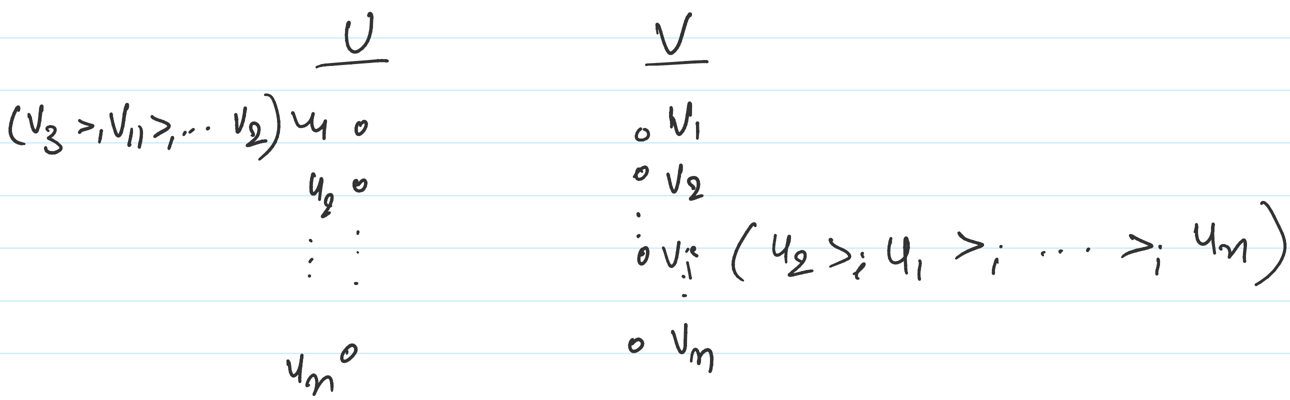
Lec 5: MD without Money Part I (Stable matching, TTC)

Tuesday, September 6, 2022 10:02 AM

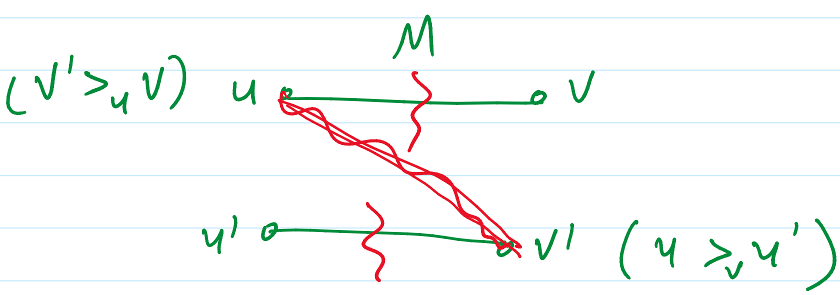
Mechanism Design.

★ Stable Matching (Marriage):

Appl^n. hospital-resident assignment
students-schools/courses.

