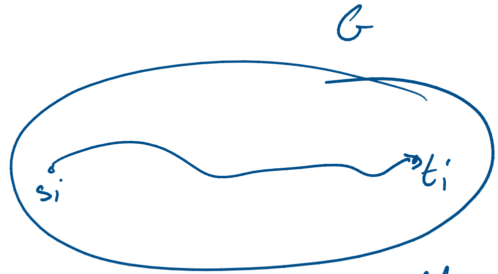
N/W Cost Sharing Games / N/W Formation Games

Thursday, November 3, 2022 10:58 AM

(Games w/ positive externalities)



- $G=(V,E)$ directed
- $N=\{1,\dots,n\}$
- $i\in N$ wants to "build" $s_i \to t_i$ path. $\mathcal{P}_i$: set of all $s_i - t_i$ paths.

$$P = \bigtimes_{i\in N} \mathcal{P}_i$$

- $P_i \in \mathcal{P}_i$ built by $i\in N$, $\bar{P} = (P_1,\dots,P_n)$.

$$\forall e,\quad f_e^{\bar{P}} = f_e = \#\text{ players who want to build edge } e \text{ in } \bar{P}$$
$$= |\{i\in N \mid e\in P_i\}|$$

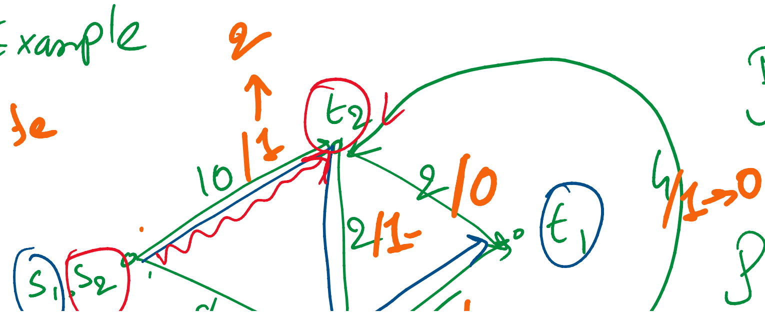
- $\forall e\in E$, $\gamma_e \geq 0$ is the cost.

$$c_i(\bar{P}) = \sum_{e\in P_i} \frac{\gamma_e}{f_e}$$

$$cost(\bar{P}) = \sum_{i\in N} c_i(\bar{P}) = \sum_{i\in N}\sum_{e\in P_i}\frac{\gamma_e}{f_e} = \sum_{\substack{e\in E:\\ f_e\geq 1}} \frac{\gamma_e}{\cancel{f_e}} \cdot \overbrace{\left(\cancel{\sum_{\substack{i\in N:\\ e\in P_i}} 1}\right)}^{f_e}$$

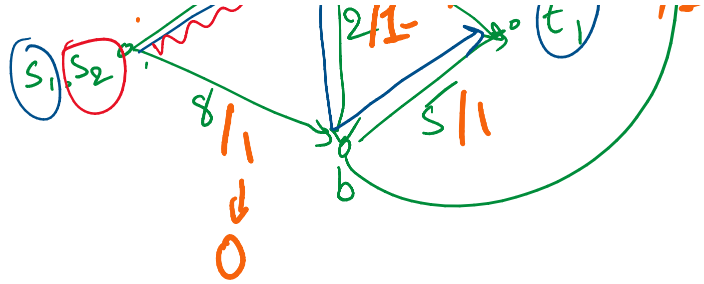
$$= \sum_{\substack{e\in E:\\ f_e\geq 1}} \gamma_e$$

* Example



$\mathcal{P}_1 = \{ s_1 - t_2 - t_1,\ s_1 - b - t_1,\ \boxed{s_1 - t_2 - b - t_1},\ s_1 - b - t_2 - t_1 \}$

$\mathcal{P}_2 : \{ \boxed{s_2 - t_2},\ s_2 - b - t_2 \}$

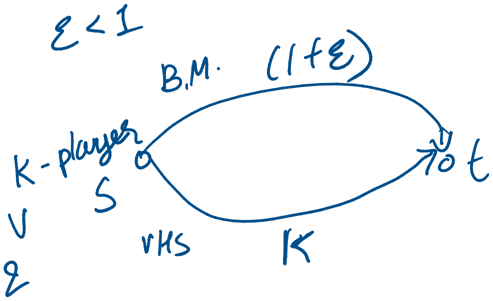


$P_2 = \{s_2 - t_2, \; s_2 - b - t_2\}$

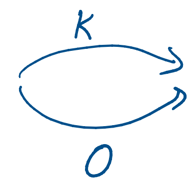
Q: PoA better or worse than routing game?

