Commitment
(Stackelberg strategies)

Commitment

Unique Nash equilibrium (iterated strict dominance solution) → top-left cell

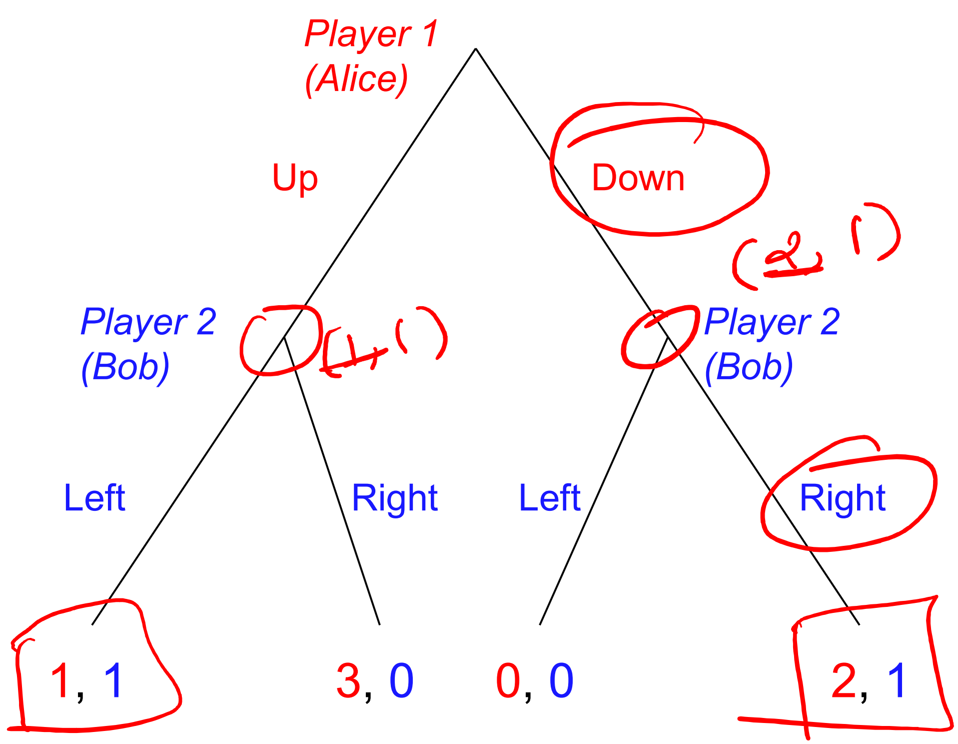
1, 1	3, 0
0, 0	2, 1

von Stackelberg

- Suppose the game is played as follows:
  - Alice **commits** to playing one of the rows,
  - Bob observes the commitment and then chooses a column
- Optimal strategy for Alice: commit to Down

Commitment: an extensive-form game

For the case of committing to a pure strategy:



