Prophet Inequalities: Extensions & Variants

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Theorem: [Krengel, Sucheston & Garling '76, Samuel-Cahn '84]
For the prophet inequality setting, let τ be the median value of the distribution of $\max_i X_i$, i.e. $\Pr[\max_i X_i \geq \tau] = 1/2$. Then, the algorithm ALG that selects the first realization where $X_i \geq \tau$, obtains value:

$$\mathbb{E}[ALG] \geq \frac{1}{2}\,\mathbb{E}[\max_i X_i].$$

- $1/2$ is tight!

$X_1 = 1$ w.p. 1, $\quad X_2 = \begin{cases} L, & \text{w.p. } 1/L \\ 0, & \text{otherwise.} \end{cases}$

→ $\mathbb{E}[ALG] = 1$, for any algorithm!

→ $\mathbb{E}[\max\{X_1, X_2\}] = L \cdot 1/L + (1 - 1/L)\cdot 1 = 2 - 1/L$

$$\frac{\mathbb{E}[ALG]}{\mathbb{E}[\max\{X_1,X_2\}]} = \frac{1}{2 - 1/L}.$$

- Multiple-Item Prophet Inequality

→ k-Unit prophet inequality / PI with cardinality constraint.

Same setting as before, but we have k items to sell.

$X_1, \ldots X_n \overset{\text{ind.}}{\sim} D_1, \ldots, D_n$, we can select $\leq k$ values.

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$$\mathbb{E}\left[\max_{S\subseteq[n]:|S|\leq k} \sum_{i\in S} X_i\right]$$

Idea: Set threshold T s.t. expected # of random variables $\geq T$ is ~~$\geq k$~~ $= k-\delta$ for some appropriately chosen $\delta>0$.

(W.h.p. Actual # of realizations $\geq T$ is between $\underline{k-2\delta}$ and $\underline{k}$.) Concentration Inequality.

Chernoff Bounds

Let $Y_1, \dots, Y_n$ be real-valued random variables, such that $Y_i \in \{0,1\}$, for all i. If $S_n = \sum_{i=1}^{n} Y_i$, then for all $t>0$:

$$\Pr[S_n \geq \underline{(1+t)\mathbb{E}[S_n]}] \leq e^{-\frac{\mathbb{E}[S_n]\cdot t^2}{2+t}}, \text{ and}$$

$$\Pr[S_n \leq \underline{(1-t)\mathbb{E}[S_n]}] \leq e^{-\frac{\mathbb{E}[S_n]\cdot t^2}{2}}$$

Theorem [Hajiaghayi, Kleinberg, Sandholm '07]

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$$\mathbb{E}[ALG] \geq \left(1 - \sqrt{\frac{8 \log k}{k}}\right) \cdot \mathbb{E}\left[\max_{S : |S| \leq k} \sum_{i \in S} X_i\right].$$

$\left(1 - O\left(\sqrt{\frac{\log k}{k}}\right) \text{-approximation}\right)$

Proof

$$S_T = \{ i \in \{1, \dots, n\} \mid X_i \in T \}.$$

$\mathbb{E}[|S_T|] = k - \delta$. Let Y_i be the r.v. that is 1 if $X_i \geq T$, and 0 otherwise.

$$(1-t) \cdot \mathbb{E}[|S_T|] = k - 2\delta \iff (1-t) \cdot k - \delta = k - 2\delta \iff$$

$$(1+t) \cdot \mathbb{E}[|S_T|] = k \quad \Big| \iff \cancel{k-\delta} - t(k-\delta) = \cancel{k-\delta} - \delta \iff$$

$$\iff \boxed{t = \frac{\delta}{k-\delta}}.$$

Set t to be this

$$\Pr[k - 2\delta \leq |S_T| \leq k] \geq 1 - e^{-\frac{\mathbb{E}[|S_T|] \cdot t^2}{2+t}} - e^{-\frac{\mathbb{E}[|S_T|] \cdot t^2}{2}}$$

