Lecture 14
Games and Nash Equilibrium

CS 580

6^{th} October 2022

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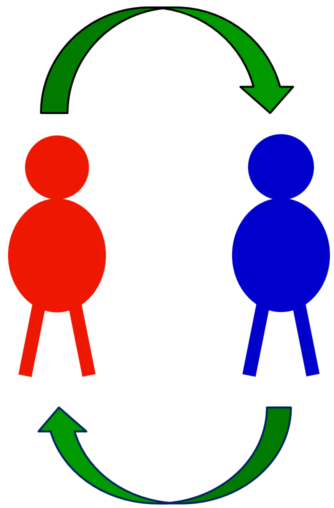
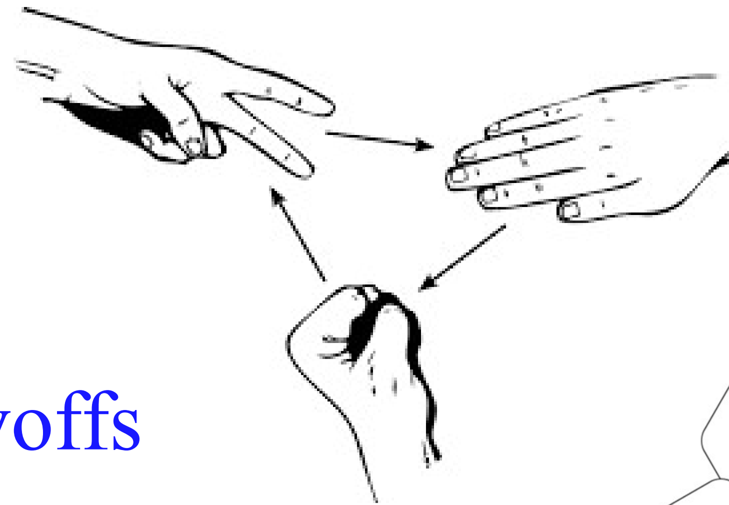
Games

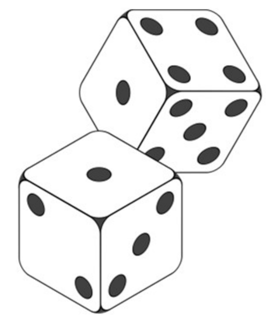


Players

Payoffs

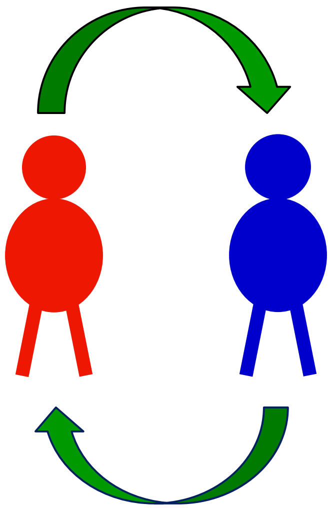


Strategies

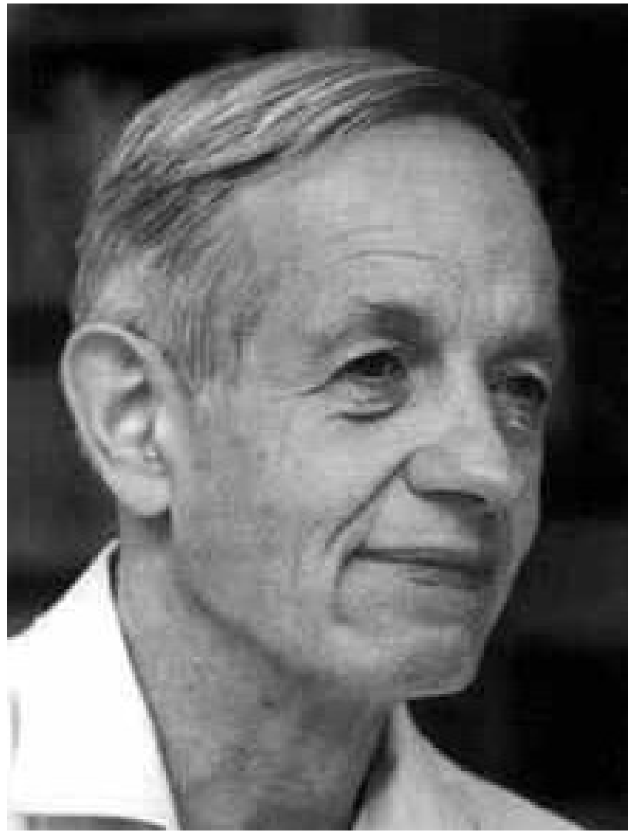


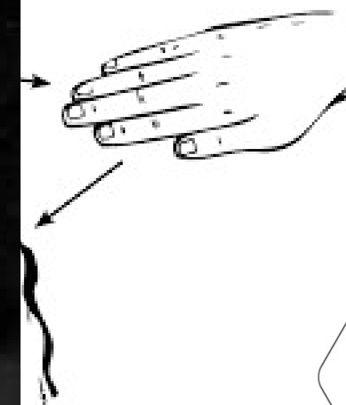
Randomize!

Games

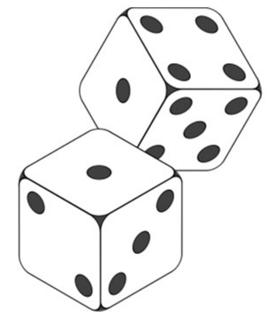


Players





tegies



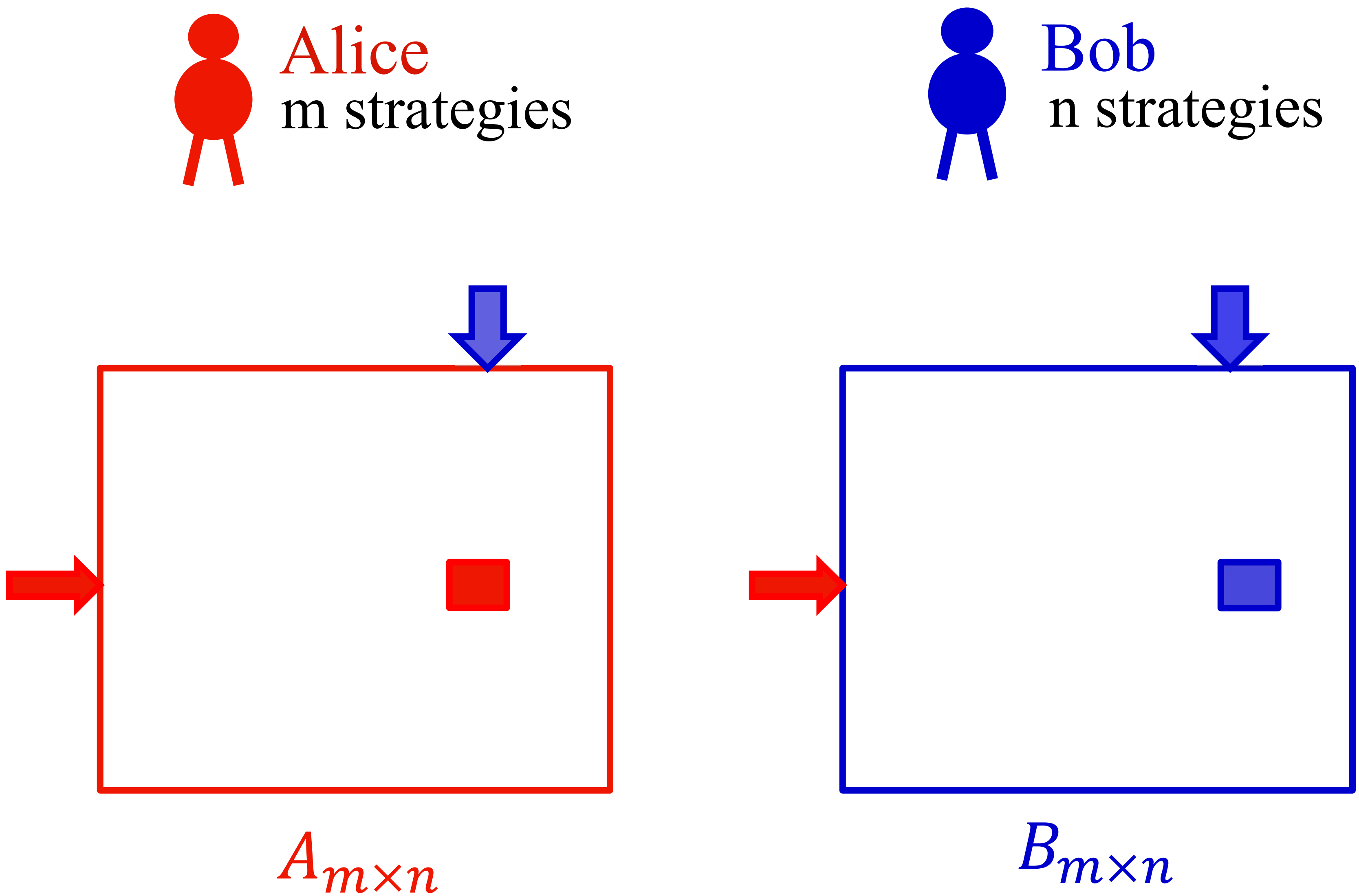
Randomize!

