Voting: Arrow's Theorem

Thursday, September 8, 2022 10:58 AM

$N = \{1, \ldots, n\}$ agents / voters.

O = set of outcomes / candidates. $|O| \geq 3$

L = all possible orderings over the candidates. $|L| = n!$

$i \in L$, $\succ_i \in L$

★ Social choice function

$$C : L^n \rightarrow O$$

$$\boxed{\text{★ Social Welfare function} \quad W : L^n \rightarrow L}$$

$O = \{A, B, C\}$

# agents	2	1	0
499	A	> B	> C
3	B	> C	> A
498	C	> B	> A

★ Voting Schemes : (Choice)

① plurality (majority): candidate placed at top by the most voters. $\longrightarrow$ A

② Plurality of elimination ("instant runoff") $\longrightarrow$ C

- Eliminate candidates one-by-one until only one remains
- Candidate of fewest top vote is eliminated in every round

(3) Bordacount. ... i) to ith candidate

A : 2×499

B : $6 + 499 + 498$

(3) Bordacount.

- Give score $(n-j)$ to j^{th} candidate
- candidate w/ highest total score wins $\Downarrow$ B

$A: 2\times 499$
$B: 6+499+498$
$C: 0+3+2\times 498$

(4) Condorset Winner:

- The candidate preferred against every other candidate in "pairwise run-off" $\longrightarrow$ B.

✱ Desired Properties.

$\succ \in L^n \qquad \succ_W = W(\succ)$

(1) Pareto Efficiency (PE):

$\forall i \in N, \quad a \succ_i b \Rightarrow a \succ_W b$

(2) Independent of Irrelevant Alternatives (IIA)

$\succ, \succ' \in L^n$, If $\forall i,\quad a \succ_i b \Rightarrow a \succ'_i b$

$b \succ_i a \Rightarrow b \succ'_i a$

Then, $\quad a \succ_W b \Rightarrow a \succ'_W b$

$b \succ_W a \Rightarrow b \succ'_W a$

(3) Non-dictatorship:

W does not have a dictator. That is

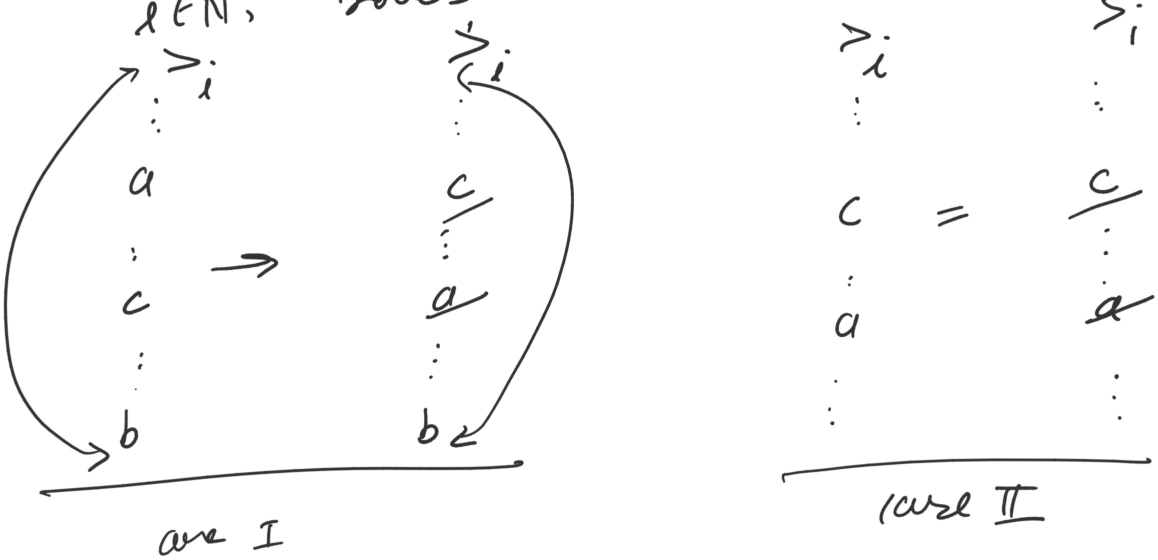
$\neg \exists i, a, b \in O \ (a \succ_i b \Rightarrow a \succ_W b)$

Arrow's Theorem: Any W that is PE & IIA, is dictatorial.