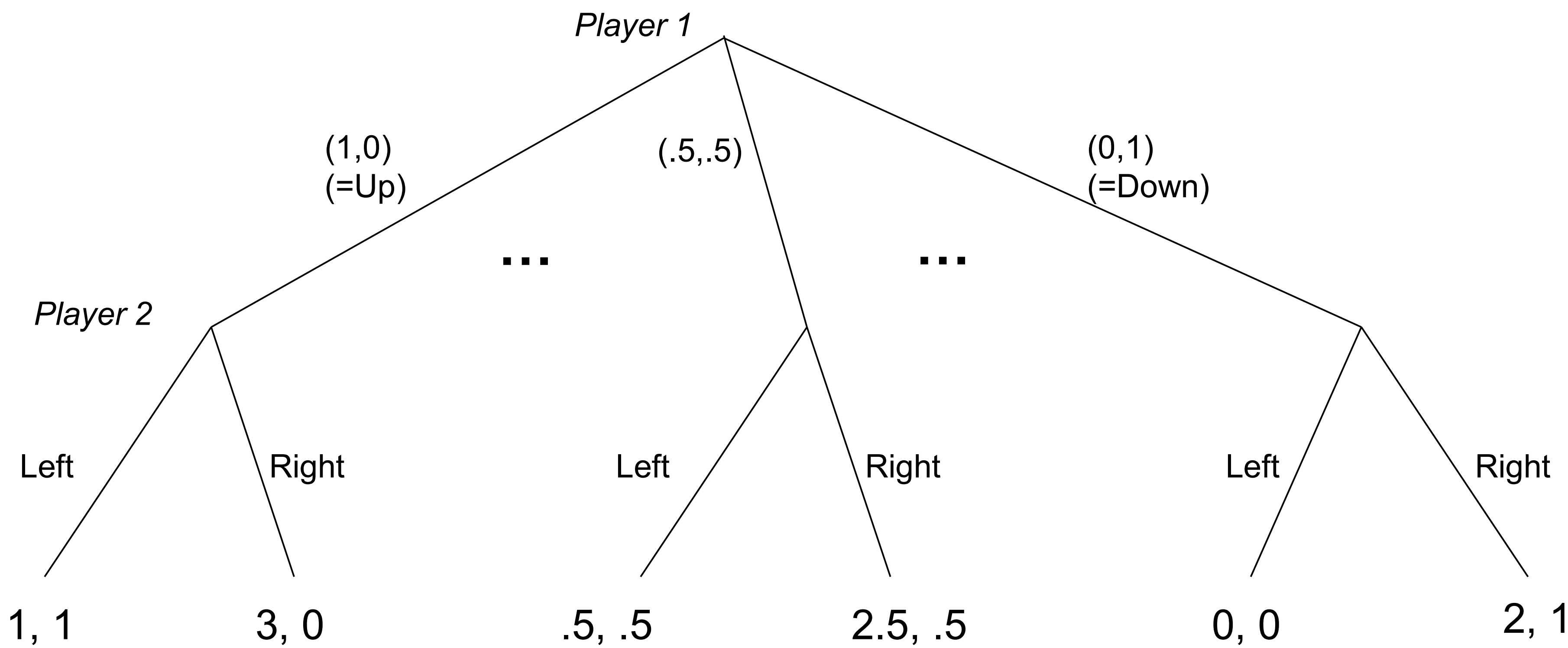
Commitment to mixed strategies

	0	1
.49	1, 1	3, 0
.51	0, 0	2, 1

Also called a Stackelberg (mixed) strategy

Commitment: an extensive-form game

- … for the case of committing to a mixed strategy:



- Economist: Just an extensive-form game, nothing new here
- Computer scientist: Infinite-size game! Representation matters

Computing the optimal mixed strategy to commit to

[Conitzer & Sandholm EC'06]

- Player 1 (Alice) is a leader.
- Separate LP for Bob's move (column) $j^* \in S_2$:

$$\text{maximize } \sum_i x_i A_{i j^*}$$

Alice's utility when Bob plays j^*

$$\text{subject to } \forall j, \ \left(x^T B\right)_{j^*} \geq \left(x^T B\right)_j$$

Playing j^* is best for Bob

$$x \geq 0, \ \sum_i x_i = 1$$

x is a probability distribution

Among soln. of all the LPs, pick the one that gives max utility.

On the game we saw before

	L	R
x_1	1, 1	3, 0
x_2	0, 0	2, 1

$$\text{maximize } 1x_1 + 0\,x_2$$

$$\text{subject to}$$

$$1\,x_1 + 0\,x_2 \geq 0\,x_1 + 1\,x_2$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0,\ x_2 \geq 0$$

