When can the 4/5 Maxmin Share be Guaranteed?

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1 Project Summary

Maxmin share (MMS) is an important concept in fair division. We study the case where there are n agents and m indivisible goods under additive valuations. An MMS allocation is that for each agent i, its value should be at least its maximum value of the least preferred bundle (denoted μ_i). The multiplicative approximation to MMS is that an $\alpha-$MMS allocation makes each agent i's value at least $\alpha\mu_i$. An MMS allocation always exists when $n = 2$. However, Procaccia and Wang [1] showed that an MMS allocation may not exist for $n \geq 3$, but they did show that an 2/3-MMS allocation always exists. Later, Ghodsi et al. [2] showed that a 3/4-MMS allocation always exists. However, the techniques require the agent's MMS value so it is not an efficient algorithm (finding the MMS value for an agent is NP-hard). Recently, Garg and Taki [3] improved that factor to 3/4 + 1/12n with a efficient polynomial algorithm by clever strategy for reduction and bag fill. Until now, 3/4 is still the best factor for $n > 4$.
In this project, I will dive into the problem of existence of 4/5-MMS allocation. I finished the proof of 4/5-MMS allocation for 4 agents using a similar pairing strategy as [3].

2 Properties of MMS Allocation

These properties of MMS allocation are already proved in the reference papers. The proof of these properties are omitted because the reference papers are given.

2.1 Definition of MMS Allocation and α-MMS Allocation

Let $\langle N, M, V\rangle$ denote a fair devision problem, and let

$$\Pi_n(M) = \{P = \{P_1, \ldots, P_n\} \mid P_i \cap P_j = \emptyset, \forall i, j; \cup_k P_k = M\}$$

to be the set of all partitions of M into n bags. Agent i is MMS value $\mu_i(M)$ is defined as

$$\mu_i(M) = \max_{P \in \Pi_n(M)} \min_{P_k \in P} v_i\left(P_k\right)$$

The allocation $A = (A_1, \ldots, A_n)$ is α-MMS ($\alpha \leq 1$) if each agent i receives the bag A_i is worth at least α times its MMS value $\mu_i(M)$, $v_i(A_i) \geq \alpha\mu_i(M)$.

2.2 Upper Bound

$$\mu_i(M) \leq \frac{v_i(M)}{n}$$

2.3 Scale Invariant

Let $A = (A_1, \ldots, A_n)$ be an α-MMS allocation for the problem $\langle N, M, V \rangle$. Then an alternative problem with $\langle N, M, V' \rangle$ where the valuations of each agent is scaled by $c_i > 0$, i.e. $\forall j \in M, v'_{ij} := c_i v_{ij}$, then $\mu'_i = c_i \mu_i$ is an α-MMS allocation for $\langle N, M, V' \rangle$.

2.4 Same Order of Preferences

We say that $\langle N, M, V \rangle$ is ordered if

$$\forall i \in N, v_{i1} \geq v_{i2} \geq \cdots \geq v_{im}$$

In other words, all the agents have the same ordered of preferences over all items. Bouveret and Lemaitre [5] showed that same order of preferences is the worst case finding the MMS-allocation. Later, Barman and Krishnamurthy [6] extended the results to α-MMS allocations.

2.5 Bag Filling

First normalize the problem $\langle N, M, V \rangle$ such that $\forall i, \mu_i(M) = 1$. Then if $\forall i \in N, \forall j \in M, v_{ij} < \beta$, we can find a $(1-\beta)$-MMS allocation by the following bag filling algorithm:

- Start with an empty bag B.
- Keep adding items to B until some agent i values it $\geq (1-\beta)$
- Assign B to i and $N \leftarrow N \backslash \{i\}, M \leftarrow M \backslash B$

Proof of correctness of the bag filling algorithm: when the bag is assigned to some agent, for all the other agents, there valuation for this bag is at most $1 - \beta + \beta = 1$. Because the total value $v_i(M) \geq |N|$, there are enough items to make every bag at least $(1-\beta)$.
In the next section, we designed a modified bag filling procedure.

2.6 Valid Reduction

The valid reduction is widely used in finding α-MMS allocation [1,2,3,4,7,8]. Here is the defintion of valid reduction: to get α-MMS allocation, the act of assigning a set $S \subseteq M$ to some agent i is a valid reduction if

$$v_i(S) \geq \alpha \cdot \mu_i^{|N|}(M)$$

$$\mu_{i'}^{|N|-1}(M \backslash S) \geq \mu_{i'}^{|N|}(M), \quad \forall i' \in N \backslash \{i\}$$

In other words, a valid reduction converts the original problem $\langle N, M, V \rangle$ to $\langle N \backslash \{i\}, M \backslash S, V \rangle$ by assigning the bag S to the agent i without decreasing the MMS value for other agents.