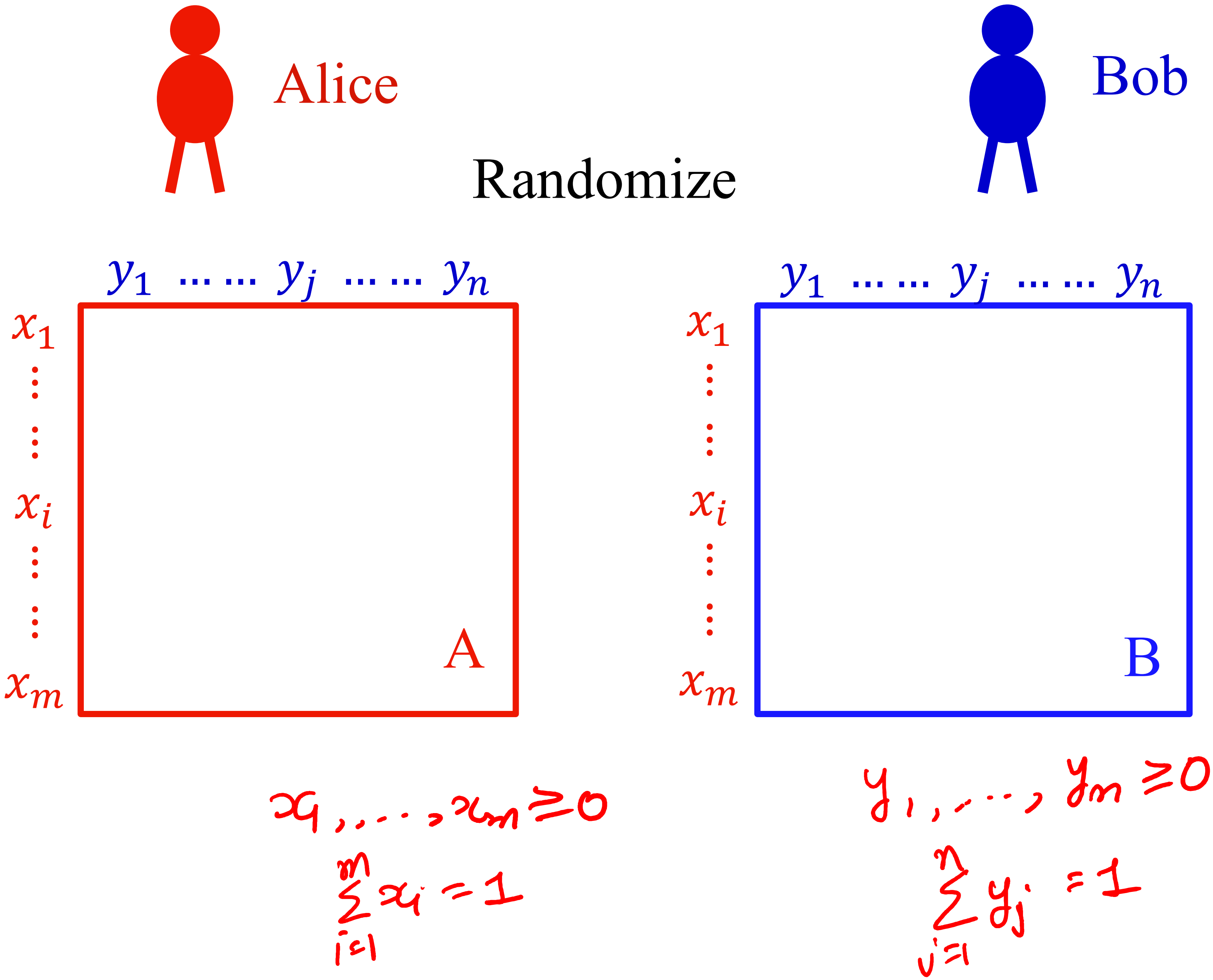
Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

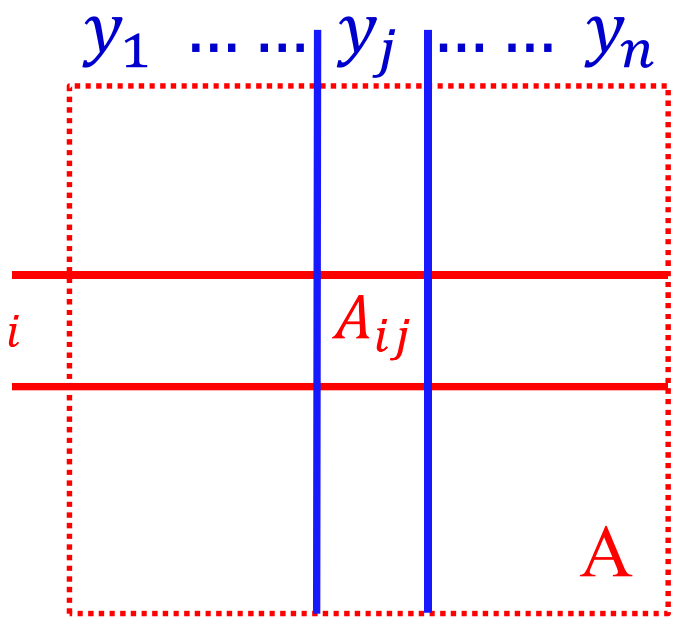
Our focus: Two-player games




Alice
Bob
Randomize
y1 …… yj …… yn
x1
xi
xm
A
y1 …… yj …… yn
x1
xi
xm
B
x1, . . . , xm ≥ 0
Σ_{i=1}^{m} xi = 1
y1, . . . , ym ≥ 0
Σ_{j=1}^{n} yj = 1