Pf: 4 claims. Suppose !! W satisfies PE, IIA.

Claim 1: Fix $b \in O$. In $\succ$, if every agent puts b either at the top or at the bottom. Then in $\succ_w$ b must be either at the top or at the bottom, where $\succ_w = W(\succ)$.

Pf: By contradiction, $\boxed{a \succ_w b \succ_w c}$ for some $a, c \in O$. Construct $\succ'$ from $\succ$, where every $i \in N$, moves c above a



case I

case II

$\succ'_w = W(\succ')$.

IIA { Since relative ordering of a, b is same for $\succ, \succ'$

$$a \succ_w b \Rightarrow a \succ'_w b$$

$a >_\omega b \Rightarrow a >'_\omega b$

IIA

Similarly for b, c

$b >_\omega c \Rightarrow b >'_\omega c$

$\Downarrow$ Transitivity.

$a >'_\omega b >'_\omega c \Rightarrow a >'_\omega c$

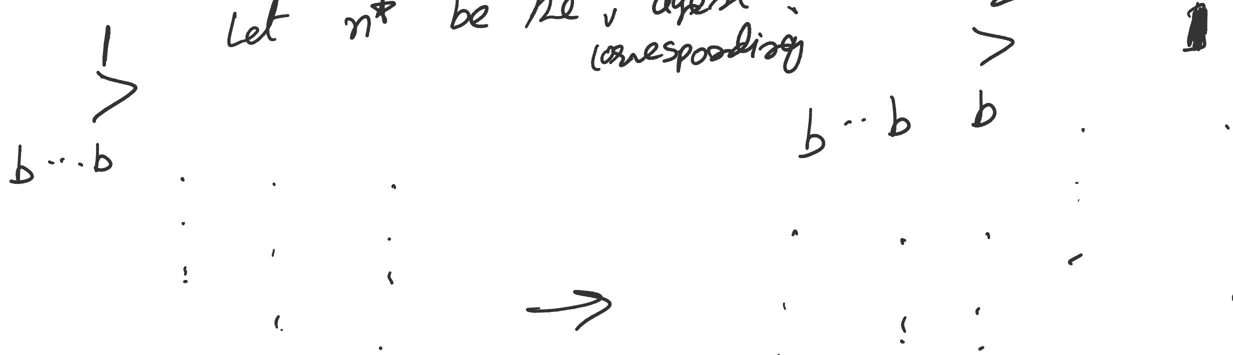
contradicts PE of W. ∎

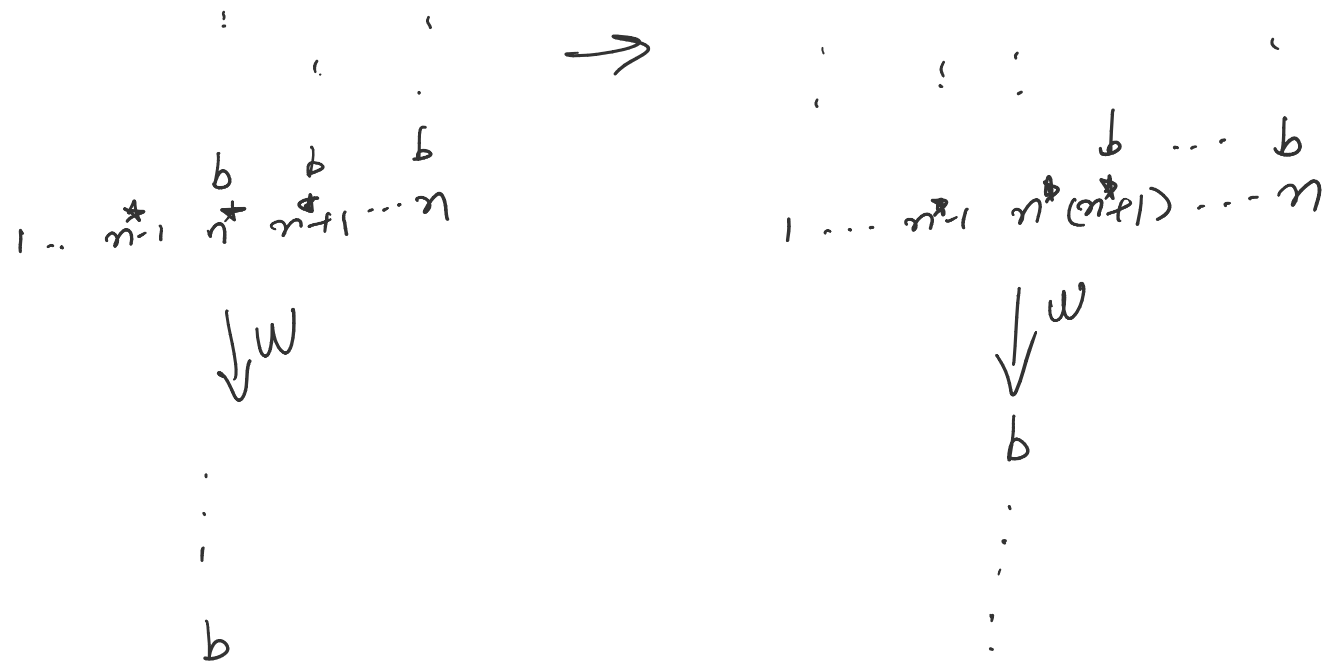
claim 2: $\exists n^* \in N$ who is "extremely pivotal": she can move a given $b \in O$ from bottom to top.