★ VHS vs. Betamax
Earlier   later but far better technology.

$\varepsilon < 1$



OPT =

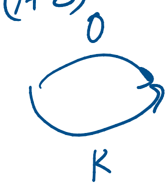


cost/person $= \frac{(1+\varepsilon)}{K}$

$\parallel$ NE1

cost(OPT) $= (1+\varepsilon)$

NE2 =



cost / person $= \frac{K}{K} = 1$

cost(NE2) $= K$.

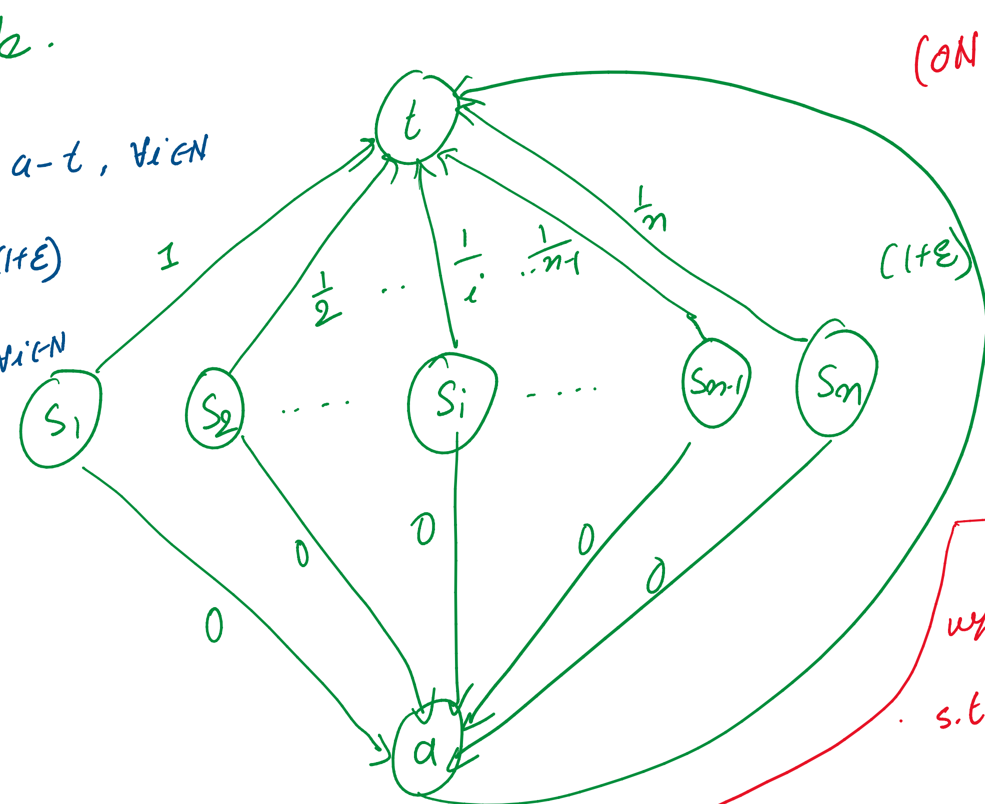
$$\text{PoA} = \frac{\text{Worst NE cost}}{\text{OPT cost}} \geq \frac{\text{cost (NE2)}}{\text{cost (OPT)}} = \frac{K}{1+\varepsilon} \approx K = \#\text{ players!}$$

!!

★ Suppose we "select" good NE by forcing coordination through mediator/ default option.

$$\text{Price-of-Stability (PoS)} = \frac{\text{Best NE cost}}{\text{OPT cost}} = \frac{\min_{P\ \text{NE}} \text{cost}(P)}{\min_{P \in \mathcal{P}} \text{cost}(P)}$$

★ Example.



