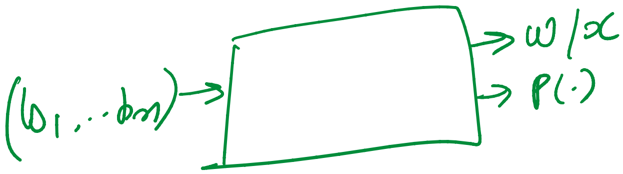
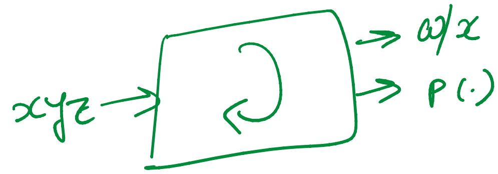
Mechanism Design (Revenue Maximization)

Thursday, September 22, 2022 11:01 AM

Direct vs Indirect





Sponsored search
Vickrey.

English
Dutch
Spectrum Auction

DSIC $\Longleftarrow$ DSIC

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① DSIC ② s.w. maximization → governments ③ poly-time computation

↓

② Revenue maximization → private companies.

Examples:

① Single item, single bidder

→ Vickrey Rev $= 0$

→ Post take-it-or-leave-it Price $P > 0$

$V \sim U[0,1]$

$$\mathbb{E}[Rev] = \max_p P \cdot \Pr[V \geq P]$$

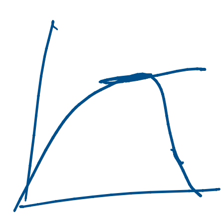
$$= \max_p P \cdot (1-P)$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

$$\frac{d}{dp}\, P - P^2 = 0$$

$$\Rightarrow 1 - 2P = 0$$

$$\Rightarrow P = \frac{1}{2}$$



$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$ $\Rightarrow p = \frac{1}{2}$

② Two bidders $v_1, v_2 \sim U[0,1]$

$\Rightarrow$ Vickrey Rev $= \mathbb{E}[\min\{v_1, v_2\}] = \frac{1}{3}$

reserve price.

$\Rightarrow$ Highest bidder wins, pays $\max\left\{\frac{1}{2}, \underbrace{\text{second-highest bid}}\right\}$

- $v_1, v_2 < \frac{1}{2} \Rightarrow \text{Rev} = 0$ (loss)
- $v_1 \geq \frac{1}{2} \geq v_2 \Rightarrow \text{Rev} = \frac{1}{2}$ (gain)
- $v_1 > v_2 \geq \frac{1}{2} \Rightarrow \text{Rev} = v_2$ (same)

$$\mathbb{E}[\text{Rev}] = \frac{5}{12} > \frac{1}{3} \, !$$

Rev. maximizing DSIC Mechanisms.

Q: Design Rev. maximizing DSIC Mechanisms.

☆ Single-Parameter Setting:

- $N = \{1, \dots n\}$
- $i \in N$, $v_i \sim D_i$ → f_i : density function $D = \times_i D_i$

private public F_i : cumulative D.F. $v = (v_1, \dots, v_n)$

$\Rightarrow X$ = set of feasible allocations.

Design space.

Myerson: (x, p) is DSIC $\Leftrightarrow$ $\boxed{x \text{ is monotone. } \&}$

p is a specific formula depending on x.

p is a specific ... depending on x.

Goal: Design an allocation rule that leads to rev. maximization.

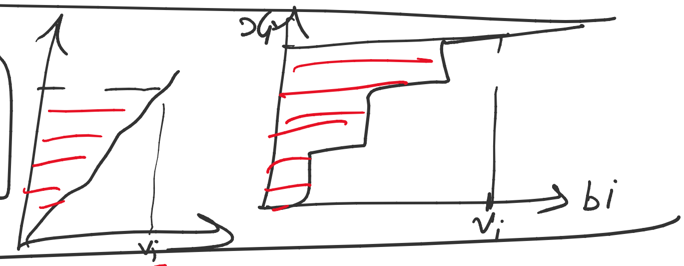
(Proof by reverse engineering & assuming DSIC) $\forall i,\ b_i = v_i$

$$\max_{(v_1,\dots,v_n)\sim \times_i D_i} \mathbb{E}[Rev] = \max \mathbb{E}_{v\sim D}\left[\sum_{i=1}^{n} p_i(v)\right]$$

$$= \max \sum_{i=1}^{n} \mathbb{E}_{v\sim D}[p_i(v)]$$

$$= \max \sum_{i=1}^{n} \mathbb{E}_{v_{-i}\sim D_{-i}} \boxed{\mathbb{E}_{v_i\sim D_i} p_i(v_i, v_{-i})} \to (1)$$

