CS 598RM: Algorithmic Game Theory, Fall 2021
HW 2 Solutions

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whatever is suitable. We will upload submission instructions on the course webpage and Piazza.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (*Nash equilibrium: existence and computation*)

   (a) (1 points) Does every zero-sum game have a Pure Nash equilibrium, where every player plays a single move with probability 1? Give a proof or a counter example to support your answer.

   (b) (2 point) Is the problem of computing an ϵ-approximate Nash equilibrium for a given $\epsilon > 0$ scale invariant? That is, if (x, y) is an ϵ-NE of game (A, B), then for $\alpha, \beta \geq 0$ and $a, b \in \mathbb{R}$ is it also an ϵ-NE of game $(\alpha A + a, \beta B + b)$? Justify your answer.

   (c) (3 points) Compute all Nash equilibria of the game whose payoff bimatrix is given in Table 1.

5,3	9,0	-8,3
0,30	5,3	10,3
5,-10	5,10	5,3

Table 1: Payoff bimatrix of a 3X3 game

(d) (4 points) Game (A, B) is said to be symmetric if $B = A^T$. First note that, both players have the same number of moves in a symmetric game. Prove that a symmetric game always has a symmetric NE, i.e., NE (x, y) such that $y = x$.

[Hint: Modify Nash's proof]

Solution.

(a) The standard rock-paper-scissors shows this is not true, as it is zero-sum but there is no pure NE:

0,0	-1,1	1,-1
1,-1	0,0	-1,1
-1,1	1,-1	0,0

(b) No. Consider even $\alpha = \beta$ and $a = b = 0$. Then, by the definition, an ϵ-approximate NE of (A, B) is an $(\epsilon \cdot \alpha)$-approximate NE of $(\alpha A, \alpha B)$.

(c) Let R denote the row player and C denote the column player. Furthermore, let r_1, r_2, r_3 denote R's strategies and c_1, c_2, c_3 denote C's strategies. Finally, let p_i (respectively q_i) denote the probability that R (respectively C) plays strategy r_i (respectively c_i), for $i = 1, 2, 3$.

One can easily check that the only pure NE of the game is (r_1, c_1). For mixed NE, R should receive the same payoff from all their strategies, so

$$5q_1 + 9q_2 - 8q_3 = 5q_2 + 10q_3 = 5(q_1 + q_2 + q_3).$$

Using the fact that $q_1 + q_2 + q_3 = 1$, we get

$$5q_1 + 9q_2 - 8q_3 = 5q_2 + 10q_3 = 5,$$

which simplifies to

$$\begin{cases} 5q_1 + 9q_2 - 8q_3 = 5 \\ q_2 = 1 - 2q_3 \\ q_1 = 1 - q_2 - q_3 \end{cases} \Leftrightarrow \begin{cases} 5q_1 + 9q_2 - 8q_3 = 5 \\ q_2 = 1 - 2q_3 \\ q_1 = q_3 \end{cases} \Leftrightarrow \begin{cases} 9q_2 - 3q_3 = 5 \\ q_2 = 1 - 2q_3 \\ q_1 = q_3 \end{cases} \Leftrightarrow$$

$$\begin{cases} 9 - 18q_3 - 3q_3 = 5 \\ q_2 = 1 - 2q_3 \\ q_1 = q_3 \end{cases} \Leftrightarrow \begin{cases} 21q_3 = 4 \\ q_2 = 1 - 2q_3 \\ q_1 = q_3 \end{cases} \Leftrightarrow \begin{cases} q_1 = \frac{4}{21} \\ q_2 = \frac{13}{21} \\ q_3 = \frac{4}{21} \end{cases}.$$