OPT: $s_i - a - t$, $\forall i \in N$

$\text{cost}(OPT) = (1+\varepsilon)$

$c_i(OPT) = \frac{1+\varepsilon}{n}$, $\forall i \in N$

$\Downarrow$

n deviates to $(s_n - t)$.

(ONLY) NE: $(s_i - t)$, $\forall i \in N$

$\text{cost}(NE) = 1 + \frac{1}{2} + \dots + \frac{1}{n} \approx H_n \sim \ln n$

$\bar P \in P$, where $A \subseteq S$ s.t. $\forall i \in A$, $s_i - a - t$ path, $\forall i \notin A$, $s_i - t$

Not a NE because

$$c_{i^*}(\bar P) = \frac{1+\varepsilon}{|A|} \geq \frac{1+\varepsilon}{i^*} > \frac{1}{i^*} = \gamma(s_{i^*} - t)$$

$i^* = \arg\max_{i \in A} i$

$i^* \geq |A|$

$$PoS = \frac{\text{cost}(NE)}{\text{cost}(OPT)} = \frac{H_n}{1+\varepsilon} \approx \ln n.$$

Thm: $PoS^{PNE} \leq H_n$. for any n/w cost sharing game.

Why $\exists$ Pure NE? Because, it's a potential game.

Recall: $\phi(\bar P) = \sum_{e \in E} \sum_{k=1}^{f_e^{\bar P}} c_e(k)$, $c_e(k) = \frac{\gamma_e}{k}$

★ Potential func^c.

$f_e^{\bar P}$

$$\phi(\bar{P}) = \sum_{e \in E} \gamma_e \sum_{k=1}^{f_e} \frac{1}{k}$$

Claim: ϕ is the potential func for CSG.

$\forall \bar{P} \in P, \ \forall i, \ \forall q_i \in \mathcal{P}_i$

$$C_i(q_i, \bar{P}_{-i}) - C_i(P_i, \bar{P}_{-i}) = \phi(q_i, \bar{P}_{-i}) - \phi(P_i, \bar{P}_{-i})$$

Pf: exe

Pf (of Thm: $\text{PoS}^{PNE} \le H_n$):

$\forall \bar{P} \in P$

$$\text{cost}(\bar{P}) \le \phi(\bar{P}) \le \sum_{\substack{e \in E \\ f_e \ge 1}} \gamma_e \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{H_n} = H_n \cdot \sum_{\substack{e \in E \\ f_e \ge 1}} \gamma_e$$

$$\text{cost}(\bar{P}) = \sum_{\substack{e \in E \\ f_e \ge 1}} \gamma_e, \qquad \phi(\bar{P}) = \sum_{e \in E} \gamma_e \sum_{k=1}^{f_e} \frac{1}{k}$$

$$= H_n \cdot \text{cost}(\bar{P})$$

$$\boxed{\text{cost}(\bar{P}) \le \phi(\bar{P}) \le H_n \, \text{cost}(\bar{P})} \rightarrow (1)$$

$\bar{P}^*$: OPT

$\bar{P}' = \arg\min_{\bar{P} \in P} \phi(\bar{P}) \Rightarrow \bar{P}'$ is a NE.

$\phi(\bar{P}')$

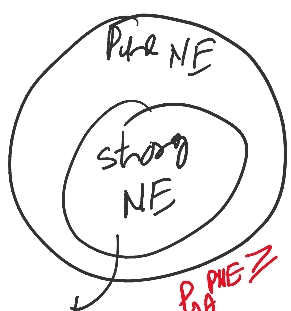
$P \in \Gamma$

$$PoS = \frac{\min_{\bar{p}:NE} cost(\bar{p})}{cost(p^*)} \le \frac{cost(\bar{p}')}{cost(\bar{p}^*)}$$

$$cost(\bar{p}') \underset{(\because ①)}{\le} \phi(\bar{p}') \underset{(\because \text{choice of } \bar{p}')}{\le} \phi(\bar{p}^*) \underset{(\because ①)}{\le} H_n \cdot cost(\bar{p}^*)$$

$$\Rightarrow PoS \le \frac{cost(\bar{p}')}{cost(p^*)} \le H_n. \qquad \blacksquare$$

