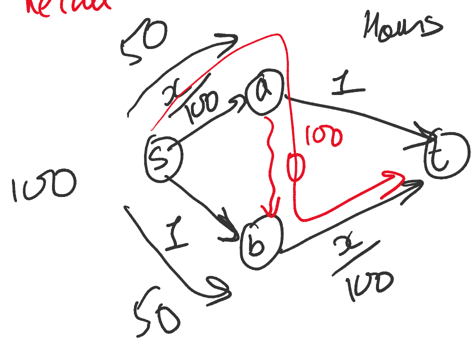
Non-atomic Selfish Routing

Thursday, October 20, 2022 10:48 AM

Recall: Braess' Paradox.



NE
$\text{cost}(50, 50) = 1.5$ hour/person
$= 1.5 \times 100 = 150.$

NE = (0, 0, 100)
$\text{cost}(NE) = 100 \times 2 = 200$

OPT = (50, 50, 0)
$\text{cost}(OPT) = 150$

$$\text{Price-of-Anarchy (PoA)} = \frac{\text{worst NE cost}}{\text{OPT cost}} = \frac{200}{150} = \boxed{\frac{4}{3}}.$$

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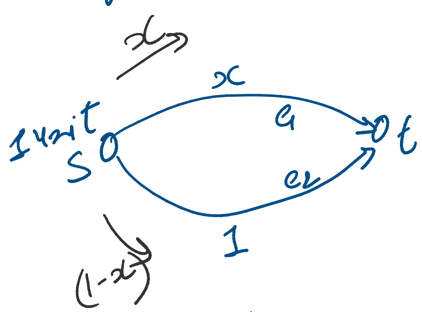
* Pigou n/w.

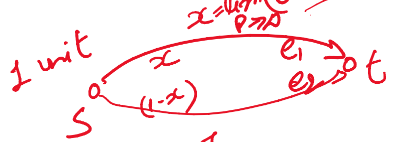


NE = (1, 0)
$\text{cost}(NE) = 1 \cdot 1 = 1$

$OPT = (x, 1-x) = \left(\frac{1}{2}, \frac{1}{2}\right)$
$\text{cost}(OPT) = \min_x x \cdot x + (1-x) \cdot 1$
$= \min_x x^2 - x + 1$
$= \frac{1}{4} - \frac{1}{2} + 1 = 1 - \frac{1}{4} = \frac{3}{4}$

$$PoA = \frac{\text{NE cost}}{\text{OPT cost}} = \frac{1}{3/4} = \boxed{\frac{4}{3}}$$

$\boxed{p \geq 1}$



$x = \lim_{p \to \infty} (0.99999999999)^p = 0$

NE = (1, 0)
$\text{cost}(NE) = 1 \cdot 1 = 1$

... 1-x)

$\text{cost (NE)} = \ldots$

$OPT = (x, 1-x)$

$$\text{cost}(OPT) = \min_{0 \le x \le 1} x \cdot x^p + (1-x) \cdot 1$$

$$= \min_x x^{p+1} - x + 1$$

$$\frac{d\, \text{cost}(OPT)}{dx} = (p+1)x^p - 1 = 0$$

$$\Rightarrow x^p = \frac{1}{p+1} \Rightarrow x = \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\lim_{p\to\infty} \text{cost}(OPT) = \lim_{p\to\infty} \left(\frac{1}{p+1}\right)^{1+\frac{1}{p}} - \left(\frac{1}{p+1}\right)^{\frac{1}{p}} + 1$$

$$= \lim_{p\to\infty} \underbrace{\left(\frac{1}{p+1}\right)}_{\to 0} \underbrace{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}_{\to 1} - \underbrace{\left(\frac{1}{p+1}\right)^{1/p}}_{\to 1} + 1$$

$$= 0 - 1 + 1 = 0$$

$$\lim_{p\to\infty} PoA = \lim_{p\to\infty} \frac{1}{\to 0} \to \boxed{\infty}$$

The degree of the cost functions matter.

Conclusion: The degree of the cost functions matter.

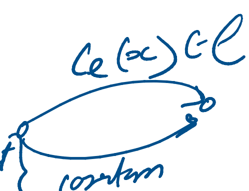
Does n/w structure also matter?

NO!! Goal.

$\mathcal{C}$: class of non-decreasing, non-negative cost functions.

Thm (informal): Given any n/w G w/ edge-costs from $\mathcal{C}$.