3 Proof of 4/5-MMS Allocation for 4 agents

In this section, I give the proof of 4/5-MMS Allocation for 4 agents. First denote the problem as $\langle N, M, V\rangle$ where N is the set of agents, M is the set of items, and V is the valuations. Then $n := |N|$ is the number of agents and $m := |M|$ is the number of items. In our problem, $n = 4$. And, use j to denote the j^{th} highest value item in M (only consider same order of preferences for all agents because it is the most difficult situation as described in 2.4).
First, since the MMS problem is scale invariant, rescale the value of items such that $\forall i, \mu_i(M) = 1$. Then from section 2.2, the total value of M is at least 4:

$$v_i(M) >= \mu_i(M)n = 1 \times 4 = 4$$

Algorithm 1 3/4-MMS Allocation

Require: $\langle N, M, V\rangle$, $n = 4$, $\forall i, v_{i1} \geq v_{i2} \geq \cdots \geq v_{im}$
1: Normalize values of items to make $\forall i, \mu_i(M) = 1$.
2: $(N, M, V) \leftarrow Initial - Assignment(N, M, V)$
3: Bag-Filling(N, M, V)

The algorithm is given by Algorithm 1. Next, we will show that Algorithm 1 guarantees the 3/4-MMS allocation. Further, denote $J_1 := \{1, 2, 3, 4\}$ as the set of first four items, $J_2 := \{5, 6, 7, 8\}$ as the set of second four items, $J_3 := \{9, 10, 11, 12\}$ as the set of third four items and $J := J_1 \cup J_2 \cup J_3$. J is the set of high value items and denote $R = M \backslash J$ as the set of low value items.

Algorithm 2 Initial-Assignment

1: For any $S \in M$, define $\Gamma(S) := \{i \in N : v_i(S) \geq \alpha\}$
2: $S_1 \leftarrow \{1\}$, $S_2 \leftarrow \{4, 5\}$, $S_3 \leftarrow \{7, 8, 9\}$, $S_4 \leftarrow \{10, 11, 12, 13\}$
3: **while** $(\Gamma(S_1) \cup \Gamma(S_2) \cup \Gamma(S_3) \cup \Gamma(S_4)) \neq \emptyset$ **do**
4: $\quad S \leftarrow$ the bag S in $\{S_1, S_2, S_3, S_4\}$ where $\Gamma(S) \neq \emptyset$
5: $\quad i \leftarrow$ an agent in $\Gamma(S)$
6: $\quad$ Assign S to agent i
7: $\quad M \leftarrow M \backslash S$, $N \leftarrow N\ \{i\}$
8: **end while**

Lemma 1. *After initial assignment,* $\forall i$, *(i)* $\forall j \in J_1, v_{ij} < \frac{4}{5}$, *(ii)* $\forall j \in J_2, v_{ij} < \frac{2}{5}$, *(iii)* $\forall j \in J_3, v_{ij} < \frac{4}{15}$, *(iv)* $\forall j \in R, v_{ij} < \frac{1}{5}$.

Proof. It is obvious that the bag S satisfies the first condition of valid reduction, i.e. $v_i(S) \geq \frac{4}{5}$. Therefore, we only need to show the second condition of valid reduction, i.e. $\mu_{i'}^{n-1}(M \backslash S) \geq \mu_{i'}^{n}(M)$: (let P denote the partition achieving $\mu_{i'}^{n}(M)$ for some i')

- $S = S_1$, removing one item from the allocation P only affects one bag, so each of the remaining $(n-1)$ bags has value at least the original MMS value.
- $S = S_2$, there exists one bag with two items of $\{1, \ldots, n, n+1\}$ from P by the pigeonhole principle. Denote this bag $B = \{j_1, j_2\}$ containing items j_1 and j_2. Clearly, for any agent i', $v_{i',j_1} \geq v_{i',n}$ and $v_{i',j_2} \geq v_{i',n+1}$. Therefore, swap j_1 and n and swap j_2 and $n+1$ and move all other items in B to other bags. Notice that now bag B only contains n and $n+1$. Therefore, assigning bag B to i do not decrease the value of other bags in P for any other agent i'.
- $S = S_3$, similar to $S = S_2$. Find the three-item bag from $\{1, \ldots, 2n, 2n+1\}$, and the proof is similar.
- $S = S_4$, similar to $S = S_2$. Find the four-item bag from $\{1, \ldots, 3n, 3n+1\}$, and the proof is similar.