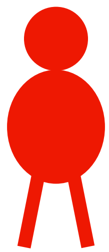
2-Nash Characterization

- For Alice, i^{th} strategy gives



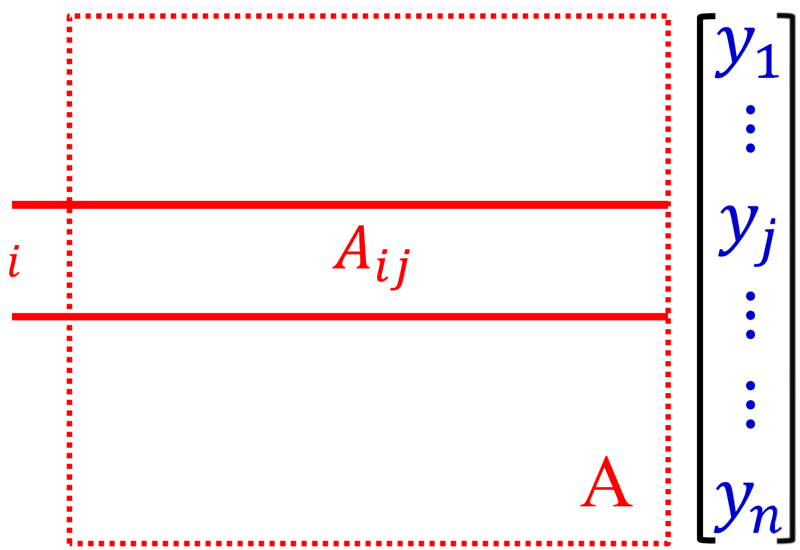
$$\longrightarrow \sum_j A_{ij} y_j$$

2-Nash Characterization



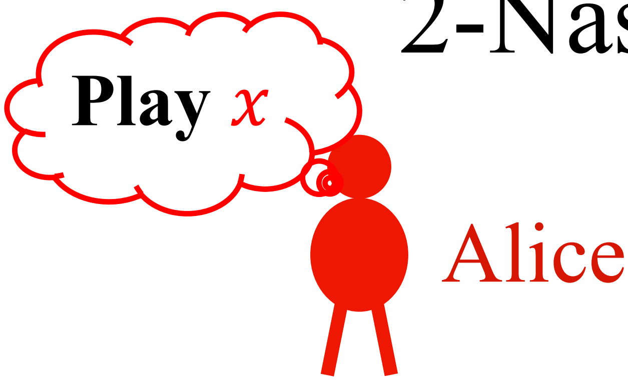
Alice

- For Alice, i^{th} strategy gives

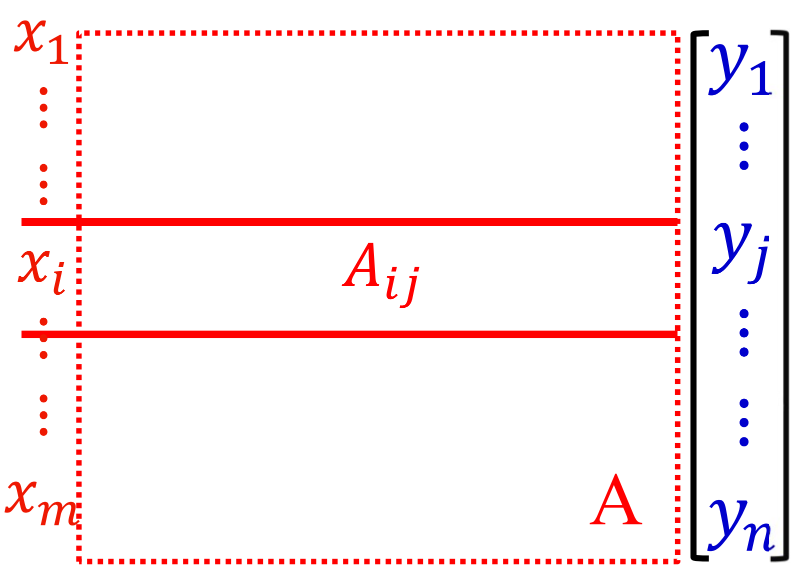


$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

2-Nash Characterization

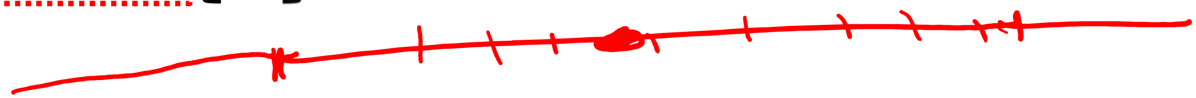


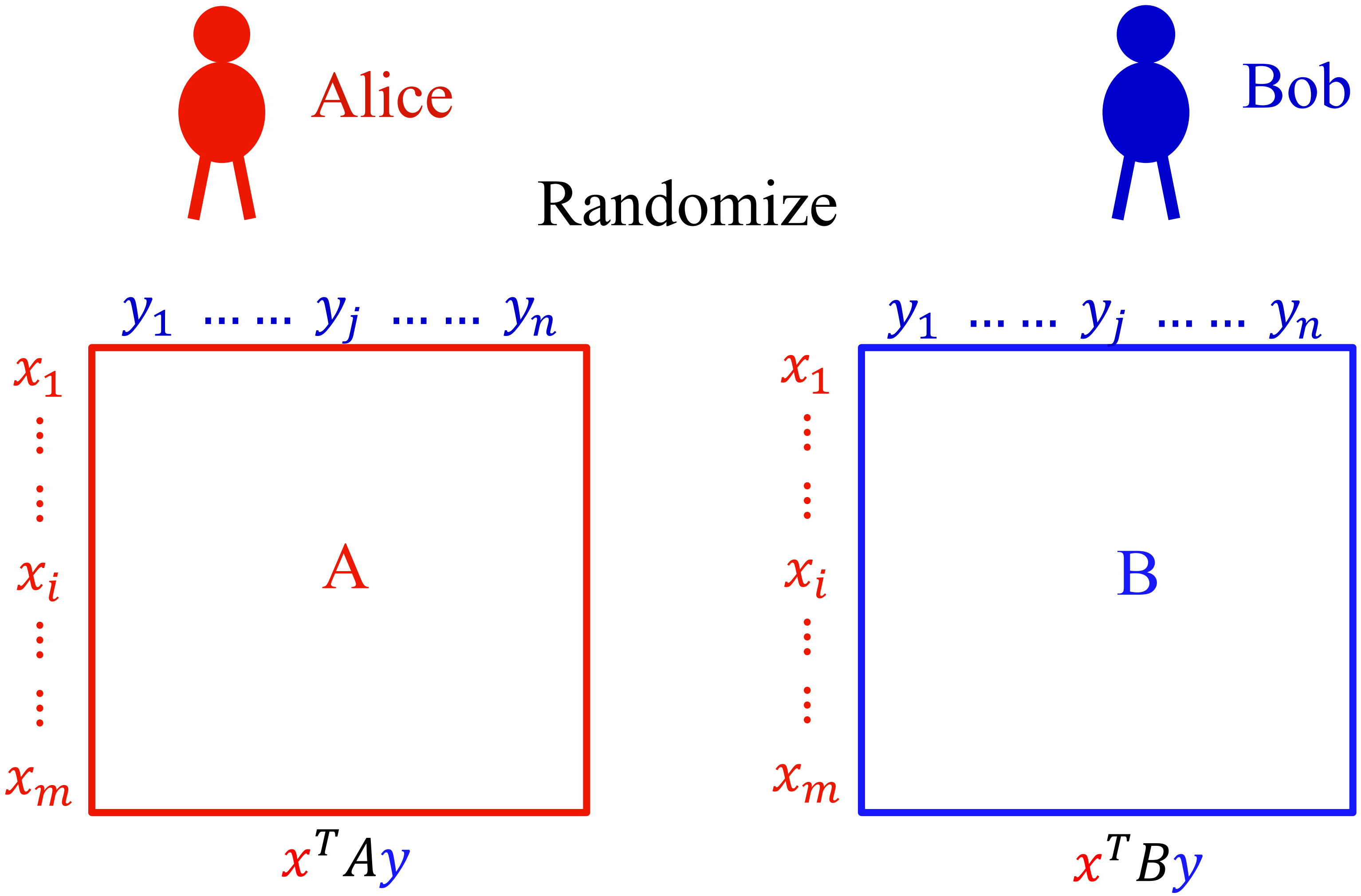
- Alice's expected payoff is



$$\sum_i x_i (Ay)_i = x^T A y$$

$\min_i (Ay)_i \le$ $\le \max_i (Ay)_i$





NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$
$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Example: Matching Pennies

