Prophet Inequality

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Simple vs Optimal Auctions.

$v_1, \dots, v_n \overset{i.i.d.}{\sim} D$ Optimal auction = Vickrey.

$\phi(v) = v - \frac{1-F(v)}{f(v)}$, F cdf of D, $f = F'$ pdf of D.

1) $v_1, v_2, \dots, v_n \overset{i.i.d}{\sim} D$, but D is unknown.

Theorem: [Bulow-Klemperer '96]

Let D be a regular distribution (ϕ is strictly increasing), and n be a positive number. Then:

$$\mathbb{E}_{v_1,\dots,v_{n+1} \sim D}\left[\mathrm{Rev}(VA_{n+1})\right] \geq \mathbb{E}_{v_1,\dots,v_n \sim D}\left[\mathrm{Rev}(OPT_{D,n})\right].$$

Proof: Let A be a third auction on $n+1$ bidders.

A:

1) Simulate $OPT_{D,n}$ on the first n bidders.
2) If step 1 does not allocate the item, give it to bidder $n+1$ <u>for free.</u>

- $\mathbb{E}[\mathrm{Rev}(A_{n+1})] = \mathbb{E}[\mathrm{Rev}(OPT_{D,n})]$
- A always allocates the item.

$\mathcal{A}'$ always allocates the item.

(WTS that VA is optimal among all auctions which always allocate the item)

By Myerson's Opt auction, $OPT_{D,n+1}$ allocates the item to the bidder w/ highest virtual welfare.

VA_{n+1} allocates the item to the bidder w/ highest welfare.

When D is regular $\Rightarrow$ ϕ is increasing.

$\hookrightarrow$ $\boxed{\text{Bidder w/ highest welfare} = \text{Bidder w/ highest virtual welfare}}$

$$\mathbb{E}[Rev(VA_{n+1})] = \mathbb{E}[Rev(OPT_{D,n+1} \text{ has to allocate})] \geq$$

$$\geq \mathbb{E}[Rev(\mathcal{A}_{n+1})] = \mathbb{E}[Rev(OPT_{D,n})] \quad \blacksquare$$

Corollary: VA_n w/ no reserve price is a $(1 - 1/n)$-approximation to OPT_n.

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2) $v_1, \dots, v_n \overset{ind.}{\sim} D_1, \dots, D_n$ and $D_1, \dots, D_n$ known distributions.

- Grocer's Auction

→ Set a price p for item.

→ Buyers arrive*, one after the other and if $v_i \geq p$, buyer i buys the item.

* in adversarial order. (worst-case)

Properties

- Trivially truthful.
- We don't have to collect bids.
- After price is set, auction runs itself.

Theorem: [Hajiaghayi et al '07, Chawla et al '10]
The grocer's auction is equivalent to the prophet inequality.

- Prophet Inequality

$X_1, X_2, \ldots, X_n \overset{ind.}{\sim} D_1, D_2, \ldots, D_n$.

At step i, we see realization z_i of X_i. We decide immediately and irrevocably, whether to accept or reject z_i.

→ If we accept z_i, game ends.

→ Else, we proceed to z_{i+1}.

Objective: Select $\max z_i$ compare ...

Objective: Select $\max_i Z_i$. Compare against prophet, who always gets $\max_i Z_i$

Prophet's value: $\mathbb{E}[\max_i X_i]$

Theorem: [Krengel, Sucheston & Garling '76, Samuel-Cahn '84]

For the prophet inequality setting, let τ be the median value of the distribution of $\max_i X_i$, i.e. $\Pr[\max_i X_i \geq \tau] = 1/2$. Then, the algorithm ALG that selects the first realization where $X_i \geq \tau$, obtains value:

$$\mathbb{E}[ALG] \geq \frac{1}{2}\mathbb{E}[\max_i X_i].$$

$(X_i - \tau)^+ = \max\{X_i - \tau, 0\}$

Proof Let $X^* = \max_i X_i$.