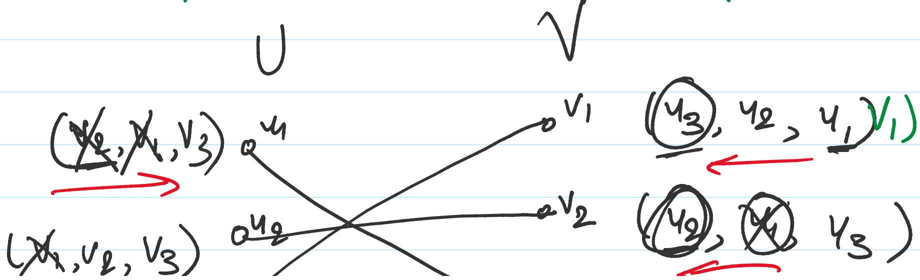


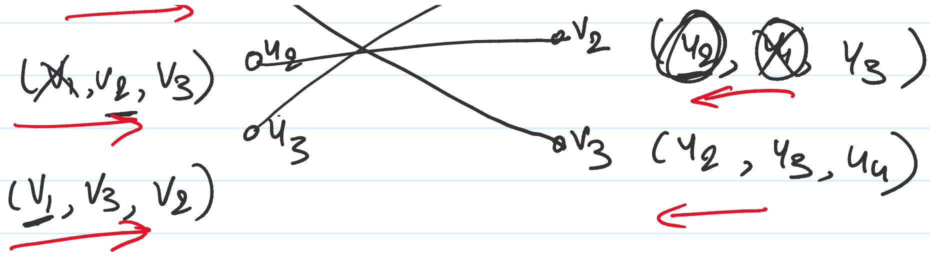
Goal: Find a "stable" matching.
No Blocking pair.



★ Gale-Shapley Differed Acceptance Algo

U: proposer, V: Disposer.





In every round the unmatched vertices on the U side proposes to it's most preferred "next" vertex on V-side.

Running time : $O(n^2)$ rounds because at least one of the u_i's goes down by one in it's preference list.

why O/P perfect matching?

Thm: GS o/p a stable perfect matching.

Pf: M: o/p GS.

Suppose, $(u, v) \notin M$ is a blocking pair.
$u \in U$, $v \in V$.

case I : suppose u never proposed to v.

$(u, v') \in M$.

$\Rightarrow (u, v)$ not a blocking pair.

$v' >_u v$ $\quad u$ ——— v'

case II : suppose u proposed to v, but got rejected by v for u'. $\Rightarrow u' >_v u$

$(u'', v) \in M$.

$\Downarrow$

$u'' >_v u' >_v u$.

$\Rightarrow (u, v)$ not a blocking pair

$\underline{u''} >_v u' >_v \underline{u}$. blocking pair

NOTE:

$OPT(u) = best\{v \mid (u,v)$ in some stable matching$\}$.

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$\forall u \in U, (u, OPT(u)) \in M$

Proposer.

Take Away: Be a proposer!

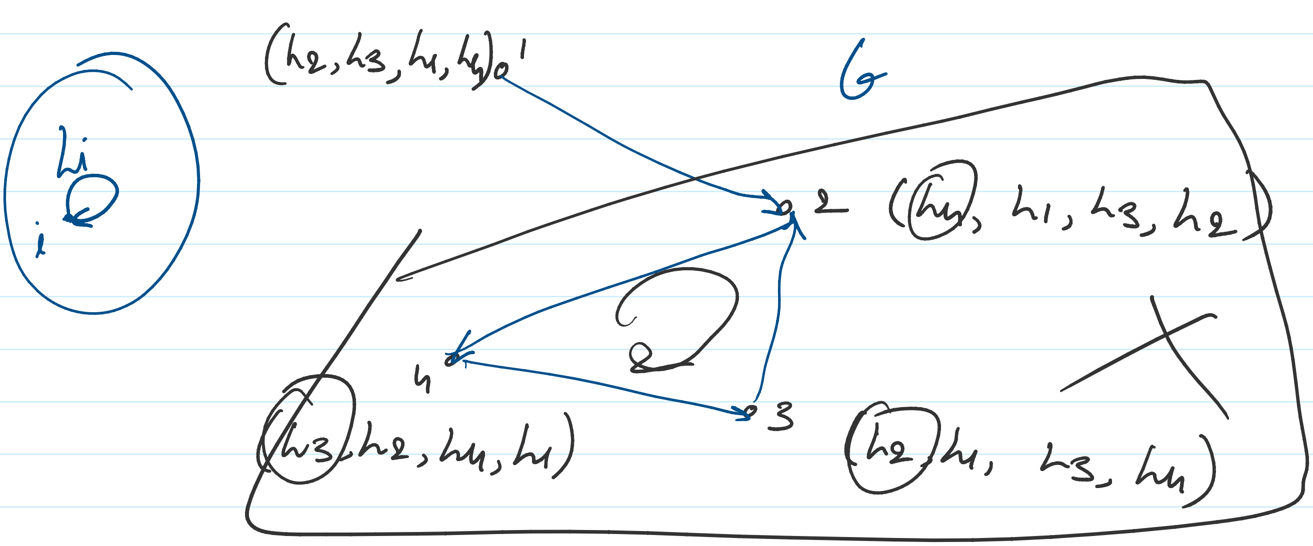
☆ House Allocation:

N: set of agents $\{1, \dots, n\}$.