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

NE: H $\frac{1}{2}$, T $\frac{1}{2}$; H $\frac{1}{2}$, T $\frac{1}{2}$

$$E^A[H] = 1 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{-1}{2}$$

$$E^A[T] = -2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{-1}{2}$$

$$E^A[(1T + 0 \cdot H)] = 1 \cdot \frac{-1}{2} + 0 \cdot \frac{-1}{2} = \frac{-1}{2}$$

$$E^B[H] = 1 \cdot -1 + 0 \cdot 2 = -1$$

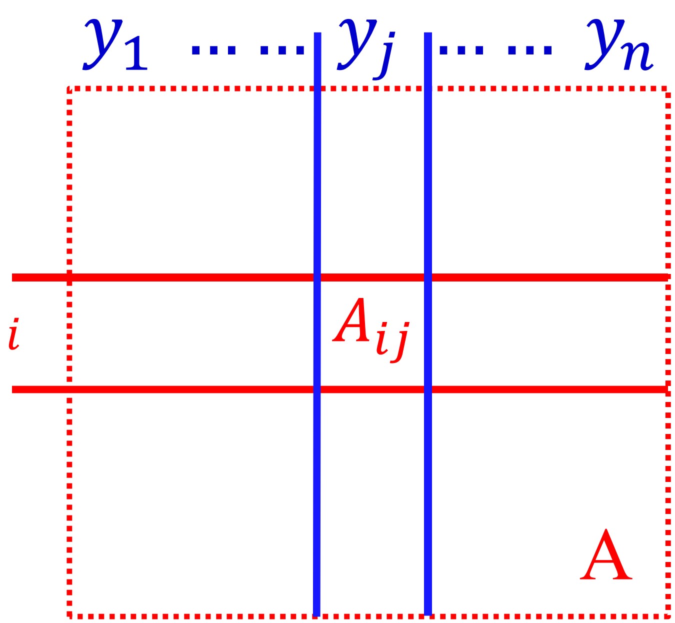
$$E^B[T] = 1 \cdot 2 + 0 \cdot (-1) = 2$$

$$E^B[(\tfrac{1}{2} \cdot H + \tfrac{1}{2} T)] = -1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{1}{2}$$

2-Nash Characterization

2-Nash Characterization

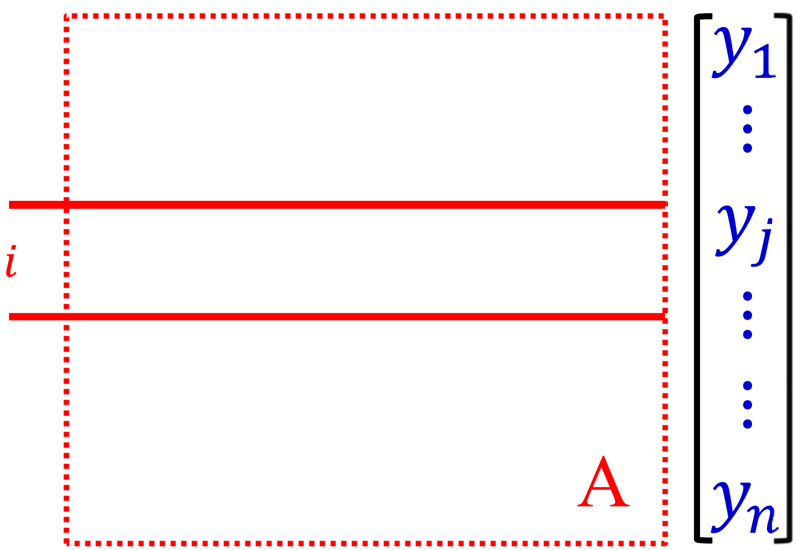
- For Alice, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j$$

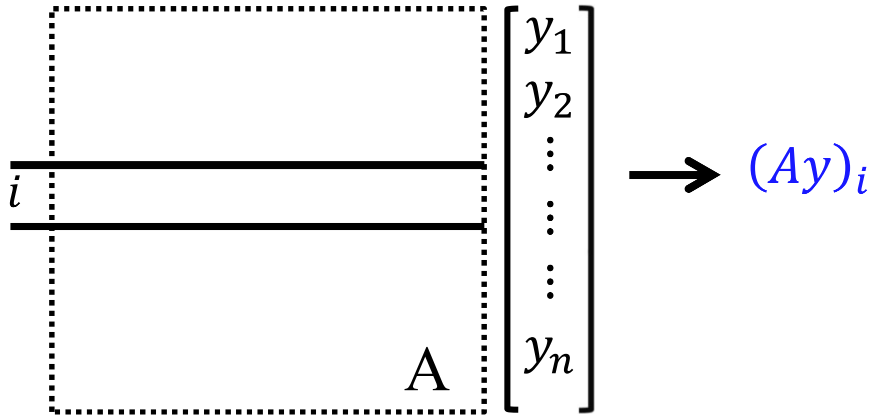
2-Nash Characterization

- For Alice, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i e_i Ay$

- x achieves max payoff iff

$$x^T Ay \geq (Ay)_i, \qquad \forall i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow k \in \operatorname{argmax}_i (Ay)_i$$

Complementarity

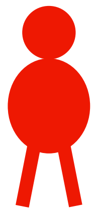
- Max possible payoff: $\max_i e_i Ay$
- x achieves max payoff iff

$$\forall i, \qquad x^T Ay \geq (Ay)_i$$
$$\equiv$$
$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

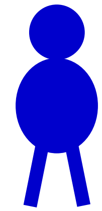
Complementarity

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

Polyhedra



$\rightarrow x$



$\rightarrow y$

max-payoff $\leq \pi_A$

max-payoff $\leq \pi_B$

P

$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

$\max_i (Ay)_i \leq \pi_A$

" max-payoff of Alice against y

$x^T A y \leq \pi_A$

$x^T B y \leq \pi_B$

$\Downarrow$

$x^T A y - \pi_A \leq 0 \quad + \quad x^T B y - \pi_B \leq 0$

$\rightarrow x^T A y + x^T B y - \pi_A - \pi_B \leq 0$

$x^T (A+B) y - (\pi_A + \pi_B)$

P

$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Sum of payoffs

At least the sum of max payoffs

$$\underbrace{x^T(A+B)y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \leq 0$$

P $\forall i, (Ay)_i \leq \pi_A$, $y \in \Delta_n$

Q $\forall j, \left(x^T B\right)_j \leq \pi_B$, $x \in \Delta_m$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Sum of payoffs

At least the sum of max payoffs

$$x^T(A+B)y - (\pi_A + \pi_B) \ \mathbf{= 0}$$

Complementarity

$\Updownarrow$

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ ✓ ✓

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A+B)y - (\pi_A + \pi_B) \le 0$. ✓

(ii) $x^T(A+B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

$\Leftarrow$ (x^*, y^*) is a NE. $\pi_A^* = x^{*T}Ay^*$, $\pi_B^* = x^{*T}By^*$

$\Downarrow$

$x^{*T}(A+B)y^* - (\pi_A^* + \pi_B^*) = 0$ ✓

$\forall i = 1, \dots, m$, $(Ay^*)_i \le x^{*T}Ay^* = \pi_A^*$, $y^* \in \Delta_n \Rightarrow (y^*, \pi_A^*) \in P$

$\forall j = 1, \dots, m$ $(x^{*T}B)_j \le x^{*T}By^* = \pi_B^*$, $x^* \in \Delta_m \Rightarrow (x^*, \pi_B^*) \in Q$.