Then we show that the initial assignment is a process of valid reduction. Then after initial assignment, we have $\forall i, \forall k \in [4], v_i(S_k) < \frac{4}{5}$. Therefore (i) $v_{i1} = v_i(S_1) < \frac{4}{5}$ and all items in J_1 is at most the value of item 1. (ii) $v_{i5} < \frac{1}{2}(v_{i4} + v_{i5}) = \frac{1}{2}v_i(S_2) < \frac{2}{5}$ and all items in J_2 is at most the value of item 5. (iii) $v_{i9} < \frac{1}{3}(v_{i7} + v_{i8} + v_{i9}) = \frac{1}{3}(v_i(S_3)) < \frac{4}{15}$ and all items from J_3 is at most the value of item 9. (iv) $v_{i13} < \frac{1}{4}(v_{i10} + v_{i11} + v_{i12} + v_{i13}) = \frac{1}{4}(v_i(S_4)) < \frac{1}{5}$ all items from R is at most the value of item 13.

If the valid reduction occurs in the initial assignment stage, the number of agents becomes three or less. We know that for two agents there always exists MMS-allocation [2] and for three agents, there exists a 7/8-MMS allocation [4]. Therefore, in the following, we only consider the case when there are still four agents. The following Bag-Filling strategy is very similar to Garg and Taki [3]. They use this strategy to prove the 3/4-MMS allocation, where we extend their strategy to prove 4/5-MMS allocation for four agents.

Algorithm 3 Bag-Filling

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1: Initialize Bags B = {B_1, B_2, B_3, B_4} where k ∈ [4], B_k = {k, 9 − k}.  ▷ See Figure 1
2: Define Γ(bag) := {i ∈ N : v_i(bag) ≥ 4/5, bag ⊂ M}
3: R' ← R ∪ J_3
4: for k=1 to 4 do
5:     T ← B_k
6:     while Γ(T) = ∅ do
7:         T ← T ∪ {j}, R' ← R'\{j}          ▷ Add one low value item to T
8:     end while
9:     Pick i ∈ Γ(T), assign T to i, N ← N\i
10: end for
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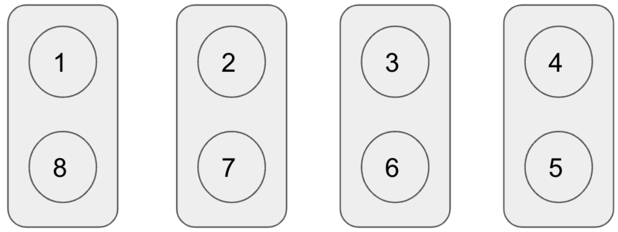

Fig. 1. Bag initialization

The proof is devided into two cases by the value of the highest item.

Case 1: $\forall i, v_{i1} <= \frac{2}{3}$

This case limits the value of the highest item to be $\frac{2}{3}$, which is a more strict limit than Lemma 1.

Lemma 2. *In this Bag-Filling algorithm, when bag* B_k *is assigned to some agent, the value of this bag has an upper bound for all agents.*

$$\forall i, v_i(B_k) < \frac{16}{15}$$

Proof. The bag B_k has three situations as the following:

- When $B_k = \{k, 9-k\}$, we know that $k \in J_1$ and $k+4 \in J_2$, so $v_i(B_k) = v_{ik} + v_{i(k+4)} < \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$ ($\frac{2}{3}$ is the assumption).
- When $B_k = \{k, 9-k\} \cup r$ where $r \subset R'$, $v_i(B_k) < \frac{4}{5}$ before the last item added. The last item j comes from J_3 or R, so $v_{ij} < \max(J_3, R) = \frac{4}{15}$. Therefore, $v_i(B_k) < \frac{4}{5} + \frac{4}{15} = \frac{16}{15}$.

Theorem 1. *After Bag-Filling, each agent* i *receives a bag* B_k *and* $v_i(B_k) \geq \frac{4}{5}$*, so the algorithm gives a 4/5-MMS allocation.*

Proof. We prove this theorem by contradiction.
Assume that $\exists i \in N, \exists B_k, v_i(B_k) < \frac{4}{5}$, then the average value of other three bags:

$$h \neq k, avg(v_i(B_h)) = \frac{1}{3}(M \backslash B_k) = \frac{1}{3}(v_i(M) - v_i(B_k)) > \frac{1}{3}(4 - \frac{4}{5}) = \frac{16}{15}$$

Then there is some bag B'_k and $v_i(B'_k) > \frac{16}{15}$, which contradicts to Lemma 2. This completes the proof of **Case 1**.