Myerson's lemma: $\boxed{p_i(v_i, v_{-i}) = \int_0^{v_i} z\, x_i'(z, v_{-i})\, dz}$

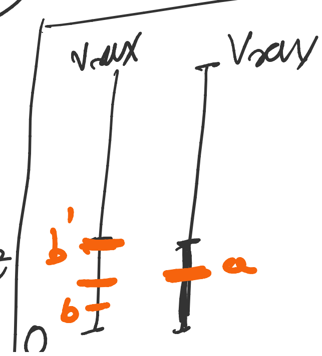


Fix i, Fix v_{-i} $\quad x = x_i \quad p = p_i \quad f = f_i \quad F = F_i$

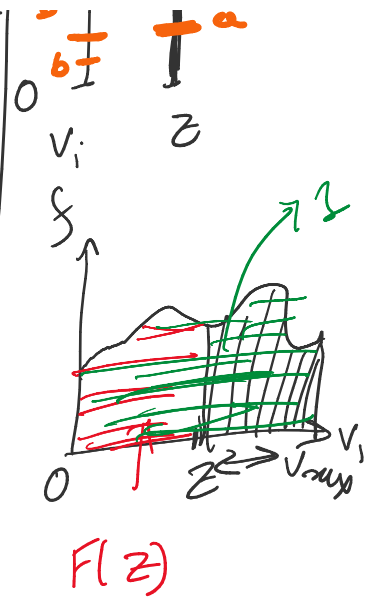
$$(1) = \mathbb{E}_{v_i\sim D_i}[p_i(v_i, v_{-i})] = \int_0^{v_{max}} p_i(v_i, v_{-i}) \cdot f(v_i)\, dv_i$$

$$= \int_0^{v_{max}} \left(\int_0^{v_i} z\, x_i'(z, v_{-i})\, dz\right) f(v_i)\, dv_i \quad (\because \text{Myerson})$$

$$= \int \left(\int^{v_{max}} f(v_i)\, dv_i\right) z\, x'(z, v_{-i})\, dz$$



$$\underset{(\because \text{swap sums})}{=} \int_0 \Big(\underbrace{\int_z f(v_i)\, dv_i}_{} \Big) z\, x'(z, v_{-i})\, dz$$



$$= \int_0^{v_{max}} \underbrace{(1 - F(z))}_{f} \cdot z \cdot \underbrace{x'(z, v_{-i})}_{g'}\, dz$$

$$\underset{\left(\int f g' = fg - \int f' g\right)}{=} \Big[(1-F(z)) \cdot z \cdot x(z, v_{-i}) \Big]_0^{v_{max}}$$

$$- \int_0^{v_{max}} \big(1 - F(z) - z f(z)\big) \cdot x(z, v_{-i})\, dz$$

$$\underset{(v_i = z)}{=} \int_0^{v_{max}} \Big[v_i f(v_i) - \big(1 - F(v_i)\big) \Big] \cdot x(v_i, v_{-i})\, dv_i \quad \cdot f(v_i)$$

$f(v_i)$

$$= \int_0^{v_{max}} \boxed{\Big[v_i - \underbrace{\Big(\frac{1 - F_i(v_i)}{f_i(v_i)} \Big)}_{} \Big]} \cdot x(v_i, v_{-i})\, f(v_i)\, dv_i$$

↳ Information rent of auctioneer.

Virtual value of agent i

$\phi_i(v_i)$

$$\int^{v_{max}} \phi_i(v_i) \cdot x_i(v_i, v_{-i}) \cdot f(v_i)\, dv_i$$

$$= \int_0^{v_{max}} \underbrace{\phi_i(v_i) \cdot x_i(v_i, v_{-i})}_{\Uparrow \text{ virtual welfare of agent } i.} \cdot f(v_i)\, dv_i$$

$$\mathbb{E}_{v_i \sim D_i}\left[p_i(v_i, v_{-i})\right] = \mathbb{E}_{v_i \sim D_i}\left[\phi_i(v_i)\, x_i(v_i, v_{-i})\right]$$

$$\max \mathbb{E}_{v \sim D}\left[Rev\right] = \max \mathbb{E}_{v \sim D}\left[\sum_{i=1}^{n} p_i(v)\right]$$

$$= \max \mathbb{E}_{v \sim D}\left[\underbrace{\sum_{i=1}^{n} \phi_i(v_i)\, x_i(v_i, v_{-i})}_{\text{virtual social welfare.}}\right]$$

$$\Updownarrow$$

Rev. max DSIC mechanism. if $x^*(\cdot)$ monotone

$$\boxed{x^*(v) = \arg\max_{x \in X} \sum_{i=1}^{n} \phi_i(v_i)\, x_i}$$

Then by Myerson's payment formula gives $p^*(\cdot)$

(x^*, p^*) is DSIC $\Longleftarrow$ x^* monotone $\Longleftarrow$ ϕ_i's are monotone.