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A+B)y - (\pi_A + \pi_B) \le 0.$

(ii) $x^T(A+B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

$x^T A y - \pi_A \le 0$, $x^T B y - \pi_B \le 0$

$(\Rightarrow)$ Let $(\tilde{y}, \tilde{\pi}_A) \in P$, $(\tilde{x}, \tilde{\pi}_B) \in Q$ s.t.

$\tilde{x}^T A \tilde{y} + \tilde{x}^T B \tilde{y} - \tilde{\pi}_A - \tilde{\pi}_B = 0$

$\Rightarrow \tilde{x}^T A \tilde{y} - \tilde{\pi}_A = 0 \Rightarrow \tilde{x}^T A \tilde{y} = \tilde{\pi}_A \ge (A\tilde{y})_i \;\; \forall i$

$\Rightarrow \tilde{x}^T A \tilde{y} = \max_i (A\tilde{y})_i$

$\Rightarrow \tilde{x}$ gives the max payoff to Alice against $\tilde{y}$

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

2-Nash

Sum of payoffs

At least the sum of max payoffs

$$\text{max: } \underbrace{x^T(A+B)y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \quad \boxed{= \mathbf{0}}$$

$$\text{s.t. } (y, \pi_A) \in P, (x, \pi_B) \in Q$$

⇕

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } x^T(A+B)y - (\pi_A + \pi_B)$$

$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

P
$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q
$$\forall j, \left(x^T B\right)_j \le \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. [von Neumann'28] (max-min = min-max) Game ~~(A, −A)~~ A
Wrt A, Alice is a maximizer and Bob minimizer

$$\max_x \min_y x^T A y \overset{\le}{=} \min_y \max_x x^T A y$$ & the max-min is NE.

Let $x^* = \arg\max_x \min_y x^T A y$, $y^* = \arg\min_y \max_x x^T A y$, then (x^*, y^*) is a NE.

Pf:

$$\max_x \min_y x^T A y = \min_y x^{*T} A y \le x^* A y^* \le \max_x x A y^* \to ① = \min_y \max_x x^T A y$$

Let $(\tilde{x}, \tilde{y})$ be a NE.

$$\min_y x^* A y \ge \min_y \tilde{x} A y = \tilde{x} A \tilde{y} = \max_x x A \tilde{y} \ge \max_x x A y^* \to ②$$

(∵ def of x^*) (∵ NE) (∵ NE) (∵ def of y^*)

$$\max_x x A y^* = x^{*T} A y^* = \min_y x^{*T} A y = \max_x \min_y x^T A y$$

$$\| $$

$$\min_y \max_x x^T A y$$

$\Rightarrow (x^*, y^*)$ is a NE.

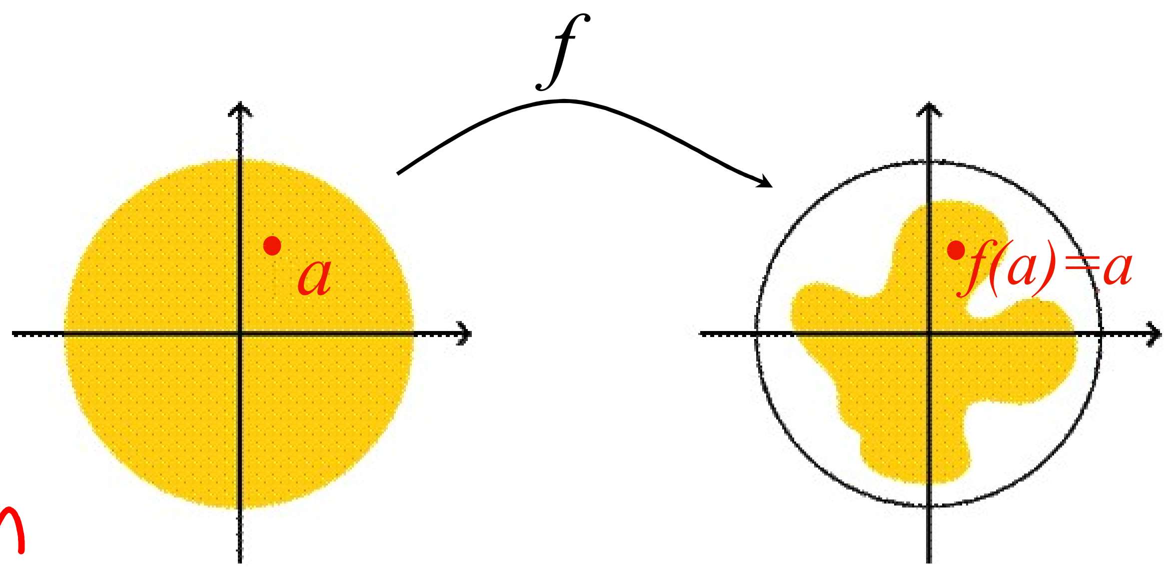
$\Rightarrow$ Convex set of NE

$\Rightarrow$ Same payoff at all NE.

$\vdots$

Computation in general?

NE existence via fixed-point theorem.



