MD: Single Parameter

Thursday, September 15, 2022 11:02 AM

Last lec:

Single Item $\to$ First Price

Single Item $\to$ second Price. $\to$

Single Item $\downarrow$ K - identical items.

Awesome Auction.

$\boxed{\begin{array}{l} \text{(1) Truthful (DSIC)} \\ \text{(2) S.W. maximizing} \\ \text{(3) Easy to implement (poly-time)} \end{array}}$

★ Sponsored search

pay-per-click keyword auctions.

agent i has value v_i/click.

bid b_i/click.

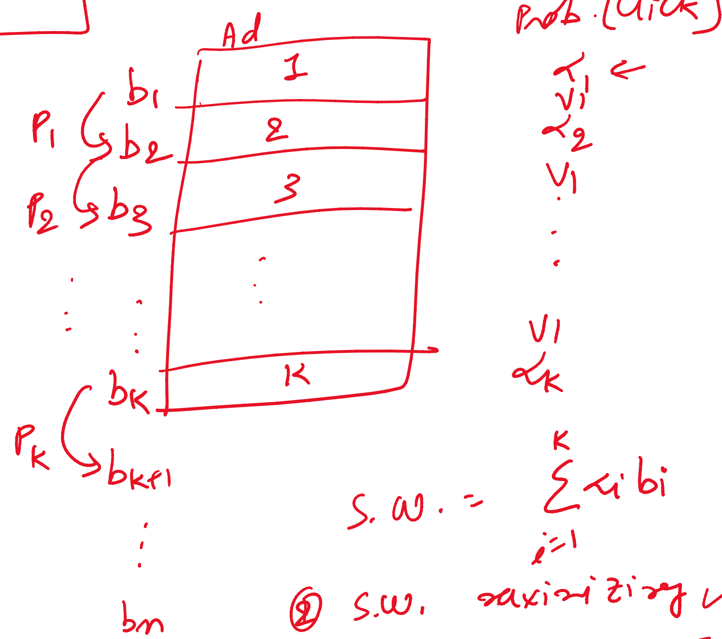
$\boxed{\text{Pizza}}$

$b_1 \geq b_2 \geq \dots \geq b_n$

i^{th} highest bidder gets i^{th} slot.

$p_i = b_{i+1} \quad i \leq k$

$p_i = 0 \quad i > k$



$$S.W. = \sum_{i=1}^{k} \alpha_i b_i$$

$i \leq k$

$U_i(b) = \alpha_i (v_i - p_i)$

$i > k \quad U_i(b) = 0$

(2) S.W. maximizing ✓

(3) Easy ✓

(1) Truthful ?

$b_1=10$	$b_2=9$
$b_2=9$	$b_1=8$
$b_3=1$	$b_3=1$

α_i: 1, 0.9

$\boxed{v_1 = 10}$

$v_2 = 9$

$v_3 = 1$

$U_1(v_1) = 1(10-9) = 1$

$U_1(b_1=8) = 0.9(10-1) = 0.9 \times 9$
$= 8.1.$

★ Single Parameter: Single "Stuff" on auction
(eg. item / clicks / impressions...)

agent i has value v_i / unit-stuff. ← Private info of agent i.

★ Goal: Design awesome auction.

★ Format:

① Each Bidder i submit bid b_i (← bid / unit-stuff.) in a sealed envelope to auctioneer. b_i need not be v_i

$\bar{b} = (b_1, \ldots, b_n)$

② Auctioneer decides $\boxed{\text{allocation that max. s.w. wrt } \bar{b}}$

$$x = (\bar{b}) = \underset{x \in X}{\arg\max} \sum_i x_i b_i$$

③ Auctioneer decides payment.

$P_i(b)$, $\forall i$

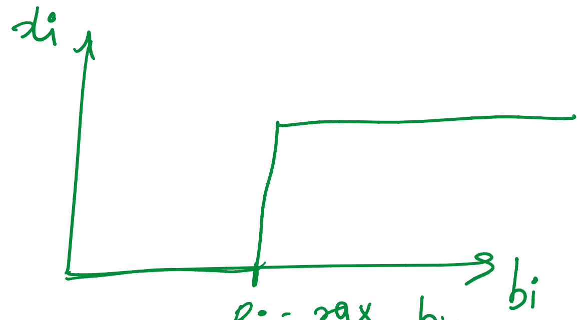
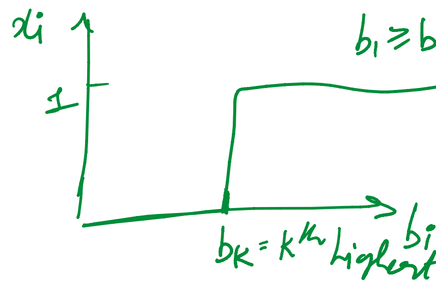
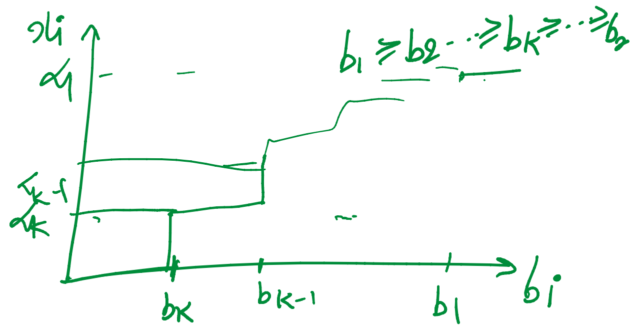
Mechanism

