CS 580RM: Algorithmic Game Theory, Fall 2021
HW 1 (due on Wednesday, 22nd Sept at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. Consider the following allocation rules for a single-parameter environment auction. For each of these, prove if they are implementable or not.
   (a) (3 points) The allocation rule that maximizes the social welfare among all feasible allocations. That is, for bids $b_1, b_2 \ldots, b_n$, $x(b) = \text{argmax}_x \sum_i b_i x_i$.
   (b) (3 points) Suppose the set of feasible allocations is $X = \{x \in [0,1]^n \mid \sum_i x_i = 1\}$. Give each agent a fraction of the item proportional to the distance of their bid from the average of all bids (normalized to 1), and additionally have $x_i = 0$ when $b_i = 0$. That is, if $b_1, b_2, \ldots, b_n$ are the bids of all agents, and if we let $b_{avg} = \sum_i b_i/n$, then the allocation $x(b)$ satisfies for all agents i, j where $x_i, x_j > 0$, $\frac{x_i}{x_j} = \frac{|b_i - b_{avg}|}{|b_j - b_{avg}|}$, and $\sum_i x_i = 1$.
2. (4 points) Recall the sponsored search (keyword) auction: k ad slots are on sale on a search website, where probability of getting i^{th} slot clicked is α_i. We know that $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$. Given bids $b_1 \geq b_2 \geq \ldots b_n$ of n agents, agent i gets slot i for $i \leq k$, and the remaining agents do not get anything. Note that this allocation rule is monotone. Compute the Myerson's payment for this sponsored search auction.
3. (VCG: Combinatorial Auction)

(a) (5 points) We want to auction m heterogeneous items among n unit demand bidders, where bidder i has value v_{ij} for item j, and her value for set $S \subset \{1, \ldots, m\}$ is $\max_{j \in S} v_{ij}$. Design a polynomial-time algorithm to implement the VCG auction.

(b) Consider combinatorial auctions for m items among n bidders. The goal of this exercise is to study special cases in which the VCG mechanism can be implemented efficiently. Suppose that each buyer sends as a bid a valuation vector of $2^m - 1$ numbers (a value for each subset of items, since $v_i(\phi) = 0$ for all i).

   i. (2 points) Suppose that $m = 2$. Show that the VCG mechanism can be implemented in time $O(n^2)$. Provide both the allocation and the payments computed by the VCG mechanism.

   ii. (3 points) Suppose that $m = k$, for some fixed $k > 0$. Show that the VCG mechanism can be implemented in time $O(n^2 \cdot f(k))$, for some function f, where $f(k)$ does not depend on n. Notice that for fixed k, this is polynomial in n. Provide both the allocation and the payments computed by the VCG mechanism.

4. (a) (5 points) This problem considers a variation of the Bulow-Klemperer theorem. Consider selling $k \geq 1$ identical items (with at most one given to each bidder) to bidders with valuations drawn iid from F, where F is a regular distribution (i.e., the corresponding ϕ^{-1} is a monotonically increasing function). Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Myersons optimal auction for F with n bidders.

   [Hint: Myersons optimal auction will be Vickrey with reserve $\phi^{-1}(0)$, i.e., discard bids below $\phi^{-1}(0)$, give the item to k highest bidders and charge them $\max(k+1)^{th}$ highest bid, $\phi^{-1}(0)$]

   (b) (5 points) Consider the reverse auction we briefly talked about in class: A denotes the set of bidders who are willing to sell the spectrum they hold. We say that set $S \subset A$ is feasible, if we can repack $A \setminus S$ in the available range given that S is acquired. Clearly, the set $F \subseteq 2^B$ of feasible sets is upward closed (i.e., supersets of feasible sets are again feasible).

   - Initialize $S = A$.
   - While there is a bidder $i \in S$ such that $S \setminus \{i\}$ is feasible:

     (*) Delete some such bidder i from S such that $S \setminus \{i\}$ is feasible.
   - Return S.