$[0,1]^n$

Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, …

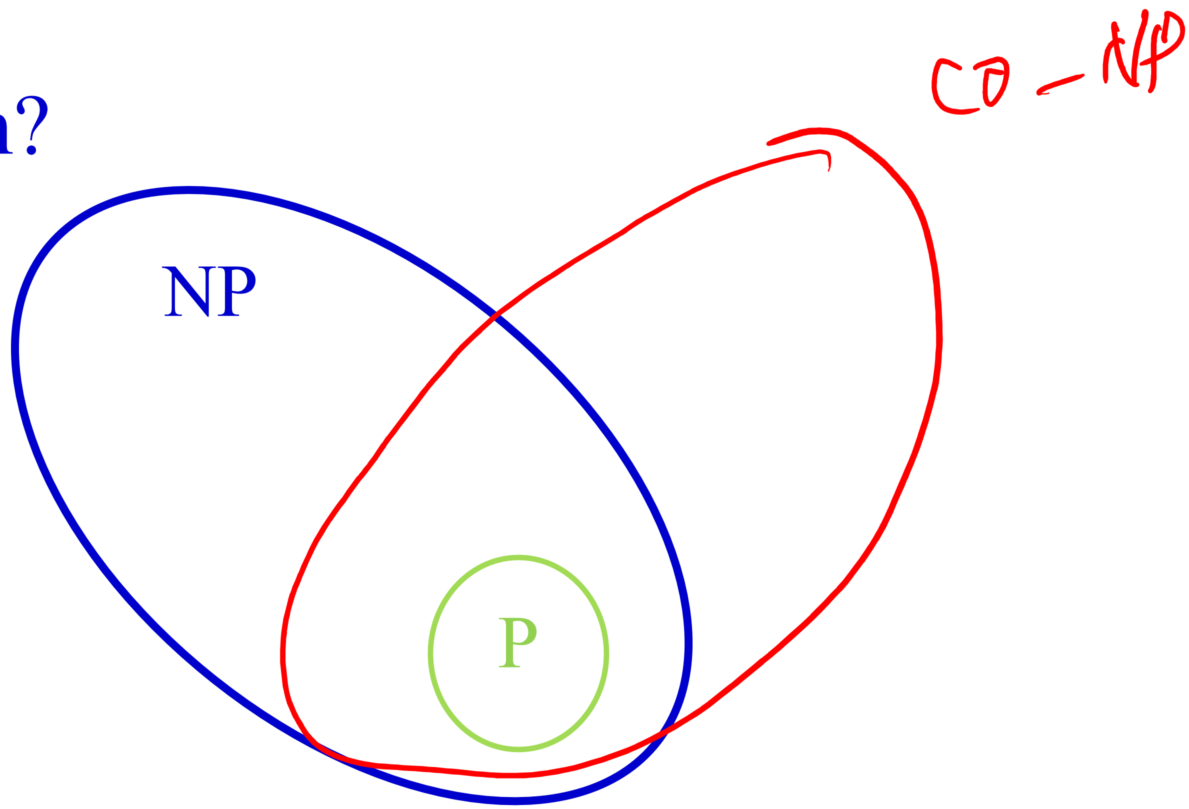
- Scarf'67: Approximate fixed-point.
  - Numerical instability
  - Not efficient!

- …

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

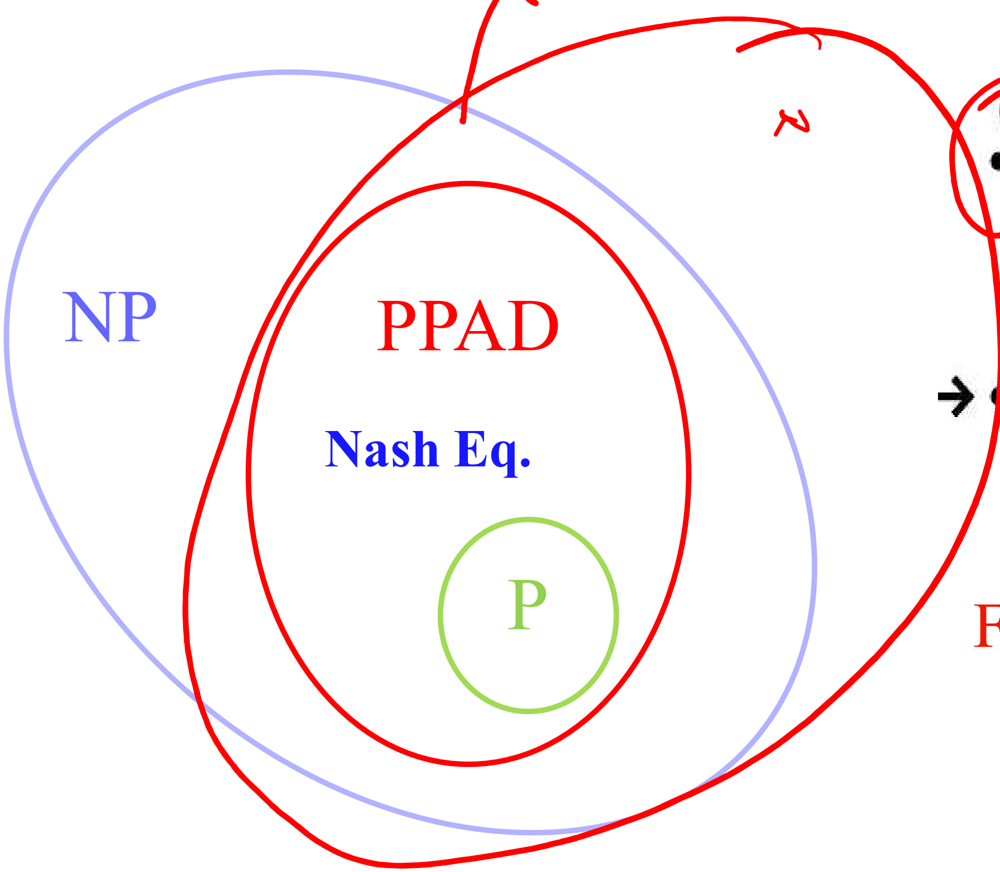
NP-hardness is ruled out!

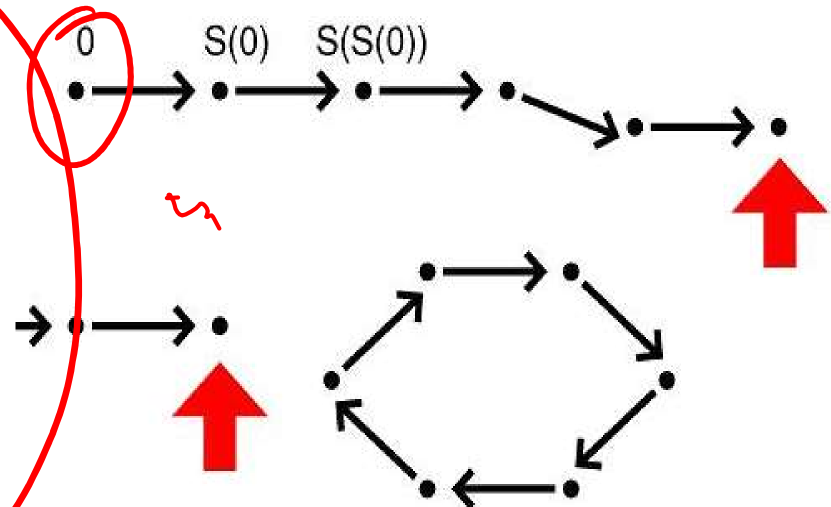
Complexity Classes

2-Nash is PPAD-complete!
[DGP'06, CDT'06]

Papadimitriou'94

PPAD Polynomial Parity Argument for Directed graph





Find an end

Brute-force Algorithm?

P
$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$

Q
$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

Let (x, y) be a NE. Suppose we know supp(x) and supp(y).

Now can we find a NE?

Can we do better?

Not so far. And may be never!

It is one of the hardest problems in PPAD.

What about special cases/approximation?

- Rank(A) or rank(B) is constant
- O(1)-approximate NE: quasi-polynomial time algorithm
- Constant rank games: rank(A+B) is a constant
  - FPTAS

P $\begin{array}{|c|} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{|c|} \forall j, \left(x^T B\right)_j \leq \pi_B \\ x \in \Delta_m \end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

Rank 1 Game

$$A + B = u \cdot v^T$$

$u \in R^m$, $v \in R^n$

2-Nash

Bilinear

$$\max: \; x^T(A+B)y - (\pi_A + \pi_B)$$

$$\text{s.t.} \quad (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank 1 Game [AGM.S'11]

$$A + B = u \cdot v^T$$

Product of two linear terms

2-Nash

$$\max: \ (x^T u)(v^T y) - (\pi_A + \pi_B)$$

$$\text{s.t.} \quad (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank 1 QP is NP-hard in general

Rank 1 Game [AGM.S'11]

$$A + B = u \cdot v^T \longrightarrow (A, u, v)$$

2-Nash

$$\text{max: } (x^T u)(v^T y) - (\pi_A + \pi_B)$$

$$\text{s.t. } P \times Q$$

$$(x^T u) v^T - x^T A$$

Think Big!