Equivalently, C should receive the same payoff from all their strategies, so

$$3p_1 + 30p_2 - 10p_3 = 3p_2 + 10p_3 = 3(p_1 + p_2 + p_3)$$

Using the fact that $p_1 + p_2 + p_3 = 1$, we get

$$3p_1 + 30p_2 - 10p_3 = 3p_2 + 10p_3 = 3,$$

which simplifies to

$$\begin{cases} 3p_1 + 30p_2 - 10p_3 = 3 \\ 3p_2 + 10p_3 = 3 \\ p_1 = 1 - p_2 - p_3 \end{cases} \Leftrightarrow \begin{cases} 3p_1 + 30 - 100p_3 - 10p_3 = 3 \\ p_2 = 1 - \frac{10}{3}p_3 \\ p_1 = \frac{7}{3}p_3 \end{cases} \Leftrightarrow \begin{cases} 110p_3 - 3p_1 = 27 \\ p_2 = 1 - \frac{10}{3}p_3 \\ p_1 = \frac{7}{3}p_3 \end{cases} \Leftrightarrow$$

$$\begin{cases} 110p_3 - 7p_3 = 27 \\ p_2 = 1 - \frac{10}{3}p_3 \\ p_1 = \frac{7}{3}p_3 \end{cases} \Leftrightarrow \begin{cases} 103p_3 = 27 \\ p_2 = 1 - \frac{10}{3}p_3 \\ p_1 = \frac{7}{3}p_3 \end{cases} \Leftrightarrow \begin{cases} p_1 = \frac{63}{103} \\ p_2 = \frac{13}{103} \\ p_3 = \frac{27}{103} \end{cases}.$$

Therefore, the NE of the game are $((1,0,0),(1,0,0))$ and $\left(\left(\frac{63}{103}, \frac{13}{103}, \frac{27}{103}\right), \left(\frac{4}{21}, \frac{13}{21}, \frac{4}{21}\right)\right)$.

(d) Let $D \coloneqq conv(S)$ be the space of mixed strategies for each player. We define a function $f : D \to D$ as follows. For all mixed strategies $x \in D$, $f(x) = y$, where for all $i \in S$

$$y_i \coloneqq \frac{x_i + \max\{(Ax)_i - x^T Ax, 0\}}{1 + \sum_{k \in S} \max\{(Ax)_k - x^T Ax, 0\}}$$

In words, function f tries to boost the probability mass that each player places on various pure strategies depending on the gains in payoff the player would get by switching to these strategies. The denominator just ensures that $\sum_{i \in S} y_i) = 1$. The term $\max\{(Ax)_i - x^T Ax, 0\}$ represents each player's increase in payoff if they were to switch from some mixed strategy x to pure strategy i, unless the increase is negative in which case the gain is taken to equal 0.

Now, it is easy to see that f is continuous. Furthermore, D is a convex, closed and bounded set, hence, Brouwer's fixed point theorem ensures the existence of a point $x \in D$ such that $x = f(x)$. Also, a strategy (x, x) is a symmetric NE if and only if

$$\forall i \in S, \quad x_i > 0 \Rightarrow (Ax)_i = max_{k \in S}(Ax)_k.$$

Claim 1. *Let x be a fixed point of f. Then (x, x) is a symmetric NE for game (A, B).*

Proof. Since x is a fixed point of f, we have $x = f(x) = y$. Assume, towards contradiction, that (x, x) is not a NE for the game. Then for some $i \in S$, $x_i > 0$ but $(Ax)_i < \max_{k \in S} (Ax)_k$. Consider such an i with the least payoff. Clearly, $(Ax)_i \leq x^T Ax$. Now let $i^* = \arg\max_{k \in S} (Ax)_k$ be a pure strategy that attains the maximum payoff. Then, $(Ax)_{i^*} > x^T Ax \geq (Ax)_i$. This implies

$$\max\{0, (Ax)_i - x^T Ax\} = 0 \text{ and } \max\{0, (Ax)_{i*} - x^T Ax\} > 0$$

Thus

$$y_i = \frac{x_i + \max\{(Ax)_i - x^T Ax, 0\}}{1 + \sum_{k \in S} \max\{(Ax)_k - x^T Ax, 0\}} = \frac{x_i}{1 + \sum_{k \in S} \max\{(Ax)_k - x^T Ax, 0\}} < x_i$$

Since the denominator is greater than 1 as there is at least one non-zero gain in the summation - the one corresponding to pure strategy i^*. Therefore, x is not a fixed point of f, and we arrive at a contradiction. It follows that (x, x) is a symmetric Nash equilibrium for (A, B). □

2. (*TFNP classes*)

   (a) (1 point) Prove that every 1-D Sperner problem instance has an odd number of solutions.