$\bar{b} = (b_1, \ldots b_n) \rightarrow$ Mechanism $\rightarrow x_i(\bar{b})$, $P_i(\bar{b})$ $\forall i$

$\boxed{\begin{aligned} &U_i(\bar{b}) \\ &= v_i x_i(\bar{b}) - P_i(\bar{b}) \end{aligned}}$

(x, P) a mechanism.

settings | set of feasible | [illegible] max allocation

settings	set of feasible allocations X	$x^*(b)$ = s.w. max. allocation $= \arg\max_{x\in X} \sum_i b_i x_i$. Fix i, fix b_{-i}, $x_i(b_i, \cdot)$ is monotone?
Single item	$X=\{x\in\{0,1\}^n \mid \sum_i x_i = 1\}$	$x^*(b)$ = highest bidder gets the item  $B_i = \max_{k\neq i} b_k$ = max bid in b_{-i}
K-item identical	$X=\{x\in\{0,1\}^n \mid \sum_{i=1}^n x_i = k\}$	$x^*(b)$ = Top k bidders. $b_1 \geq b_2 \geq \dots \geq b_k \geq \dots \geq b_n$  b_k = k^{th} highest bid in b_{-i}
Sponsored Search	$X=\{x\in\mathbb{R}^n \mid x_i\in\{\alpha_1,\dots,\alpha_k,0\},\ \forall j\leq k,\ \sum \mathbb{1}_{[x_i=\alpha_j]} \leq 1\}$	$x^*(b)$ = i^{th} highest gets i^{th} slot. $b_1 \geq b_2 \geq \dots \geq b_k \geq \dots \geq b_n$ 

claim: $x(\cdot)$ max. s.w. $\Rightarrow$ $x(\cdot)$ is monotone (exe).

Claim: $x(\cdot)$ max. s.w. $\Rightarrow$ $x(\cdot)$ is monotone (exe).

Def 1: Allocation $x(\cdot)$ is "implementable" if $\exists$ $p(\cdot)$ s.t. (x, p) is DSIC.

Def 2: Allocation rule $x(\cdot)$ is monotone if $\forall i$, $\forall b_{-i}$ $x_i(b_i, b_{-i})$ is monotone wrt b_i.

Myerson's Lemma:

1. $x(\cdot)$ is implementable iff it is monotone.
2. If $x(\cdot)$ is monotone then $\exists$ unique $p(\cdot)$ s.t. (x, p) is DSIC.
3. There is an explicit formula for $p(\cdot)$.

Claim 1: If $x(\cdot)$ is implementable $\Rightarrow$ $x(\cdot)$ is monotone.

Pf: $x(\cdot)$ implementable $\Rightarrow$ $\exists p(\cdot)$ s.t. (x, p) is DSIC.

Fix i, fix b_{-i}

Now on,

$$\underline{x}(b_i) = x_i(b_i, b_{-i})$$

$$p(b_i) = p_i(b_i, b_{-i})$$

$z \neq y \in \mathbb{R}^+$

$U_i(b_i) = v_i\, x(b_i) - p(b_i)$

$$\text{DSIC} \iff \begin{cases} b_i = v_i = z & \qquad b_i = y,\ v_i = z \\ z\cdot x(z) - P(z) \geq & z\, x(y) - P(y) \quad \rightarrow ① \\ & \\ & \qquad b_i = z,\ v_i = y \\ b_i = v_i = y & \\ y\, x(y) - P(y) \geq & y\, x(z) - P(z) \quad \rightarrow ② \end{cases}$$

$$z(x(y) - x(z)) \overset{①}{\leq} \underline{P(y) - P(z)} \overset{②}{\leq} y\,\underline{(x(y) - x(z))} \rightarrow ★$$