Case 2: $\exists i, v_{i1} > \frac{2}{3}$

To show the correctness of the Bag-Filling strategy in this case, we need to show that there are enough items in R' to add to B_k so that each agent gets at least $\frac{4}{5}$. Then we divide the agent into two types:

$$N^1 := \{i \in N \mid v_i(B_k) \leq 1, \forall k\} \text{ and } N^2 := N \backslash N^1$$

If N^2 is empty, then when the bag is assigned to some agent, the value of the bag is at most $\frac{4}{5} + \frac{4}{15} = \frac{16}{15}$ (similar to Lemma 3.2). Then each agent must gets a bag at least $\frac{4}{5}$ (see Theorem 3.3). Therefore, we need to deal with the situation when N^2 is not empty.
For an agent $i \in N^2$, define

$$L_i := \left\{B_k : v_i(B_k) < \frac{4}{5}\right\}; \quad l_i := |L_i|$$
$$H_i := \{B_k : v_i(B_k) > 1\}; \quad h_i := |H_i|$$

L_i is the set of bags that values less than $\frac{4}{5}$, H_i is the set of bags that values greater than $\frac{4}{5}$. l_i and h_i are the number of bags of L_i and H_i.

Lemma 3. *For an agent $i \in N^2$, $l_i > 0$ and $h_i > 0$.*

Proof. If $h_i = 0$, then the agent $i \in N^1$, not N^2. Consider the bag $B_4 = \{4, 5\}$, in the Initial-Assignment we show that $v_{i4} + v_{i5} < \frac{4}{5}$, then $B_4 \in L_i$, so $l_i > 0$.

Because the proof for different l_i is strongly different, we show the proof in three subcases.

Case 2.1: $l_i = 1$

If $l_i = 1$, then $L_i = \{B_4\}$, denote $x_i = \frac{4}{5} - v_i(B_4)$ to be the difference value. Denote $J' := J_1 \cup J_2$, now we only need to show that

$$v_i(M \backslash J') \geq x_i \tag{1}$$

Showing (1)

Lemma 4. *(i) $x_i < \frac{2}{5}$, (ii) $v_{i4} < \frac{3}{5}$*

Proof. We know that $h_i > 0$, so for $B_k \in H_i$, $v_{i,9-k} = v_i(B_k) - v_{ik}$. By the definition of H_i, $v_i(B_k) > 1$. From Lemma 1, $v_{ik} < \frac{4}{5}$. Then $v_{i,9-k} = v_i(B_k) - v_{ik} > 1 - \frac{4}{5} = \frac{1}{5}$. We know that $9 - k > 5$ and the value is ordered, so $v_{i4} \geq v_{i5} \geq v_{i,9-k} = \frac{1}{5}$, then we have (i) $x_i = \frac{4}{5} - v_i(B_4) < \frac{4}{5} - \frac{1}{5} - \frac{1}{5} = \frac{2}{5}$ and (ii) $v_{i4} = v_i(B_4) - v_{i5} < \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$.

Denote $P_i(M) = \{P \in \Pi_n(M) : \min_{P_k \in P} v_i(P_k) \geq 1\}$, notice $P_i(M)$ contains the MMS partition of the agent i, so it is not empty.

Lemma 5. *For every $i \in N^2$, if there exists a partition $P = \{P_1, P_2, P_3, P_4\} \in P_i(M)$ such that there are some $P_k \in P$ which contains two or more items j with $v_{ij} > \frac{3}{5}$, then $v_i(M \backslash J') \geq \frac{2}{5}$.*

Proof. If the P_k contains more than three items valued more than $\frac{3}{5}$, we can simply remove one item and add it to other bag, and P_k is still greater than 1, so we only need to consider the case that P_k contains exactly two large value items. Notice that only the first three items can have value more than $\frac{3}{5}$ by Lemma 1 and Lemma 4(ii). Therefore, there is at most one bag contains two items greater than $\frac{3}{5}$, then denote this bag P_k where $P_k \cap J_1 = \{k_1, k_2\}$ where $k_1, k_2 \in \{1, 2, 3\}$. Then denote $k_3 := \{1, 2, 3\} \backslash \{k_1, k_2\}$ and P'_k where $k_3 \in P'_k$.
Case 1: If $P'_k \cap J' = \{k_3\}$, then $v_i(P'_k \backslash J') > \frac{1}{5}$ because $v_{i,k_3} < \frac{1}{5}$. For the remaining five items $\{4, 5, 6, 7, 8\}$, there exists one P''_k where P''_k constains two items from $\{4, 5, 6, 7, 8\}$ and denote these two items j_1 and j_2. $v_i(P''_k \backslash J') = 1 - (v_{i,j_1} + v_{i,j_2}) \geq 1 - (v_{i4} + v_{i5}) > 1 - \frac{4}{5} = \frac{1}{5}$ (from Lemma 1). Then there are two bags contain more than $\frac{1}{5}$ for the value of low value items, and therefore $v_i(M \backslash J') > \frac{2}{5}$.
Case 2: If P'_k contains k_3 and one or more items from $\{4, 5, 6, 7, 8\}$, the remaining two bags contains four or less items from $\{4, 5, 6, 7, 8\}$. The value of these four items is smaller than $v_{i4} + v_{i5} + v_{i6} + v_{i7} \leq 2 \times (v_{i4} + v_{i5}) = 2 \times \frac{4}{5} = \frac{8}{5}$, so the value of the low value items is greater than $2 - \frac{8}{5} = \frac{2}{5}$.