   (b) (3 points) Prove that PPAD $\subseteq$ PPP. That is show that the canonical problem of PPAD reduces to the canonical problem of PPP (see lecture slides).

   (c) (6 points) The colorful Caratheodory theorem (CCT) is as follows. In d-dimensions, we are given points colored with one of the $\{1, \ldots, (d+1)\}$ colors. Furthermore, let S_i be the set of points with color i, then $|S_i| = (d+1)$ and $\mathbf{0}$ is in the convex hull of S_i. We say that set S is colorful if it has exactly one point from each S_i. CCT proves that there exists a colorful set S whose convex hull contains $\mathbf{0}$.

   Look up the proof of CCT and using it show that finding such a colorful set S is in PLS.

Solution.

(a) In the 1-D case of the Sperner Lemma, the k vertices of $[0, k-1]$ are labelled with either red or blue, with the stipulation that the vertices at 0 and $k-1$ are labelled differently. Let us denote the subintervals of the form $[i, i+1] \subset [0, k-1]$ by the symbol σ_i. They are the 1-simplices in the triangulation of $[0, k-1]$. Let the (a, b)-simplices denote the σ_i which are labelled a on one vertex and b on the other, with no restriction on ordering. For each σ_i, let $F(\sigma_i)$ be the number of endpoints of σ_i which are labelled red. F takes the values 0,1 and 2. The proof proceeds by calculating $\sum_i F(\sigma_i)$ in two ways.

First we note that this value is equal to the number of (red, blue)-simplices, plus twice the number of (red, red)-simplices. In particular, it is the number of (red,blue)-simplices plus an even number. We can compute the sum in a different way by noticing that every vertex in the interior of $[0, k-1]$ which is labelled red contributes 2 to the sum. Therefore, the sum is also twice the number of vertices labelled red inside the interval, plus one for the one boundary vertex which is labelled red. Since this second method of counting shows that the sum is odd, the number of (red, blue)-simplices must also be odd.

(b) To show that PPAD $\subseteq$ PPP, it suffices to reduce End-Of-Line to Collison. In End-Of-Line, the input is a directed graph G, where each vertex has at most one successor and at most one predecessor, represented by two polynomial-time computable predecessor and successor circuits P and N respectively. We're given that $P(0^n) = 0^n$, and we need to find another node $x \neq 0^n$ such that $P(x) = x$ or $x = N(x)$.

We create an instance of Collision as follows: Define a circuit C whose input is a vertex x, and whose output is $N(x)$ if there is one, or x if there is not. Notice that we cannot get a vertex x such that $C(x) = 0^n$, because this implies that 0^n is also a sink. Therefore, assume that we get a collision in C, i.e. two vertices $x \neq y$ such that $C(x) = C(y)$. Notice that we cannot have $N(x) = N(y)$, since every node has at most one predecessor, and thus, either $x = N(x)$ or $y = N(y)$ and we have found a sink in G.

(c) In order to show that CCt is in PLS, it suffices to show that CCT is reducible to FindSink. Given an instance of CCT, we create an instance of FindSink as follows: The vertices of the DAG correspond to panchromatic simplices, i.e. a set $\{p_1, \ldots, p_{d+1}\}$ where $|\{p_1, \ldots, p_{d+1}\} \cap S_i| = 1$ for all $i = 1, \ldots, d+1$. The neighbors of a vertex