///

D_i's are "regular"

Example: Single item, n bidders.

assume DSIC ($\forall i$, $b_i = v_i$).

$$X = \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} x_i \le 1 \right\}$$

$$\left(v_i - \frac{(1-F(v_i))}{f(v_i)} \right)$$

$$x^*(v) = \arg\max_{x \in X} \sum_{i=1}^{n} \boxed{\phi_i(v_i)}\, x_i$$

= give item to the agent w/ highest virtual value.

if their $\phi_i(v_i) \ge 0$.

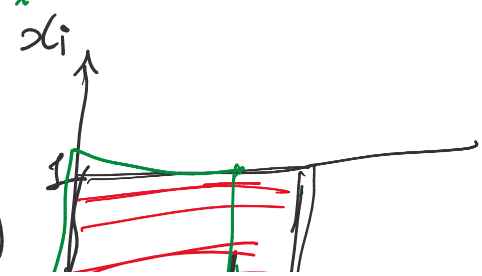
$$i^* = \arg\max_i \phi_i(v_i)$$

Give item to i^* if $\phi_{i^*}(v_{i^*}) \ge 0$.

① Suppose $D_i = D$ regular

$\phi_i = \phi$ monotone.

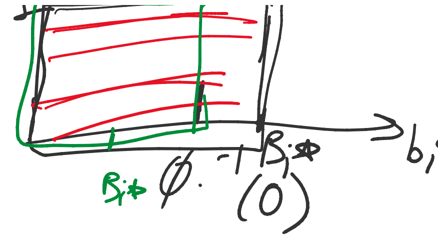
$$i^* = \arg\max_i \phi(v_i) = \arg\max_i v_i$$



i^* gets item. if

$\phi(v_{i^*}) \ge 0 \Rightarrow v_{i^*} \ge \phi^{-1}(0)$

$-\ \phi(v_{i^*}) \geq 0 \Rightarrow v_{i^*} \geq \phi^{-1}(0)$



$\forall j \neq i \quad \phi(v_{i^*}) \geq \phi(v_j) \overset{\forall j}{\Leftrightarrow} v_{i^*} \geq v_j$

B_{i^*} = highest bid in V_{-i^*}

$$P_{i^*} = \max\{\phi^{-1}(0),\ B_{i^*}\}$$

$\phi^{-1}(0)$: reserve price.

B_{i^*}: second-highest bid

② D_i's are different but regular

ϕ_i's " " " monotone.

$\phi_{i^*}(v_{i^*}) \geq \phi_j(v_j) \quad \forall j \neq i^*$

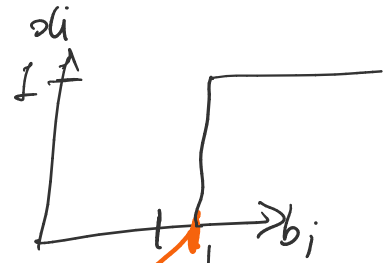
$\cancel{\Downarrow}$ $\Rightarrow$ item may not go to highest value bidder!

$v_{i^*} \geq v_j$?

$\underbrace{\phi_1(v_1)}_{\text{winner.}} \geq \phi_2(v_2) \geq \dots \geq \phi_n(v_n)$

1 gets the item

① $\phi_1(v_1) \geq 0 \Rightarrow v_1 \geq \phi_1^{-1}(0)$

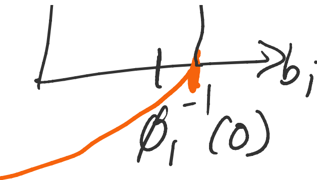


① $\phi_1(v_1) \geq 0$

&

② $\phi_1(v_1) \geq \phi_2(v_2) \Rightarrow$

$$v_1 \geq \boxed{\phi_1^{-1}(\phi_2(v_2))}$$



$$P_1 = \max\left\{ \underbrace{\phi_1^{-1}(0)}_{\text{reserve price of agent 1.}},\ \underbrace{\phi_1^{-1}(\phi_2(v_2))}_{\text{virtual second highest bid wrt } \phi_1} \right\}$$