   Suppose we implement the step (*) using a scoring rule, which assigns a number to each bidder i. At each iteration, the bidder with the largest score (whose deletion does not destroy feasibility of S) gets deleted. The score assigned to a bidder i can depend on is bid, the bids of other bidders that have already been deleted, the feasible set F, and the history of what happened in previous iterations. (Note a score is not allowed to depend on the value of the bids of other bidders that have not yet been deleted.) Assume that the scoring rule is increasing – holding everything fixed except for b_i, is score is increasing in its bid b_i. Then, show that the allocation rule above is monotone: for every i and b_{-i}, if i wins with bid bid b_i then she will keep winning with any bid less than b_i.

5. (Public good game) Consider a game among n agents, where each agent has some initial money endowment, say agent i has $\$m_i$. There is a magic-pot (government) to which if

they put certain amount of money (taxes) it gets multiplied by a factor $\beta > 1$ and gets redistributed among the agents equally (benefits of government facilities). That is if agent i puts $g_i \in [0, m_i]$ in the pot, then out comes $\beta \sum_{i=1}^n g_i$ and every agent is allocated $\frac{\beta}{n} \sum_{i=1}^n g_i$. Strategy of each agent is to choose $g_i \in [0, m_i]$, the amount to contribute to the magic-pot. When $g = (g_1, \ldots, g_n)$ is the strategy profile played, the net utility of the agents is:

$$U_i(g) = (m_i - g_i) + \frac{\beta}{n} \sum_{i=1}^{n} g_i$$

(a) (3 points) What is the social welfare maximizing strategy, *i.e.*, $g^* \in \operatorname{argmax}_{g: g_i \in [0, m_i]} \sum_{i=1}^n U_i(g)$.

(b) (2 points) Suppose $\beta < n$ and strategies of all the agents except i is fixed to g_{-i}. What is the best response strategy of agent i?

(c) (3 points) For $\beta < n$, using the previous part, construct a Nash equilibrium. Is it a dominant strategy Nash equilibrium?

(d) (2 points) Construct a Nash equilibrium of this game when $\beta > n$.

6. (Bonus questions)

(a) In the VCG auction question, give an asymptotic upper bound on the number of items m (with respect to the number of agents n) such that the VCG mechanism can be implemented in time that is polynomial in n. In particular, show that the VCG mechanism can be implemented in time $O(n^2 \cdot f(m))$, for some function f, where $f(m)$ does not depend on n, and give an asymptotic upper bound on m such that this running time is polynomial in n.

[Hint: Use dynamic programming]

(b) (Iterated Public goods games) Consider the following iterated variant of the public goods game (Players play a single shot pubic goods game for multiple rounds). Find the Nash equilibrium strategy of each player for each round, and reason whether their strategies for different rounds will be different. We assume that after each round, the payoff to all players together is twice the total investment. That is, if they invest amounts $x_1, x_2, \cdots, x_n$, then the total payoff is $2 \sum_i x_i$. Again, let m_i be the money endowment/money of agent i before the game starts.

i. The total payoff of each round is distributed equally only over the players who invested more than 3/4-value of the average of all investments, and the remaining players get back their investment. That is, if $x_1, x_2, \cdots, x_n$ are the investments of all players, then every player gets back their investment, and the additional payoff $\sum_i x_i$ is distributed equally among all players i whose investment x_i is equal or higher than $(3/4) \cdot \sum_i x_i$.

ii. In every round, the payoff is distributed in proportion to their total wealth at the beginning of the round. That is, if players start with endowments $w_1, w_2 \cdots, w_n$ and invest $x_1, x_2 \cdots, x_n$, then their wealth after the round is $w_i' = w_i - x_i + 2 \frac{w_i}{\sum_{k=1}^n w_k} \cdot (\sum_i x_i)$ for all i. Assume that all players start with equal endowments, and play more than one round.