$\{p_1, \ldots, p_{d+1}\}$ are defined as the set of all panchromatic simplices $\{p'_1, \ldots, p'_{d+1}\}$, where $p_i = p'_i$ for all i except for exactly one, say j. Then, we require p_j and p'_j to have the same color. In other words, two simplices are connected via an edge only if you can get one simplex form the other by swapping one point from another in the same color class. Notice that if this was the definition of the edge set, then the resulting graph is not a DAG, because it is essentially undirected. To remedy this, we impose another condition for the existence of an edge between two vertices Δ_1 and Δ_2. Specifically, there exists an edge (Δ_1, Δ_2) if the above condition is true and also $dist(\mathbf{0}, \Delta_1) > dist(\mathbf{0}, \Delta_2)$, where *dist* corresponds to the minimum distance from a point to any of the faces of the simplex. Furthermore, we define the potential function $F(\Delta)$ to be exactly this distance $dist(\mathbf{0}, \Delta)$ if it is positive, and 0 it is negative.

Notice that, because the colorful Caratheodory theorem is true, for every hyperplane P, if there exists a point $p \in S_i$ on one side of the hyperplane P and $\mathbf{0}$ is on the other side, then there must exist another point $p' \in S_i$ on the same side of the hyperplane as $\mathbf{0}$, as otherwise $\mathbf{0} \notin \text{conv}(S_i)$. This ensures that, as long as $dist(\mathbf{0}, \Delta) > 0$, then there exists a color class S_i such that we can swap the point of Δ which is in S_i with another point in S_i and get a new simplex Δ' where $dist(\mathbf{0}, \Delta') < dist(\mathbf{0}, \Delta)$. To be more precise, if we project the minimum distance ray from $\mathbf{0}$ to Δ, it will hit a face f of Δ, defined by d of the points that constitute Δ. Then we can swap the only point of Δ which is not in this face, with another point, which lies on the other side of the hyperplane defined by f, and decrease the distance to $\mathbf{0}$. When we find a sink in the DAG, we know that we are at a simplex Δ where we can no longer swap any points in Δ and decrease the distance to $\mathbf{0}$, which implies that $\mathbf{0} \in \Delta$, and thus we have found a solution of CCT. We omit the details as to how the computation of this potential function can be done in polynomial time.

Therefore, CCT can be reduced to FindSink in polynomial-time, and thus CCT is in PLS.

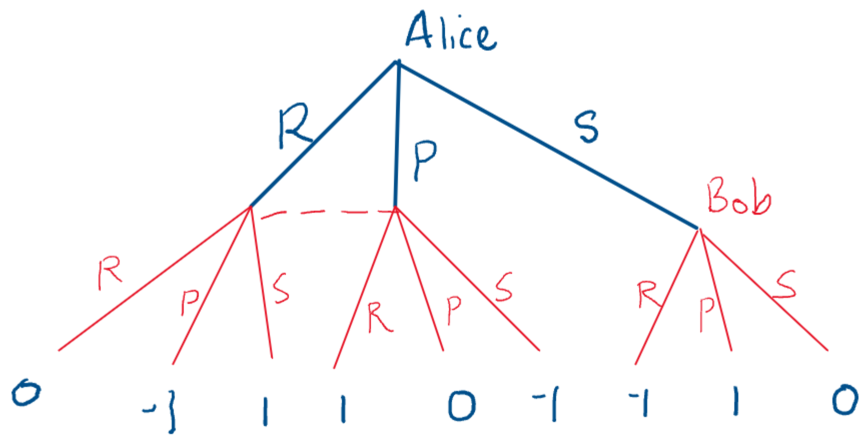

Figure 1: Extensive Form Game

3. (*Other game and equilibrium notions*)

(a) (2 points) Compute all Nash equilibria of the game shown in Table 2.

-1.4	1,-8	3,2	3,-2
4,-5	5,-2	5,-6	5,-8
-3,-2	3,-5	8,-1	7,-5
3,1	-3,1	3,0	7,-2

Table 2: Payoff bimatrix of a 4X4 game

[Hint: Apply iterated dominance.]