Consider game space $S = (A, *, v)$

2-Nash

$$\text{max: } (x^T *)(v^T y) - (\pi_A + \pi_B)$$

$$\text{s.t. } \quad P \times Q$$

$$(x^T *)v^T - x^T A$$

Think Big!

Consider game space $S = (A, *, v)$ All NE of S

Complementarity

Captures

LP(λ)

max: $\lambda(v^T y) - (\pi_A + \pi_B)$

s.t. $P \times Q$

$\lambda v^T - x^T A$

Solutions of LP(λ) $\forall \lambda \in \mathbb{R}$

Claim. For any $\lambda \in \mathbb{R}$, optimal value of LP(λ) is zero.

max:	$\lambda(v^T y) - (\pi_A + \pi_B)$
s.t.	$P \times Q$
	$\downarrow$
LP(λ)	$\lambda v^T - x^T A$

λ

↓

LP(λ)

↓

(y, π_R, x, π_C)

Goal: NE of $(R, \boldsymbol{u}, v)$

⇓

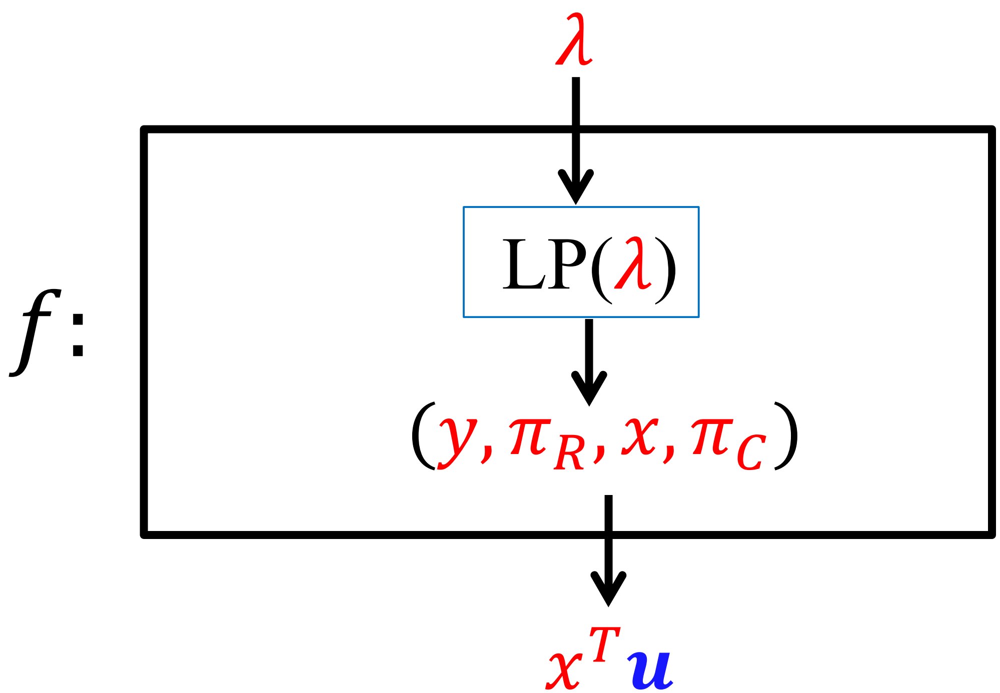
If $\boldsymbol{u}$ one of them then done!

(m-1)-dimensional space in S

Claim: $\forall \boldsymbol{c}$ s.t. $x^{\mathrm{T}}\boldsymbol{c} = \lambda$,

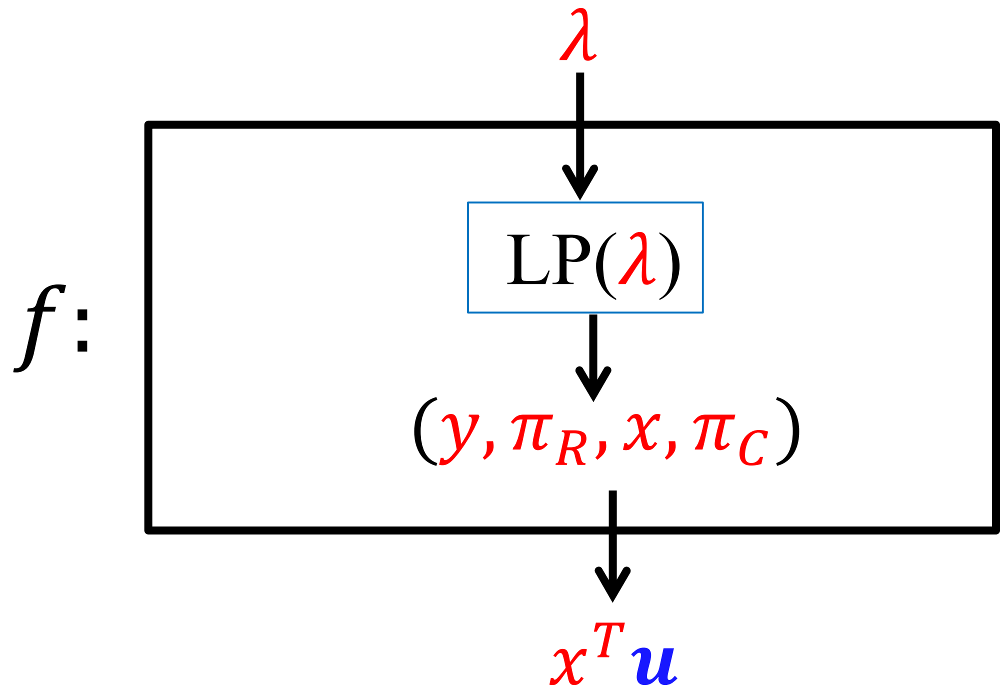
(x, y) is a NE of game $(R, \boldsymbol{c}, v)$

Goal: NE of game $(R, \boldsymbol{u}, v)$



If $x^T \boldsymbol{u} = \lambda$ then done!