In the situation of Lemma 5, $v_i(M \backslash J') \geq \frac{2}{5} > x_i$ (Lemma 4(i)), then we are done.
Therefore, we only need to deal with the situation when $P = \{P_1, P_2, P_3, P_4\} \in P_i(M)$ such that each $P_k \in P$ has at most one item j with $v_{ij} > \frac{3}{5}$.
Let a be an agent in N^2 and $P = (P_1, P_2, P_3, P_4) \in \mathcal{P}_a(M)$ be the partition satisfying the condition that each $P_k \in P$ has at most one item j with $v_{ij} > \frac{3}{5}$. We can manipulate P as followings: first, for each $B_i = \{j, j'\}$ with $\frac{4}{5} \leq v_a(B_i) \leq 1$, if $j \in P_k$ and $j' \in P_{k'}$, we can make P_k and $P_{k'}$ into two new bags $\{j, j'\}$ and $((P_k \cup P_{k'}) \setminus \{j, j'\})$ and it is obviously to see that $v_a(((P_k \cup P_{k'}) \setminus \{j, j'\})) \geq 1$. Therefore, we can assume that for each $B_i \notin L_a \cup H_a$, there exists a $P_k = B_i$ and all other bags value at least 1. Then we re-enumerate the bags in P such that

- $P_1, \ldots, P_t$: each has an item j of value more than $\frac{3}{5}$ and $v_a(B_j) > 1$ (Notice that $t = h_a$).
- P_{t+1}: the remaining bag.
- $P_{t+1}, \ldots, P_4$: each $P_k = B_{k'}$ for some k' and $\frac{4}{5} \leq v_a(B_{k'}) \leq 1$.

In the bags H_a, there are exactly $2h_a$ items from J'. However, in $\{P_1, \ldots, P_t\}$, we know that each bag has one item with value $> \frac{3}{5}$ but we don't know the exactly number of items from J'. Then we define z as the possible difference: $z := \max \left\{ 2h_a - \left| \bigcup_{k=1}^{t} P_k \cap J \right|, 0 \right\}$.

Lemma 6. $v_a(M \backslash J') \geq \max \left\{ x_a + \frac{1}{5} - \frac{2}{5}z + \frac{z}{5}, \frac{z}{5} \right\}$

Proof. The $\frac{z}{5}$ part comes from when $z > 0$, there are at least z bags in $\{P_1, \ldots, P_t\}$ with exactly one item from J', and each of these bags need at least $\frac{1}{5}$ to become 1.
For the first part, if $z = 0$, there are at least $2h_a$ items from J' in $\{P_1, \ldots, P_t\}$ and 2 items from J' in P_{t+1}. Then the bag P_{t+1} need at least $x_a + \frac{1}{5}$ from $M\backslash J'$ to become 1. If $z > 0$, the bag P_{t+1} needs at least $x_a + \frac{1}{5} - \frac{2}{5}z$ because each of the z items has value at most $v_{a5} < \frac{2}{5}$. Therefore, in total, there are at least $x_a + \frac{1}{5} - \frac{2}{5}z + \frac{1}{5}z$ value of items from $M\backslash J'$ in $\{P_1, \ldots, P_t, P_{t+1}\}$. Notice that if the value of the first part $x_a + \frac{1}{5} - \frac{2}{5}z + \frac{1}{5}z$ is negative when z is large, we can just count the small value items in $\{P_1, \ldots, P_t\}$, which gives us at least $\frac{1}{5}z$. Therefore, $v_a(M\backslash J') \geq \max\left\{x_a + \frac{1}{5} - \frac{2}{5}z + \frac{z}{5}, \frac{z}{5}\right\}$.