$$\Downarrow$$

$$0 \leq (y - z)\,(x(y) - x(z))$$

$$\Downarrow$$

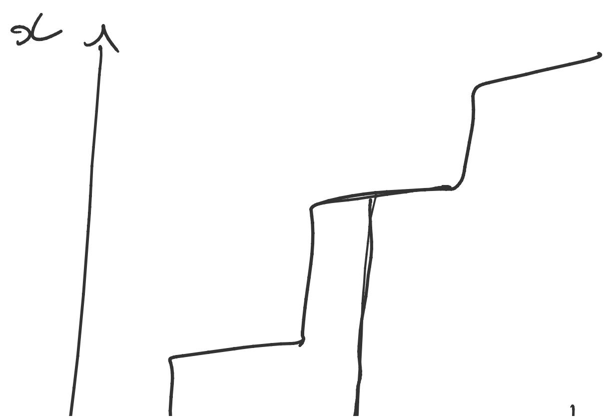
$$\left.\begin{array}{l} y > z \Rightarrow x(y) \geq x(z) \\ z > y \Rightarrow x(z) \geq x(y) \end{array}\right\} \equiv x \text{ is monotone} \quad \blacksquare$$

Claim 2: If $x(\cdot)$ is monotone then $\exists$ unique $P(\cdot)$ formula s.t. (x, P) is DSIC.

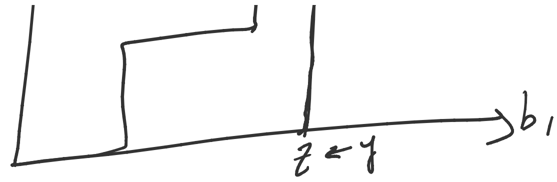
Pf: $z \leftarrow y$

Case I: z in betⁿ a seg

$$\lim_{y \to z} x(y) = x(z)$$



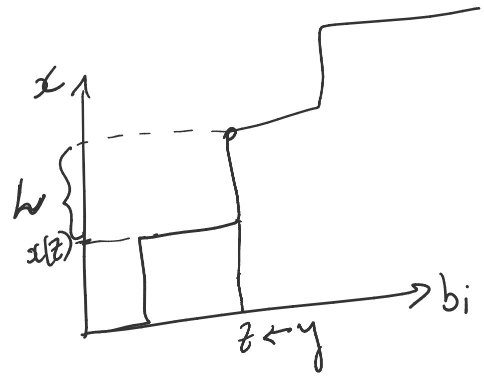
$y \to z$



$$\textcircled{\star} \Rightarrow \lim_{y \to z} 0 \le P(y) - P(z) \le 0$$

$$\Rightarrow \quad P(y) = P(z)$$

Case II: z at a break point.



$$\lim_{\substack{y \to z \\ \text{above}}} x(y) = x(z) + h$$

$$\Downarrow$$

$$x(y) - x(z) = h$$

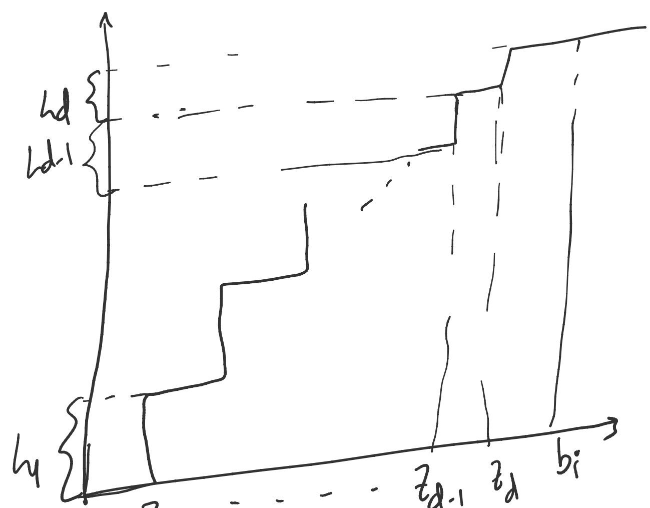
$$\textcircled{\star} \Rightarrow \lim_{\substack{y \to z \\ \text{above}}} z \cdot h \le P(y) - P(z) \le y \cdot h$$

$$\Vert$$

$$z \cdot h \le P(y) - P(z) \le z \cdot h \quad \Rightarrow \quad P(y) - P(z) = z \cdot h$$

$$p(b_i) = \overbrace{P(b_i) - P(z_d)}^{z_d h_d}$$
$$+ \overbrace{P(z_d) - P(z_{d-1})}^{z_{d-1} h_{d-1}}$$
$$+ \underbrace{P(z_{d-1}) - P(z_{d-2})}$$
$$\vdots$$
$$+ \overbrace{P(z_2) - P(z_1)}^{z_1 h_1}$$
$$+ \underbrace{P(z_1) - P(0)}_{=0}$$
$$+ \underbrace{P(0)}_{=0}$$