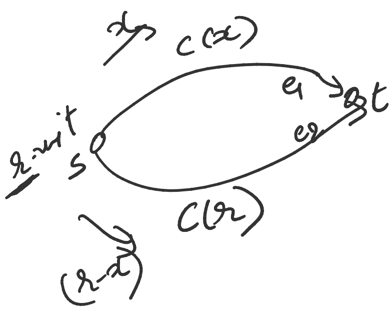
PoA of G $\le$ PoA of Pigou n/w w/ costs from $\mathcal{C} \cup \{\text{constant func}\}$



$\underbrace{\qquad\qquad\qquad}_{\alpha(\mathcal{C})}$

$$\alpha(\mathcal{C}).$$

★ $\alpha(\mathcal{C})$: $c \in \mathcal{C}$



$NE = (r, 0)$

$\text{cost}(NE) = r \cdot c(r)$

$OPT = (x, r-x)$

$$\text{cost}(OPT) = \inf_{0 \le x \le r} x \cdot c(x) + (r-x) \cdot c(r)$$

$$= \inf_{x \ge 0} x \cdot c(x) + (r-x)\, c(r)$$

$$PoA = \sup_{x \ge 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

$$\frac{d}{dx}(\cdot) = c(x) + \underbrace{x \cdot c'(x)}_{\ge 0} - c(r) = 0$$

$$c(x) + x \cdot c'(x) = c(r)$$

$$\Rightarrow c(x) \le c(r) \Rightarrow x \le r.$$

$$\alpha(\mathcal{C}) = \sup_{\substack{c \in \mathcal{C} \\ r \ge 0}} \sup_{x \ge 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

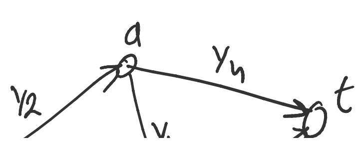
Thm: Given $G = (V, E)$, $s, t \in V$, $\forall e$: $c_e \in \mathcal{C}$, $r \ge 0$
Given $\mathcal{C}$, where r units of flow goes from s to t,

$$PoA(G, s, t, r) \le \alpha(\mathcal{C})$$

★ **Definitions:**

→ n/w $G = (V, E)$, $s, t \in V$

→ $r \ge 0$, r units going from s to t.



→ n/w $G = (V, E)$,

→ $r > 0$, r units going from s to t.

→ $\mathcal{C}$ class of functions.

$\forall e$: $c_e \in \mathcal{C}$.

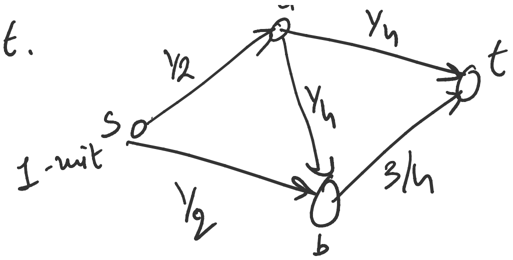
* f: Valid s-t flow of r units

→ flow on edge e: f_e

$\mathcal{P}$: set of all st paths.

→ flow on path P: f_P.

$$f_e = \sum_{P \in \mathcal{P}:\, e \in P} f_P$$



$f_{(a,b)} = \frac{1}{4}$, $f_{(s,a)} = \frac{1}{2}$

⋮

$\mathcal{P} = \{s\text{-}a\text{-}t,\ s\text{-}b\text{-}t,\ s\text{-}a\text{-}b\text{-}t\}$

$f_{s\text{-}a\text{-}t} = \frac{1}{4}$, $f_{s\text{-}b\text{-}t} = \frac{1}{2}$

$f_{s\text{-}a\text{-}b\text{-}t} = \frac{1}{4}$

Cost on edge e: $c_e(f_e)$

cost on path P: $c_P(f) = \sum_{e \in P} c_e(f_e)$

NE: No infinitisimale flow can change their path & reduce their cost.

$\equiv$

$\forall P$: $f_P > 0 \Rightarrow c_P(f) \le c_q(f)$ $\forall q \in \mathcal{P}$.

$$\text{cost}(f) = \sum_{e \in E} f_e \cdot c_e(f_e) = (\text{exe})$$

$$= \sum_{P \in \mathcal{P}} f_P \cdot c_P(f)$$

Proof of Thm: Let f: NE —

f^*: OPT —

Proof of Thm: Let f : NE —
f^* : OPT. —

Claim: $\sum_{e\in E} (f_e - f_e^*)\, c_e(f_e) \le 0$

Pf: f is NE $\Rightarrow$ $\sum_{p\in \mathcal{P}} f_p\, c_p(f) \le \sum_{p \in \mathcal{P}} f_p^*\, c_p(f)$

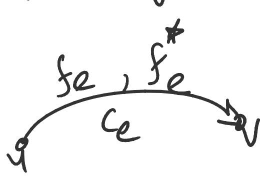
$$\Rightarrow \sum_{e\in E} f_e\, c_e(f_e) \le \sum_{e\in E} f_e^*\, c_e(f_e)$$