☆ Strong NE :



$PoA^{PNE} \ge PoA^{SNE} \ge PoS^{SNE} \ge PoS^{PNE}$

$PoS^{PNE} \le H_n$

Def : No "benefitial coalition"

$\boxed{\bar{P}_S = (P_i)_{i \in S}}$

Def : Benefitial coalition (B.C.) wrt $\bar{P}(P_1, \ldots, P_n)$

$A \subseteq N$ s.t. $\exists\ q_A \in \bigtimes_{i \in A} P_i$

$\forall i \in A, \quad c_i(q_A, P_{-A}) \le c_i(\bar{P})$

& at least one of the inequalities is strict.

Thm: $PoA^{SNE} \leq H_n$.

Notation $\bar{P}_S = (P_i)_{i \in S}$

Pf: $\bar{P}^*$: OPT

$\bar{P}$: strong NE.

Q_n: why $A_n = \{1, \ldots, n\}$ does not want to deviate to $\bar{P}^*$?

$\because \exists i \in A_n$ s.t. $c_i(\bar{P}^*) \geq c_i(\bar{P})$

Denote this agent by n.

$Q_{(n-1)}$: why $A_{n-1} = \{1, \ldots, n-1\}$ does not want to deviate to $P^*_{A_{n-1}}$

$\because \exists i \in A_{n-1}$ s.t. $c_i(P^*_{A_{n-1}}, P_n) \geq c_i(\bar{P})$

Denote this agent by $(n-1)$

$\vdots$

Q_k: why $A_k = \{1, \ldots, k\}$ does .... to $P^*_{A_k}$

$\because \exists i \in A_k$ s.t. $c_i(P^*_{A_k}, P_{-A_k}) \geq c_i(\bar{P})$

$\vdots$

$$\text{cost}(\bar{P}) = \sum_{k \in N} c_k(\bar{P}) \leq \sum_{k \in N} c_k(P^*_{A_k}, P_{-A_k})$$

$$\leq \sum c_k(P^*_{A_k})$$

$$\leq \sum_{k \in N} c_k(p^*_{A_k})$$

↳ remove all agents in $-A_k = N \setminus A_k$ from the game & A_k plays as per $p^*_{A_k}$.

$$\boxed{\begin{aligned} &\text{Def : } f_e^k = \#\text{ agents in } p^*_{A_k} \text{ taking edge } e \\ &\qquad\quad = \left|\{ i \in A_k \mid e \in p_i^* \}\right| \\ &c_k(p^*_{A_k}) = \sum_{e \in p^*_k} \frac{\gamma_e}{f_e^k} \ (\text{exe}) = \phi(p^*_{A_k}) - \phi(p^*_{A_{k-1}}) \end{aligned}}$$

$$\sum_{k=1}^{n} c_k(p^*_{A_k}) = \sum_{k=1}^{n} \phi(p^*_{A_k}) - \phi(p^*_{A_{k-1}})$$

$$= \cancel{\phi(p^*_{A_1})} - \underbrace{\phi(p^*_{A_0})}_{A_0 = \{\}} = 0$$

$$+ \cancel{\phi(p^*_{A_2})} - \cancel{\phi(p^*_{A_1})}$$

$$\vdots$$

$$+ \cancel{\phi(p^*_{A_{n-1}})} - \cancel{\phi(p^*_{A_{n-2}})}$$

$$+ \phi(p^*_{n}) - \cancel{\phi(p^*_{n-1})}$$

$$+ \phi(P^*_{A_n}) - \cancel{\phi(P^*_{n-1})}$$

$$= \phi(P^*_{A_n}) - 0$$

$$= \phi(P^*) \underset{(\because \text{①})}{\leq} H_n \, \mathrm{cost}(P^*)$$

$$\Rightarrow \mathrm{cost}(\bar{P}) \leq H_n \cdot \mathrm{cost}(P^*)$$

$$\Rightarrow \mathrm{PoA}^{SNE} = \frac{\mathrm{cost}(\bar{P})}{\mathrm{cost}(P^*)} \leq H_n.$$

∎