$$\left(\Pr[|S_T| \leq k - 2\delta] + \Pr[|S_T| \geq k] \leq \quad + \quad \right)$$

$$= 1 - e^{-\frac{(k-\delta) \cdot \frac{\delta^2}{(k-\delta)^2}}{2 + \frac{\delta}{k-\delta}}} - e^{-\frac{(k-\delta) \frac{\delta^2}{(k-\delta)^2}}{2}}$$

$$= 1 - e^{-\frac{\delta^2}{2(k-\delta)+\delta}} - e^{-\frac{\delta^2}{2(k-\delta)}}$$

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Recall that $\delta = \sqrt{2k\log k}$. Then:

$$\Pr\left[k - 2\delta \le |S_T| \le k\right] \ge 1 - e^{-\underbrace{\frac{2k\log k}{2k - \frac{1}{2}\sqrt{2k\log k}}}_{\ge \log\frac{k}{2}}} - e^{-\underbrace{\frac{2k\log k}{2k - 2\sqrt{2k\log k}}}_{\ge \log k}}$$

$$= 1 - e^{-\log k/2} - e^{-\log k}$$

$$= 1 - \frac{2}{k} - \frac{1}{k} = 1 - \frac{3}{k}$$ ($\Rightarrow$ with high probability).
(w.h.p.)

$$\sum_{i\in S_T} X_i = \sum_{i\in S_T}\left(T + X_i - T\right) = T\cdot|S_T| + \sum_{i\in S_T}(X_i - T)$$

$|S_T| \ge k - 2\delta$. $T\cdot|S_T| \ge (k-2\delta)\cdot T$.

Let S^* be the optimal set selected by the prophet.

$$OPT = \sum_{i\in S^*} X_i = \sum_{i\in S^*}\left(T + X_i - T\right) \le k\cdot T + \sum_{i=1}^{n}(X_i - T)^+ \quad (1).$$

Since $|S_T| \le k$, we accepted every value above T. Thus:

$$\sum_{i \in S_T} X_i - T = \sum_{i=1}^{n} (X_i - T)^+ \overset{(1)}{\geq} OPT - k \cdot T =$$

$$\frac{k-2\delta}{k}(OPT - k\cdot T) = \left(1 - \frac{2\delta}{k}\right) OPT - (k - 2\delta) T$$

$$\sum_{i \in S_T} X_i \geq (k-2\delta)\cdot T + \sum_{i \in S_T} X_i - T \geq \cancel{(k-2\delta)}T + \left(1 - \frac{2\delta}{k}\right) OPT - \cancel{(k-2\delta)}T$$

$$= \left(1 - \frac{2\delta}{k}\right) OPT = \left(1 - \frac{2\sqrt{2k\log k}}{k}\right)\cdot OPT$$

$$= \left(1 - \sqrt{\frac{8\log k}{k}}\right) OPT.$$

[Alaei '11] You can get $\left(1 - \frac{1}{\sqrt{k+3}}\right)$-approx. by an adaptive threshold algorithm.

[Jiang, Ma, Zhang '22] Tight ratios for all k!

- Random Order instead of adversarial.

→ Prophet Secretary problem: $\geq 1 - 1/e \simeq 0.632$ [Esfandiari et al '15].

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$\geq 1 - 1/e + 1/400$ [Azar et al '18]

≥ 0.669 [Correa et al '20].

$\leq \sqrt{3} - 1 \simeq 0.732$ — // —

→ Selling k items?

Set threshold T s.t. $\mathbb{E}[X_i \geq T] = k \cdot \gamma_k$.

$$\boxed{\gamma_k = 1 - e^{-k} \frac{k^k}{k!} \simeq 1 - \frac{1}{\sqrt{2\pi \cdot k}}.}$$

• IID Prophet Inequality

$$D_1 = D_2 = \dots = D_n = D.$$

$= 0.745.$

• Cost Minimization

Very negative results, basically no bounded approximation in general.