$$= \sum_{k=1}^{d} z_k h_k$$



K=1

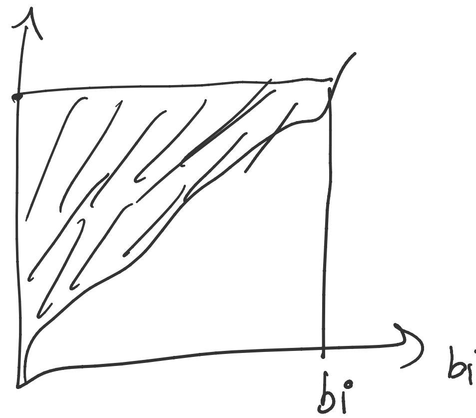
★ Continuous:

$$\lim_{y \to z} \frac{z(x(y)-x(z))}{y-z} \leq \frac{p(y)-p(z)}{y-z} \leq \frac{y(x(y)-x(z))}{y-z}$$

$$\downarrow x'(z) \qquad \downarrow p'(z) \qquad \downarrow x'(z)$$

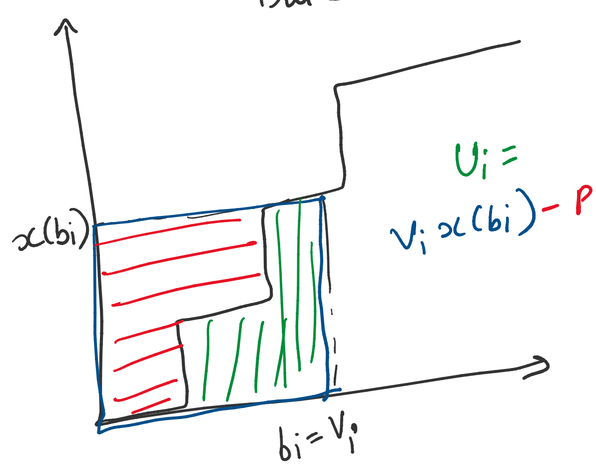
$$\Rightarrow \quad p'(z) = z\, x'(z)$$

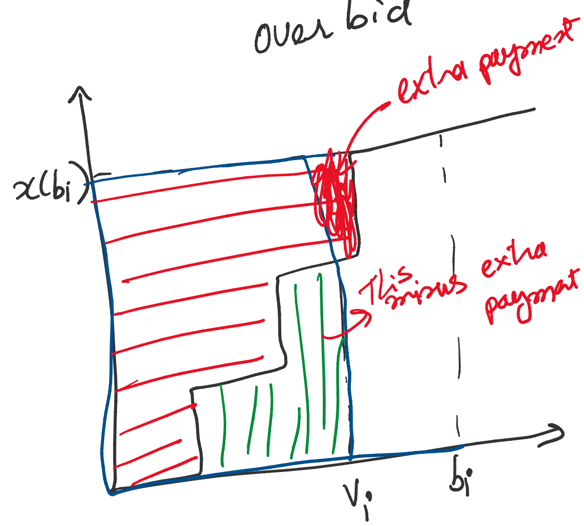
$$\therefore \quad p(b_i) = \int_0^{b_i} p'(z)\,dz = \int_0^{b_i} z\, x'(z)\,dz$$



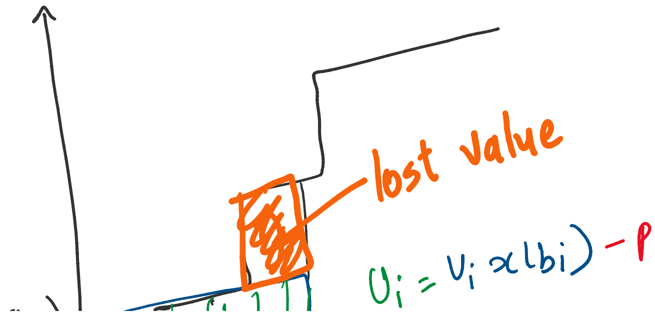
claim 3: (x, p) is DSIC.

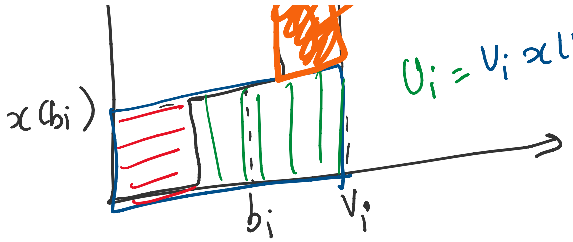
Pf: (By pictures)





$U_i = v_i\, x(b_i) - p$





$$U_i = v_i\, x(b_i) - p$$