$$\text{maximize } 3\,x_1 + 2\,x_2$$

$$\text{subject to}$$

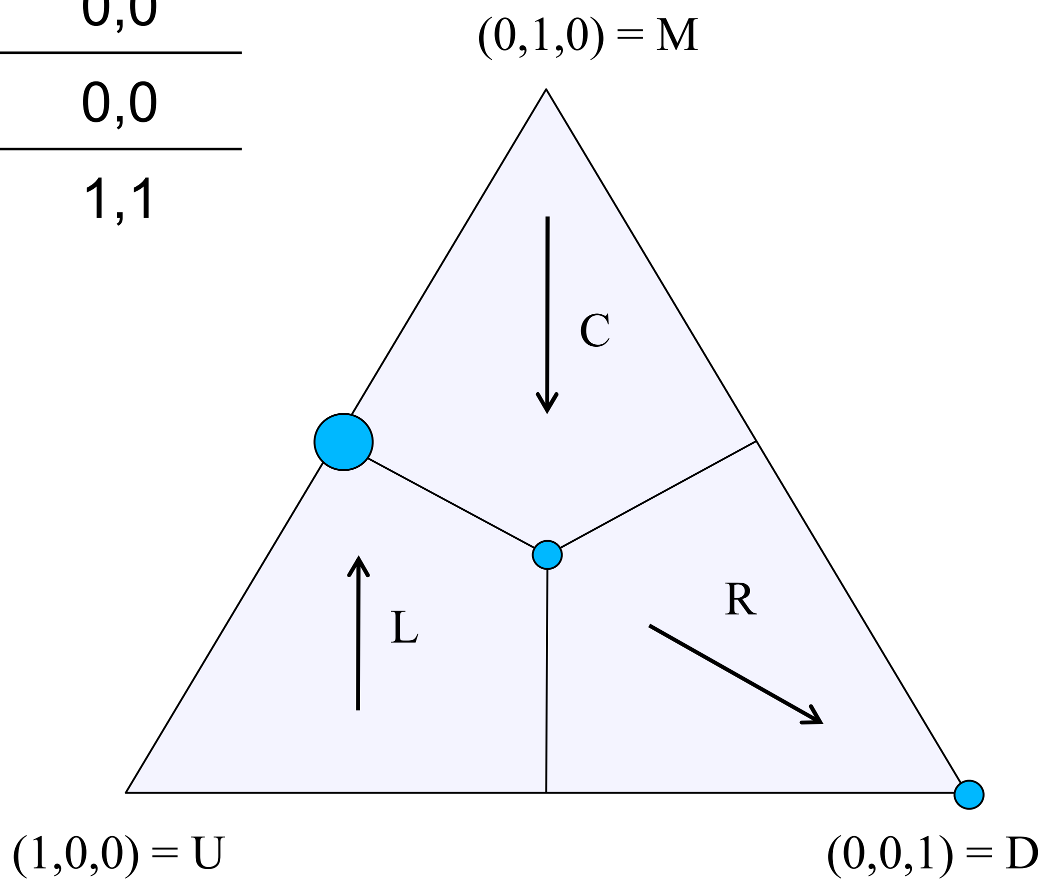
$$0\,x_1 + 1\,x_2 \geq 1\,x_1 + 0\,x_2$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0,\ x_2 \geq 0$$

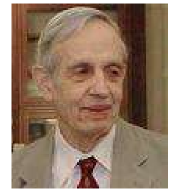
Visualization

	L	C	R
U	0,1	1,0	0,0
M	4,0	0,1	0,0
D	0,0	1,0	1,1

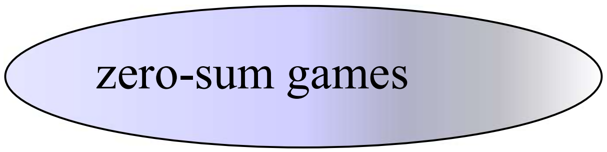


Generalizing beyond zero-sum games

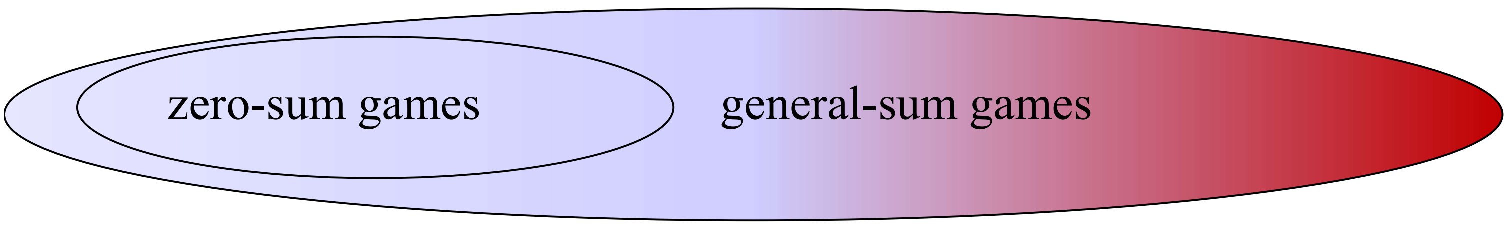
Minimax, Nash, Stackelberg all agree in zero-sum games