$i \in N$, ① owns a house h_i.

② has a complete preference over $\{h_1, \dots, h_n\}$

$$h_3 >_i h_5 >_i h_1 >_i \dots >_i h_i >_i \dots$$



Top Trading Cycle (TTC):

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① $A = \{1, \ldots, n\}$
② while $A \neq \emptyset$

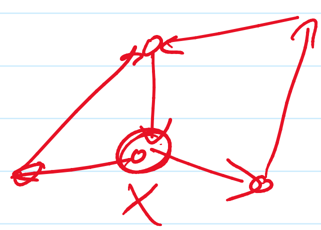
round j

(2.1) Each agent $i \in A$, points to the agent their most preferred house among the ones in A. $\Downarrow$ G

(2.2) $\mathcal{C}$ = set of cycles in G
(no sink $\Rightarrow$ not a DAG $\Rightarrow$ has a cycle).

(2.3) For each cycle $C \in \mathcal{C}$, exchange houses along C. $A = A \setminus$ agents in $\mathcal{C}$. $\rightarrow N_j$

OBS: Each agent can be part of at most one cycle in graph G. Because each has exactly one outgoing edge



Claim 0: Every agent i gets at least as preferred as h_i.

$$\ldots > h_i > \ldots$$

DSIC: Dominant Strategy Incentive Compatible.

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For each $i \in N$, reporting their true preference is the best no matter what others do.

Thm: TTC is DSIC.

Pf: Induction

Let N_j: set of agents assigned houses in round j.

Base case: $i \in N_1$, then i gets her most prefered house. Hence cannot feign and improve.

Induction Hypothesis: $j \geq 2$, $i \in \underbrace{N_1 \cup N_2 \cup \dots \cup N_{j-1}}_{N'}$

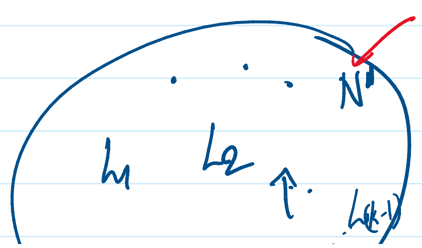
Induction: suppose $i \in N_j$. i gets assigned house h_k.

$$h_1 >_i h_2 >_i \cdots >_i h_{k-1} >_i h_k > \dots$$

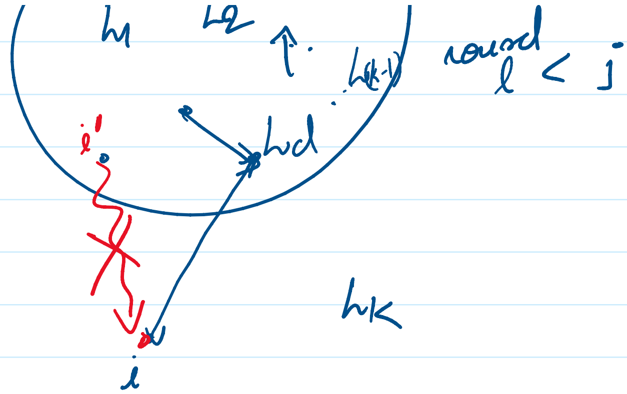
Q: can i report un-truthfully & get one of the $h_1, h_2, \dots, h_{k-1}$? NO. Because,

OBS-I: By being untruthful, i can change only outgoing edges in every round, & NOT incoming edges.

all the agents that are part of a cycle in rounds $1 \dots (j-1)$



round $l < j$



OBS-2: $i' \in N'$ is not pointing to i in any of the $1 \cdots (j-1)$ rounds. Because if it was then i' will keep pointing to i until i's house is assigned, which happens in round j. ∎