Theorem 2. $v_a(M\backslash J') \geq x_a$

Proof. If $z \geq 2$, $v_a(M\backslash J') \geq \frac{z}{5} \geq \frac{2}{5} > x_a$ (Lemma 4(i)).
If $z < 2$, which means $z \leq 1$, and then

$$v_a(M\backslash J') \geq x_a + \frac{1}{5} - \frac{2}{5}z + \frac{z}{5} = x_a + \frac{1}{5} - \frac{1}{5}z \geq x_a + \frac{1}{5} - \frac{1}{5} = x_a$$

This completes the proof of **Case 2.1**.

Case 2.2: $l_i = 2$

Lemma 7. *If $\exists B_k \in L_i, \forall B_{k'} \in H_i, k > k'$, then we have enough value to add to the bags in L_i to make them $\frac{4}{5}$*

Proof. First denote $x := \max_t(v_{it})$ where $t \in [4]$ and $B_t \in H_i$, then $k > t$, so $v_{ik} \geq v_{it} = x$. We also have $v_i(B_k) < \frac{4}{5}$ because $B_k \in L_i$, then $v_{i9} \leq v_{i,9-k} < \frac{4}{5} - x$. Therefore, the value needed for two bags in L_i when the algorithm is finished is (total value):

$$\frac{4}{5} + \frac{4}{5} + \frac{4}{5} - x = \frac{12}{5} - x$$

If $h_i = 1$, the value of the other two bags when the algorithm is finished is smaller than $x + \frac{2}{5} + 1$. If $h_i = 2$, the value of the other two bags when the algorithm is finished is smaller than $2(x + \frac{2}{5})$. Then, the value for the bags in L_i is greater than:

$$\begin{aligned}
\text{Value of two bags in } L_i &\geq 4 - \max(x + \frac{2}{5} + 1, 2(x + \frac{2}{5})) \\
&= 4 - 2(x + \frac{2}{5}) \text{ , because } x > \frac{2}{3} \\
&= \frac{16}{5} - 2x \\
&\geq \frac{12}{5} - x \text{ , because } x < \frac{4}{5}
\end{aligned}$$

Then we have enough value of items to make two bags $\frac{4}{5}$ in this situation.

By Lemma 7, we only need to deal with the situation that $\forall B_k \in L_i, \forall B_{k'} \in H_i, k < k'$, which means that the bags in L_i and H_i are always ordered.
In this situation, also denote $x_i = \frac{4}{5}l_i - \sum_{B_k \in L_i} v_i(B_k)$ be the difference to achieve valid bags. Because the last value added to the bag in L_i is smaller or equal v_{i9}, we need to show that

$$v_i(M \backslash J') \geq x_i + v_{i9} \tag{2}$$

Similar to Lemma 4(ii), the bags are ordered, so the high value item in B_k where $B_k \in L_i$ is smaller than $\frac{3}{5}$.

Showing (2)

Lemma 8. *For every $i \in N^2$, if there exists a partition $P = \{P_1, P_2, P_3, P_4\} \in P_i(M)$ such that there are some $P_k \in P$ which contains two or more items j with $v_{ij} > \frac{3}{5}$, then $v_i(M \backslash J') \geq x_i + v_{i9}$.*

Proof. Notice that only two items could have value more than $\frac{3}{5}$. In this situation, let $P_k = 1, 2$ contains two items valued more than $\frac{3}{5}$. And, in this situation, $L_i = \{B_3, B_4\}$, so we care about which bags are items 3 and 4 in.
Case 1: Item 3 and 4 are in the same bag. Then consider the other two bags in P,

$$\begin{aligned} v_i(\text{other two bags}) \leq v_i(J_2) &= v_{i5} + v_{i6} + v_{i7} + v_{i8} \leq v_{i3} + v_{i4} + v_{i5} + v_{i6} \\ &= v_i(B_3) + v_i(B_4) = v_{i3} + v_{i4} + v_i((B_3 \cup B_4) \backslash J') \end{aligned}$$

Since the value of each bag in P is at least 1, we have

$$v_i(M \backslash J') \geq x_i + \frac{2}{5} > x_i + \frac{4}{15} > x_i + v_{i9}$$

Case 2: Item 3 and 4 are in different bags.
Case 2.1: If these two bags contains four or less items, the value of these four items is smaller or equal to the value of B_3 and B_4, and the proof is the same as case 1.
Case 2.2: If these two bags contains five items, then denote the lowest value of these five items y, the value of low value items in these two bags must have $x_i + \frac{2}{5} - y$. Then consider the last bag, it only contains one item from J', and the value is smaller than $\frac{3}{5}$, so the value of low value items in the last bag is greater or equal to $\frac{2}{5}$. Then we have (because $y < \frac{2}{5}$):

$$v_i(M \backslash J') \geq x_i + \frac{2}{5} - y + \frac{2}{5} > x_i + \frac{2}{5} > x_i + v_{i9}$$