(b) (3 points) Alice and Bob are playing a game (A, B) in rounds where in t^{th} round they update their strategies as follows, starting at $x(0)$ and $y(0)$ that are uniform strategies of Alice and Bob respectively.

$$\forall i, \qquad x_i(t) = x_i(t-1)\frac{(Ay)_i}{x^T Ay}$$
$$\forall j, \qquad y_j(t) = y_j(t-1)\frac{(x^T B)_j}{x^T By}$$

Show that $(x(t), y(t)) = (x(t-1), y(t-1))$ if and only if $(x(t), y(t))$ is a Nash equilibrium.

(c) (3 points) Given a game (A, B), show that each of it's correlated equilibrium is also a coarse correlated equilibrium.

(d) (2 points) Write the normal form representation of the extensive form game shown in fig 1. The game is zero-sum, and Alice's payoffs are shown in the figure.

Solution.

(a) Let R denote the row player and C denote the column player. Furthermore, let r_1, r_2, r_3, r_4 denote R's strategies and c_1, c_2, c_3, c_4 denote C's strategies. Finally, let

p_i (respectively q_i) denote the probability that R (respectively C) plays strategy r_i (respectively c_i), for $i = 1, 2, 3, 4$.

First, notice that it is never in R's interest to play r_1, since r_1 is stictly dominated by r_2. Once we remove r_1, notice that it is never in C's interest to play c_4, since c_4 is strictly dominated by c_1. Once we remove c_4, notice that it is never in R's interest to play r_4, since r_4 is stictly dominated by r_2.

Thus, the remaining 2×3 game is:

4,-5	5,-2	5,-6
-3,-2	3,-5	8,-1

It is easy to see that (r_2, c_2) and (r_3, c_3) are the only pure NE of this game. For mixed NE, if R plays r_2, they get a payoff of $4q_1 + 5q_2 + 5q_3$, whereas if R plays r_3, they get a payoff of $-3q_1 + 3q_2 + 8q_3$. Using the fact that $q_1 + q_2 + q_3 = 1$, we get that R's payoff from strategy r_2 is $4q_1 + 5q_2 + 5 - 5q_1 - 5q_2 = 5 - q_1$, whereas R's payoff from strategy r_3 is $-3q_1 + 3q_2 + 8 - 8q_1 - 8q_2 = 8 - 11q_1 - 5q_2$. At NE, R's payoff from both strategies will be equal, and thus

$$5 - q_1 = 8 - 11q_1 - 5q_2 \Longrightarrow 2q_1 + q_2 = \frac{3}{5}.$$

Similarly, if C plays c_1, they get a payoff of $-5p_2 - 2p_3$, if C plays c_2, they get a payoff of $-2p_2 - 5p_3$ and if C plays c_3, they get a payoff of $-6p_2 - p_3$. Using the fact that $p_3 = 1 - p_2$, we get that C's payoff from strategy c_1 is $-2 - 3p_2$, from c_2 it is $-5 + 3p_2$ and from c_3 it is $-1 - 5p_2$. At NE, C's payoff from all three strategies will be equal, and thus

$$-2 - 3p_2 = -5 + 3p_2 = -1 - 5p_2 \Longrightarrow p_2 = \frac{1}{2} \text{ and } p_3 = \frac{1}{2}.$$

Therefore, the mixed equilibria are $\{(0, 1/2, 1/2, 0), (x, 3/5 - 2x, 2/5 + x, 0)\}$, where $0 \leq x \leq \frac{3}{10}$.

(b) First, assume that $(x(t), y(t)) = (x(t-1), y(t-1))$. We need to show that $(x(t), y(t))$ is a NE. We have for all i, j

$$\begin{cases} x_i(t-1) = x_i(t-1)\frac{(Ay)_i}{x^T Ay} \\ y_j(t-1) = y_j(t-1)\frac{(x^T B)_j}{x^T By} \end{cases} \Leftrightarrow \begin{cases} (Ay)_i = x^T Ay \\ (x^T B)_j = x^T By \end{cases}.$$

which implies that all of Alice's strategies achieve the same payoff and also all of Bob's strategies achieve the same payoff, and therefore neither of them has an incentive to unilaterally deviate, which implies that $(x(t), y(t))$ is a NE.