Pf: Consider $> \in L^n$ where each $i \in N$ puts b at the bottom. $\Rightarrow$ In $>_\omega$ also b is at the bottom.

Now move b from bottom to top one agent at a time. By claim 1, at some point b will move from bottom to top in the final outcome.

Let n^* be the agent corresponding. ∎





Goal: To show that no matter the actual preference order of agents, n^* decides relative ordering for every pair of candidates.

$\overset{\text{final}}{>} \in L^n$.

<u>claim 3</u>: n^* is a dictator for any pair not involving b.

<u>pf</u>: Pick pair $a, c \neq b$, $a, c \in O$. Suppose $a \overset{\text{final}}{>}_{n^*} c$.

Construct $\overset{3}{>}$ from $\overset{2}{>}$ by

① move a to the top of n^*'s pref.

↳ $a \overset{3}{>}_{n^*} b \ldots \overset{3}{>}_{n^*} c$

② $\forall i \neq n^*$ rearrange a, c arbitrarily but not changing b's position.

② $\forall i \neq n^*$ rearrange a, c arbitrarily without changing b's position.

↳ can match with final $\succ_i$ relative ordering of a, c.

→ Relative ordering of a & b did not change betn $\succ^1$ & $\succ^3$

$\Downarrow$ IIA

$$a \succ^1_w b \Rightarrow a \succ^3_w b$$

→ Relative ordering of b & c didn't change betn $\succ^2$ & $\succ^3$.

$\Downarrow$ IIA

$$b \succ^2_w c \Rightarrow b \succ^3_w c$$

$$a \succ^3_w b \succ^3_w c \Rightarrow a \succ^3_w c$$

* Construct $\succ^4$ from $\succ^3$ by

① Move b arbitrarily for all agents.

↳ put b in it's final position in final $\succ_i$

② For n^* move a, c arbitrarily while keeping it above c.

[illegible]

above c.

Relative position of a, c did not change bet^n $\succ^3$ & $\succ^4$ for any of the agents.

$$\Downarrow \text{IIA}$$

$$a \overset{3}{\underset{w}{\succ}} c \Rightarrow a \overset{4}{\underset{w}{\succ}} c$$

Claim 4: n^* decides ordering for pair ab for any a.

Pf: Consider any third candidate $c \neq a, b$.
Now, apply claim 2 for c instead of b & let n^{**} be the respective dictator. Then by claim 3, n^{**} decides ordering of ab for every pref. profile. But, in claim 2 we saw a pair of profiles $\overset{1}{\succ}$ & $\overset{2}{\succ}$ where n^* decides position of b & thereby relative ordering of ab. Hence, it must be that n^{**} & n^* are the same agents. ∎