Case 2.3: If these two bags contains six items, which are all the remaining items. Consider the last bag, it does not have any items from J', then we have

$$v_i(M \backslash J') \geq x_i + \frac{2}{5} - v_{i7} - v_{i8} + 1 > x_i + \frac{3}{5} > x_i + v_{i9}$$

After Lemma 8, we only need to deal with the situation when $P = \{P_1, P_2, P_3, P_4\} \in P_i(M)$ such that each $P_k \in P$ has at most one item j with $v_{ij} > \frac{3}{5}$.
Similar to case 1, let a be an agent in N^2 and $P = (P_1, P_2, P_3, P_4) \in \mathcal{P}_a(M)$ be the partition satisfying the condition that each $P_k \in P$ has at most one item j with $v_{ij} > \frac{3}{5}$. We can manipulate P as followings: first, for each $B_i = \{j, j'\}$ with $\frac{4}{5} \le v_a(B_i) \le 1$, if $j \in P_k$ and $j' \in P_{k'}$, we can make P_k and $P_{k'}$ into two new bags $\{j, j'\}$ and $((P_k \cup P_{k'}) \setminus \{j, j'\})$ and it is obviously to see that $v_a(((P_k \cup P_{k'}) \setminus \{j, j'\})) \ge 1$. Therefore, we can assume that for each $B_i \notin L_a \cup H_a$, there exists a $P_k = B_i$ and all other bags value at least 1. Then we re-enumerate the bags in P such that

- $P_1, \ldots, P_t$: each has an item j of value more than $\frac{3}{5}$ and $v_a(B_j) > 1$ (Notice that $t = h_a$).
- P_{t+1}, P_{t+2}: the remaining bags.
- $P_{t+2}, \ldots, P_4$: each $P_k = B_{k'}$ for some k' and $\frac{4}{5} \le v_a(B_{k'}) \le 1$.

And similarly, denote $z := \max\left\{2h_a - \left|\bigcup_{k=1}^{t} P_k \cap J\right|, 0\right\}$.
And also denote $y := \max_{k'}(v_{a,9-k'})$ where $B_{k'} \in H_i$.

Lemma 9. *(i)* $x_a \le 2(\frac{4}{5} - 2y)$.
(ii) $v_a(M \backslash J') \ge \left\{x_a + \frac{2}{5} - yz + \frac{z}{5}, \frac{z}{5}\right\}$, *where* $z = \max\left\{2h_a - \left|\bigcup_{k=1}^{t} P_k \cap J\right|, 0\right\}$.

Proof. (i) By Lemma 7, we only need to deal with the situation that $\forall B_k \in L_a, \forall B_{k'} \in H_a, k < k'$. Then $\forall B_k \in L_a, v_a(B_k) = v_{ak} + v_{a,9-k} \ge 2y$, this implies $x_a = \left(\frac{4}{5}\right) l_a - \sum_{k: B_k \in L_a} B_k \le 2\left(\frac{4}{5} - 2y\right)$.
(ii) The proof is the same as Lemma 6 except that we use a more strict bound y to estimate the value of z items.

Case 2.3: $l_i = 3$

Lemma 10. *If* $\exists B_k \in L_i, \forall B_{k'} \in H_i, k > k'$*, then we have enough value to add to the bags in* L_i *to make them* $\frac{4}{5}$*.*

Proof. Firstly, $v_{ik'} > \frac{3}{5}$ because $v_{i,9-k'} < \frac{2}{5}$ and $v_i(B_k) > 1$. Then $v_{ik} \ge v_{ik'} > \frac{3}{5}$, so $v_{i,9-k} < v_i(B_k) - \frac{3}{5} < \frac{1}{5}$. Because $v_{i,9} \le v_{i,9-k} = \frac{1}{5}$, the value of each in L_i when the algorithm is finished is smaller than $\frac{4}{5} + \frac{1}{5} = 1$. Therefore, the last bag filled is at least

$$4 - (\frac{4}{5} + \frac{2}{5}) - 2 = \frac{4}{5}$$

Therefore, we have enough value to add to the bags in L_i to make them $\frac{4}{5}$.