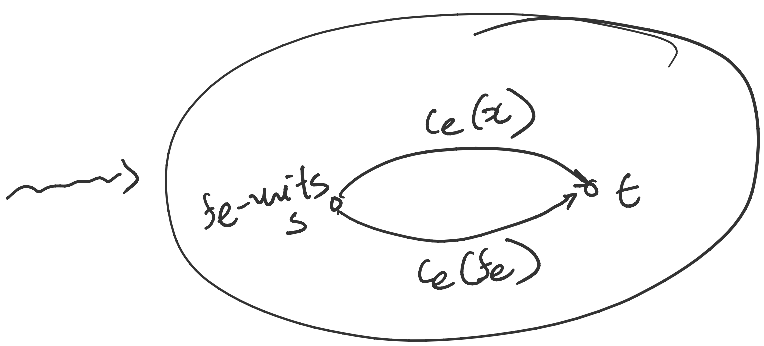
$$\Rightarrow \sum_{e\in E} (f_e - f_e^*)\, c_e(f_e) \le 0 \qquad \square$$

TPT: PoA $= \dfrac{\sum_{e\in E} f_e\, c_e(f_e)}{\sum_{e\in E} f_e^*\, c_e(f_e^*)} \le \alpha(\mathcal{C})$

$\Leftarrow$ $\forall e$: $\dfrac{f_e\, c_e(f_e)}{\alpha(\mathcal{C})} \le f_e^*\, c_e(f_e^*)$.

Pick any edge e.





$$\alpha(\mathcal{C}) \geq PoA = \sup_{x \geq 0} \frac{f_e \cdot c_e(f_e)}{x \cdot c_e(x) + (f_e - x)\, c_e(f_e)}$$

$$\geq \frac{f_e \cdot c_e(f_e)}{f_e^* \, c_e(f_e^*) + (f_e - f_e^*)\, c_e(f_e)} \quad (\text{set } x = f_e^*)$$

$$\Rightarrow \boxed{\frac{f_e \cdot c_e(f_e)}{\alpha(\mathcal{C})} \leq f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*)\, c_e(f_e) \quad \forall e}$$

$$\Rightarrow \frac{\sum_{e \in E} f_e \, c_e(f_e)}{\alpha(\mathcal{C})} \leq \sum_{e \in E} f_e^* \cdot c_e(f_e^*) + \underbrace{\sum_{e \in E} (f_e - f_e^*)\, c_e(f_e)}_{\leq 0 \; (\because \text{claim 1})}$$

(sum over all e)

$$\leq \sum_{e \in E} f_e^* \cdot c_e(f_e^*)$$

$$\Rightarrow PoA = \frac{\sum_{e \in E} f_e \, c_e(f_e)}{\sum_{e \in E} f_e^* \, c_e(f_e^*)} \leq \alpha(\mathcal{C}) \quad \blacksquare$$

G = Chicago n/w. s, t

$\forall e: c_e \in \mathcal{C}$.

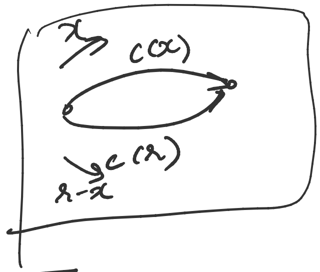
$\mathcal{C} = \{ax + b \mid a, b \geq 0\}$

r = 100,000 units of flow from s to t.

$\mathcal{C} = \{ax+b \mid a,b \geq 0\}$ $r = 100{,}000$ ... from s to t.

What is PoA?

$PoA \leq \alpha(\mathcal{C})$



$$\alpha(\mathcal{C}) = \sup_{\substack{c \in \mathcal{C} \\ r \geq 0}} \sup_{x \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

($c(x) = ax+b$)

$$= \sup_{\substack{a,b \geq 0 \\ r \geq 0}} \frac{r \cdot (ar+b)}{\inf_{x \geq 0} x \cdot (ax+b) + (r-x)(ar+b)}$$

$$\frac{d}{dx}(\cdot) = (ax+\cancel{b}) + x \cdot a - ar - \cancel{b} = 0$$

$$2ax - ar = 0$$

$$\Rightarrow x = \frac{ar}{2a} = \frac{r}{2} = \arg\inf_{x \geq 0}$$

$$= \sup_{\substack{a,b \geq 0 \\ r \geq 0}} \frac{\cancel{r}\,(ar+b)}{\frac{\cancel{r}}{2}\left(a\frac{r}{2}+b\right) + \frac{\cancel{r}}{2}(ar+b)}$$

$$= \sup_{\substack{a,b \geq 0 \\ r \geq 0}} \frac{1}{\frac{1}{2} \times \frac{1}{2}\left(1 + \frac{b}{ar+b}\right) + \frac{1}{2}}$$

$\frac{b}{ar+b}$ → minimized $b=0,\ a=1,\ r=1$

$$= \frac{1}{\ldots) + \frac{1}{2}} = \frac{1}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3/4}$$

$$= \frac{1}{\frac{1}{4}(1+0)+\frac{1}{2}} = \frac{1}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3/4}$$

$$= \boxed{\frac{4}{3}}$$