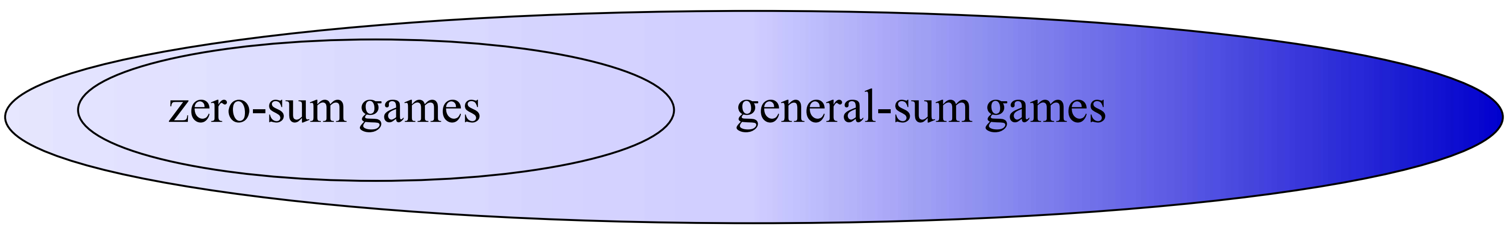
0, 0	-1, 1
-1, 1	0, 0



minimax strategies



Nash equilibrium



Stackelberg mixed strategies

Other nice properties of commitment to mixed strategies

- No equilibrium selection problem

0, 0	-1, 1
1, -1	-5, -5

- Leader's payoff at least as good as any Nash eq. or even correlated eq. (von Stengel & Zamir [GEB '10])

$\geq$

Nash Bargaining

Nash Bargaining: Dividing Utilities

Two agents: 1, 2
Outside option utilities: c_1, c_2
Feasible set of utilities: $U \subseteq R^2$ (convex),

$$(c_1, c_2) \in U$$

Goal: define a bargaining function $\mathrm{f}(c_1, c_2, U) \in U$ satisfying certain good properties

Nash Bargaining: Axioms

Two agents: 1, 2
Outside option with utilities: c_1, c_2
Feasible set of Utilities: $U \subseteq R^2$ (convex), $(c_1, c_2) \in U$

Let $(u_1^*, u_2^*) = f(c_1, c_2, U)$

Goal: $\mathrm{f}(c_1, c_2, U) \in U$ that is

1. Scale free: If $c_1' = ac_1 + b$, $U' = \{(au_1+b, u_2) \mid (u_1, u_2) \in U\}$, $a \geq 0$, $b \in \mathbb{R}$
   Then $f(c_1', c_2, U') = (au_1^* + b, u_2^*)$
2. Symmetric: If $c_1 = c_2$ & $(u_1, u_2) \in U \Leftrightarrow (u_2, u_1) \in U$
   Then $u_1^* = u_2^*$
3. Pareto Optimal: $\nexists (v_1, v_2) \in U$ s.t.
   $v_1 \geq u_1^*$ & $v_2 \geq u_2^*$ & at least one is strict.
4. Independent of Irrelevant Alternatives (IIA):
   If $U' \subset U$ & $(u_1^*, u_2^*) \in U'$ then $f(c_1, c_2, U') = (u_1^*, u_2^*)$
5. Individually Rational: $u_1 \geq c_1$, $u_2 \geq c_2$.

Nash Bargaining: Theorem

Two agents: 1, 2
Outside option with utilities: c_1, c_2
Feasible set of Utilities: $U \subseteq R^2$ (convex), $(c_1, c_2) \in U$
Goal: $\mathrm{f}(c_1, c_2, U) \in U$ that is

1. Scale free
2. Symmetric
3. Pareto Optimal ✓
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational

Theorem (Nash'50). f satisfies the 5 axioms if and only if, $f(c_1, c_2, U)$ is

$$argmax\ (u_1 - c_1)(u_2 - c_2)$$
$$\text{s.t.} \quad (u_1, u_2) \in U$$
$$u_1 \geq c_1 \ \& \ u_2 \geq c_2$$

Nash Bargaining: Theorem

Theorem (Nash'50). f satisfies the 5 axioms if and only if, $f(c_1, c_2, U)$ is

$$\begin{aligned} &argmax\ (u_1 - c_1)(u_2 - c_2) \\ &\text{s.t.} \quad (u_1, u_2) \in U, \quad u_1 \geq c_1,\ u_2 \geq c_2 \end{aligned}$$

Proof. ($\Leftarrow$)

1. Scale free

   $argmax: (a u_1 + b - a c_1 - b)(u_2 - c_2)$
   $= \text{''} \quad a(u_1 - c_1)(u_2 - c_2)$

2. Symmetric $\therefore$ $argmax: (a * b)$ s.t. $a + b = 2$ is $a = 1, b = 1$.

3. ✓ Pareto Optimal

4. ✓ Independent of Irrelevant Alternatives (IIA)

5. ✓ Individually Rational by $u_1 \geq c_1$, $u_2 \geq c_2$ constraint.