The we only need to deal with the situation that $\forall B_k \in L_i, \forall B_{k'} \in H_i, k < k'$, which means that the bags in L_i and H_i are always ordered. Also denote denote $x_i = \frac{4}{5} l_i - \sum_{B_k \in L_i} v_i(B_k)$. Since there are three bags in L_i, we need to show that

$$v_i(M \backslash J') \ge x_i + 2v_{i9} \tag{3}$$

Showing (3) If $l_i = 3$, then $h_i = 1$, so there is only one item can have value $> \frac{3}{5}$. Then there exists $P = \{P_1, P_2, P_3, P_4\} \in P_i(M)$ such that each $P_k \in P$ has at most one item j with $v_{ij} > \frac{3}{5}$. Similarly, denote the agent a, using the same strategy described above re-enumerate the bags in P such that

- P_1: has an item j of value more than $\frac{3}{5}$ and $v_a(B_j) > 1$.
- P_2, P_3, P_4: the remaining bags.

And similarly, denote $z := \max\left\{2h_a - \left|\bigcup_{k=1}^{t} P_k \cap J\right|, 0\right\}$.
And denote $y := \max_{k'}(v_{a,9-k'})$ where $B_{k'} \in H_i$. Notice that $y = v_{a8}$ in this situation because the bags in H_i and L_i are ordered. Similar to Lemma 9, we have

$$v_a(M \backslash J') \geq \left\{x_a + \frac{3}{5} - yz + \frac{z}{5}, \frac{z}{5}\right\}, \text{ where } z = \max\left\{2h_a - \left|\bigcup_{k=1}^{t} P_k \cap J\right|, 0\right\}$$

Notice that $h_a = 1$, so $z \leq h_i = 1$. When $z = 0$, we have

$$v_a(M \backslash J') \geq x_a + \frac{3}{5} > x_a + \frac{8}{15} \geq x_a + 2v_{a9}$$

Therefore, we only need to deal with the situation when $z = 1$.

Lemma 11. *When $z = 1$, $v_a(M \backslash J') > x_a - \frac{1}{5} + 2y$.*

Proof. Firstly, when $z = 1$, item 1 is the only item from J' in bag P_1. Then other three bags have the remaining seven items from J'. By the pigeonhole principle, there is one bag contains at least three items from J', and denote this bag P_2, then P_3 and P_4 contains at most four items from J'. Here is one example:

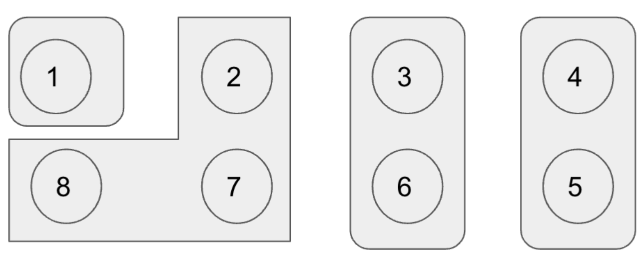

Fig. 2. An example of $z = 1$, where $P_1 = \{1\}$, $P_2 = \{2, 7, 8\}$, $P_3 = \{3, 6\}$, $P_4 = \{4, 5\}$

In this situation, we have (recall that $x_a = \frac{4}{5} * 3 - \sum_{k=2}^{7} v_{ak}$)

$$\begin{aligned} v_a(M\backslash J') &> \frac{1}{5} + (2 - v_a(P_3 \cup P_4)) \\ &\geq \frac{1}{5} + (2 - v_{a2} - v_{a3} - v_{a4} - v_{a5}) \\ &= \frac{1}{5} + x_a - (\frac{2}{5} - v_{a6} - v_{a7}) \\ &\geq x_a - \frac{1}{5} + 2v_{a8} \\ &= x_a - \frac{1}{5} + 2y \end{aligned}$$

Theorem 3. *When $z = 1$, $v_a(M\backslash J') > x_a + 2v_{a9}$.*

Proof. We have $v_a(M\backslash J') \geq x_a + \frac{3}{5} - y + \frac{1}{5} = x_a + \frac{4}{5} - y$ and $v_a(M\backslash J') > x_a - \frac{1}{5} + 2y$. Then we have

$$v_a(M\backslash J') > \frac{1}{3}(2(x_a + \frac{4}{5} - y) + x_a - \frac{1}{5} + 2y) = x_a + \frac{7}{15}$$

Then if $v_{a9} \leq \frac{7}{30}$, we are done. In the following, we only consider the case when $v_{a9} > \frac{7}{30}$. We prove it by contradiction. Assume that $x_a + \frac{4}{5} - y < x_a + 2v_{a9}$ and $x_a - \frac{1}{5} + 2y < x_a + 2v_{a9}$, then we have $\frac{4}{5} - 2v_{a9} < y < \frac{1}{10} + v_{a9}$. We have $v_{a9} > \frac{7}{30}$, then we get $\frac{1}{3} < y < \frac{1}{3}$, which leads to contradiction. Therefore, when $z = 1$, $v_a(M\backslash J') > x_a + 2v_{a9}$. This completes the proof of **Case 2.3**.

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