ln) forms a satisfying assignment for SAT Φ, then it must also be ESS.

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- For any satisfying assignment, every clause contains a literal that is played, therefore,

$$U(c \mid s) = \sum_{l \in L'} s(l)u(c,l) = (-1) \sum_{l \in L' \cap c} s(l) + \frac{n-1}{n} \sum_{l \in L' - c} s(l) < \frac{n-1}{n} = U(s \mid s)$$

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- Suppose t is a best response to s, then it can be shown that U(s | t) = U(t | s) = U(s | s) = (n-1)/n
- As proven in a lemma, the following equality holds only when t = s

$$\frac{n-1}{n} \geq t^T A\, t = U(t \mid t)$$

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- Then, there is a clause c such that none of its literal is played, then

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- Then U(c | c) = -1 = U(s | c), so s is not ESS, a contradiction.

ESS complexity classes

- NP-hard
- coNP-hard
- #P-hard
- Σ_2^p (the second level of the polynomial time hierarchy)

Concluding Remark

- Currently there are still disagreements between the dynamic and static notion of ESS
- ESS application is mostly philosophical (for explaining phenomena), rather than practical (for predicting and designing systems)