$$\mathbb{E}[ALG] = \tau \cdot \Pr[X^* \geq \tau] + \sum_{i=1}^{n} \Pr[\text{We "reach" } X_i] \cdot \mathbb{E}[(X_i - \tau)^+]$$

$$= \frac{1}{2}\tau + \sum_{i=1}^{n} \Pr[\max_{j=1}^{i-1} X_j < \tau] \cdot \mathbb{E}[(X_i - \tau)^+]$$

$$\geq \frac{1}{2}\tau + \sum_{i=1}^{n} \Pr[\max_{j=1}^{n} X_j < \tau] \cdot \mathbb{E}[(X_i - \tau)^+]$$

$$= \frac{1}{2}\tau + \sum_{i=1}^{n} \Pr[X^* < \tau] \cdot \mathbb{E}[(X_i - \tau)^+]$$

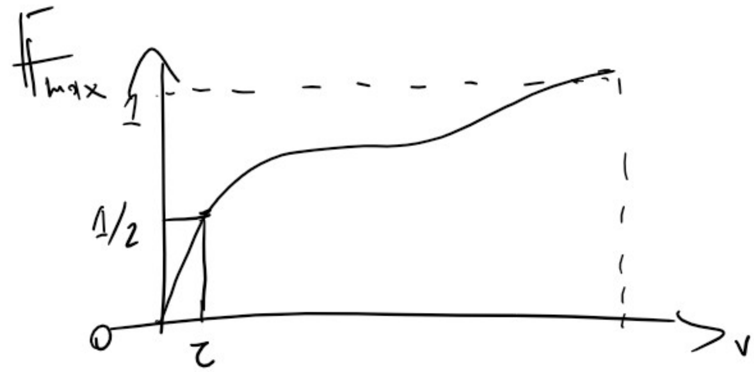
$$= \frac{1}{2}\tau + \sum_{i=1}^{n} 1/2 \cdot \mathbb{E}[(X_i - \tau)^+]$$

$$= \frac{1}{2}\tau + \sum_{i=1} \ldots /2 \ldots$$

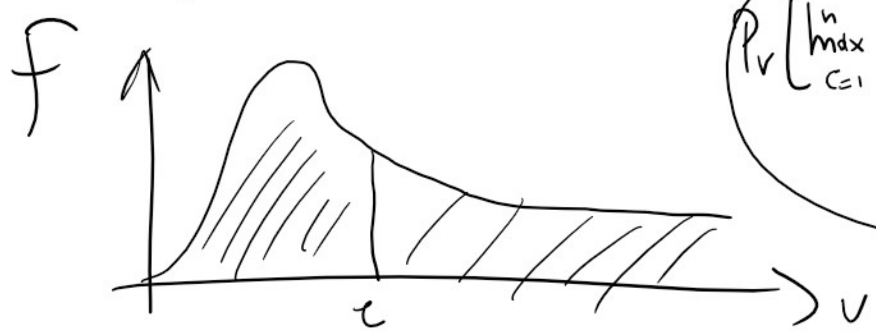
$$\geqslant \frac{1}{2}\tau + \frac{1}{2}\,\mathbb{E}[(X^* - \tau)^+]$$

$$\geqslant \frac{1}{2}\tau + \frac{1}{2}\,\mathbb{E}[X^* - \tau]$$

$$= \frac{1}{2}\left(\tau + \mathbb{E}[X^* - \tau]\right) = \frac{1}{2}\mathbb{E}[X^*] = \frac{1}{2}\mathbb{E}[\max_{i=1}^{n} X_i]. \quad \blacksquare$$



$$F_{\max}(v) = \Pr[X \le v].$$



$$\Pr[\max_{i=1}^{n} X_i \le v] = \prod_{i=1}^{n} \Pr[X_i \le v]$$

$$\Pr[X_i \le v] \to F_i(v)$$

$$= F_1(v) \cdot F_2(v) \cdot \ldots \cdot F_n(v).$$

[Robinstein et al '20]

[Kleinberg-Weinberg '12]: Just set $\tau = \frac{1}{2}\mathbb{E}[\max_{i=1}^{n} X_i]$