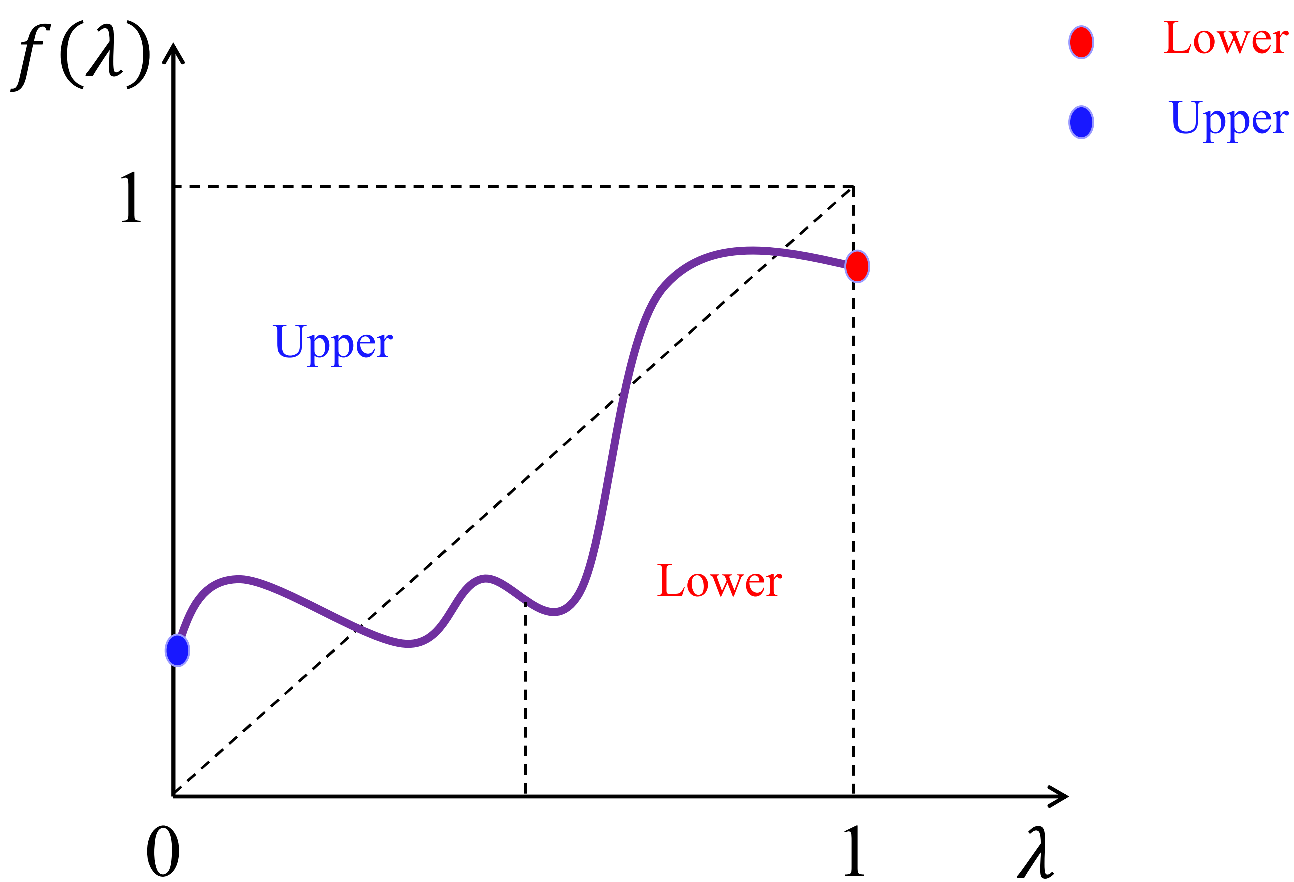
Goal: NE of game $(R, \boldsymbol{u}, v)$



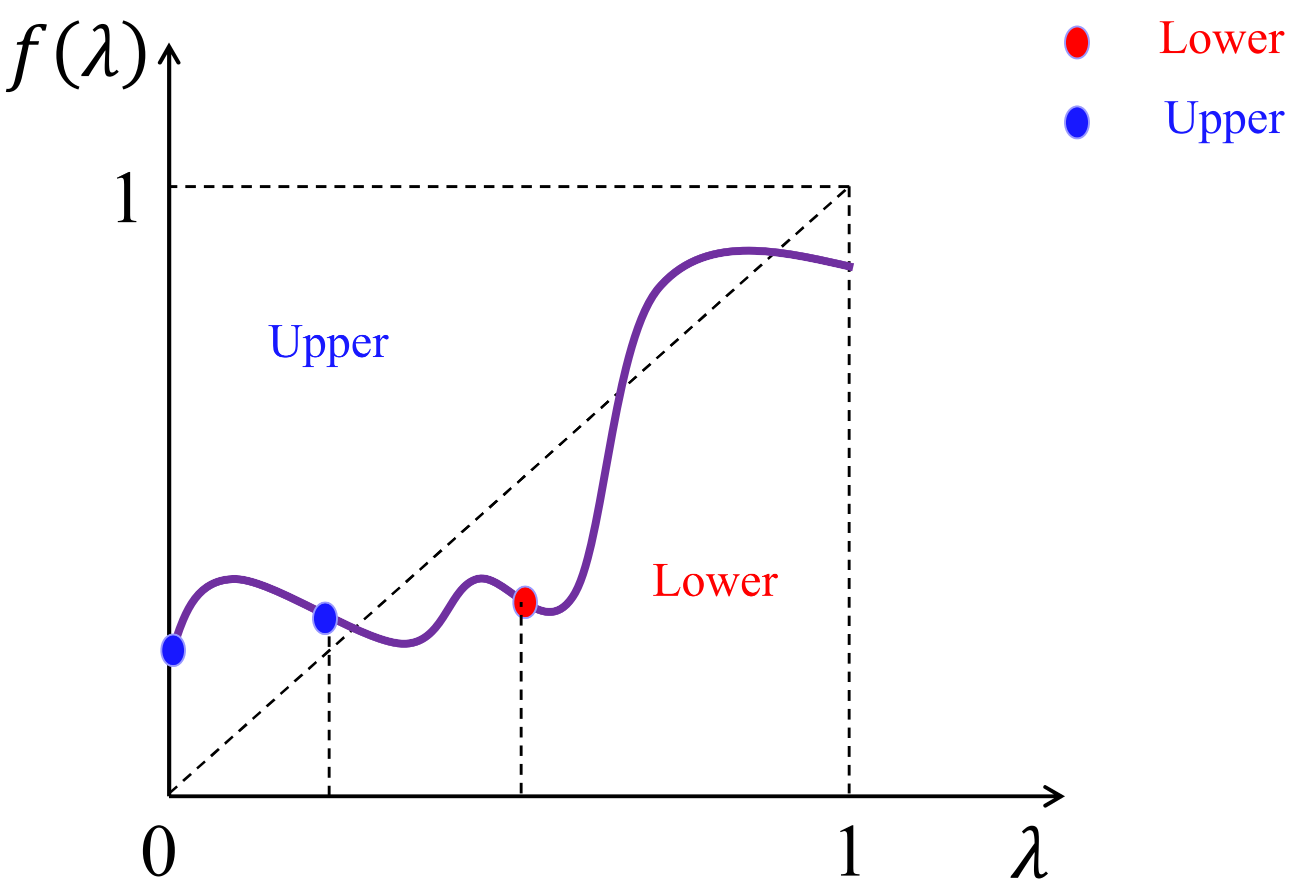
NE of game $(R, \boldsymbol{u}, v)$ $\leftrightarrow$ Fixed points of f

If $x^T \boldsymbol{u} = \lambda$ then done!

1-D Fixed Point



1-D Fixed Point



And so on until the difference becomes small enough

What about rank-2 or more?

Rank-0 (zero-sum) games rank(A+B)=0 → (Von Neumann (1928)) → LP

Rank-1 games rank(A+B)=1 →

λ

$\underline{x} = \text{LP}(\lambda)$

$\underline{c}^T \underline{x}$

Rank-2 games rank(A+B)=2 →

λ_1 λ_2

$\underline{x} = \text{LP}(\lambda_1, \lambda_2)$

$\underline{c}^T \underline{x}$ $\underline{d}^T \underline{x}$

⋮

⋮

Rank-0 (zero-sum) games → (Von Neumann (1928)) → LP

Rank-1 games → 1-D Fixed Point

In P → (LP, 1-D Fixed Point)

Rank-2 games → 2-D Fixed Point

⋮

Rank-0 (zero-sum) games → LP
Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games → 2-D Fixed Point

⋮ ⋮

Rank-0 (zero-sum) games → LP
Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games ← **?** 2-D Fixed Point

⋮

⋮

Rank-0 (zero-sum) games → LP

Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games ← 2-D Fixed Point

[M.’14, COPY’16]

⋮