Next, assume that $(x(t), y(t))$ is a NE. We need to show that $(x(t), y(t)) = (x(t-1), y(t-1))$. Since $(x(t), y(t))$ is a NE, we know that, for all x' and y'

$$\begin{cases} x'^T Ay \leq x(t)^T Ay(t) \\ x^T By' \leq x(t)^T By(t) \end{cases}$$

since neither player can gain by unilaterally deviating. In particular, for any i, j, we have

$$\begin{cases} (Ay)_i \leq x(t)^T Ay(t) \\ (x^T B)_j \leq x(t)^T By(t) \end{cases}$$

This implies that, for all i, j

$$\begin{cases} x_i(t) = x_i(t-1)\frac{(Ay)_i}{x^TAy} \le x_i(t-1) \\ y_j(t) = y_j(t-1)\frac{(x^TB)_j}{x^TBy} \le y_j(t-1) \end{cases}$$

However, x and y are probability distributions, and thus $\sum_i x_i(t) = \sum_i x_i(t-1) = \sum_j y_j(t) = \sum_j y_j(t-1) = 1$, which, along with the inequalities above, imply that $(x(t), y(t)) = (x(t-1), y(t-1))$.

(c) Consider a normal form game, (A, B). Let S_i denote the strategy sets of player $i = 1, 2$. If player 1 plays $i \in S_1$ and player 2 plays $j \in S_2$ then their respective payoffs are $A(i, j)$ and $B(i, j)$. For a joint distrubution p on $S = S_1 \times S_2$, the payoffs are:

$$\alpha(p) = \sum_{(i,j)\in S} A(i,j)p(i,j), \qquad \beta(p) = \sum_{(i,j)\in S} B(i,j)p(i,j)$$

Let σ_i and τ_j denote the marginal probability of $i \in S_1$ and $j \in S_2$, *i.e.*, $\sigma_i = \sum_{j\in S_2} p(i, j)$ and $\tau_j = \sum_{i\in S_1} p(i, j)$. Note that, for p to be a CCE, $\alpha(p)$ should be at least the expected payoff player 1 gets by playing any fixed strategy $t \in S_1$ while player 2 follows the suggestion, and similarly for player 2. Thus, the following claim holds.

Claim 2. *p is a CCE if and only if the following holds,*

$$\forall t \in S_1, \quad \alpha(p) \ge \sum_{j\in S_2} A(t,j)\tau_j; \qquad \forall u \in S_2, \quad \beta(p) \ge \sum_{i\in S_1} B(i,u)\sigma_i \tag{1}$$

Now, if p is a CE then whenever player 1 is asked to play i, her expected payoff from playing i is no worse than playing any other strategy $i' \in S_1$ given that player 2 follows the suggestion, and similarly for player 2. Formally,

$$\begin{gathered} \forall i, i' \in S_1, \quad \frac{1}{\sigma_i}\sum_{j\in S_2} A(i,j)p(i,j) \ge \frac{1}{\sigma_i}\sum_{j\in S_2} A(i',j)p(i,j) \\ \text{implies} \\ \sum_{j\in S_2} A(i,j)p(i,j) \ge \sum_{j\in S_2} A(i',j)p(i,j) \\ \text{and} \\ \forall j, j' \in S_2, \quad \frac{1}{\tau_j}\sum_{i\in S_1} B(i,j)p(i,j) \ge \frac{1}{\tau_j}\sum_{i\in S_1} B(i,j')p(i,j) \\ \text{implies} \\ \sum_{i\in S_1} B(i,j)p(i,j) \ge \sum_{i\in S_1} B(i,j')p(i,j) \end{gathered} \tag{2}$$

For any fixed strategy $t \in S_1$ for player 1, summing up the first inequalities above for all i while $i' = t$.

$$\alpha(p) \geq \sum_{i \in S_1} \sum_{j \in S_2} A(t,j)p(i,j) = \sum_{j \in S_2} A(t,j) \sum_{i \in S_1} p(i,j) = \sum_{j \in S_2} A(t,j)\tau_j \quad (3)$$

Clearly, the r.h.s. is the payoff of player 1 if she always plays t, while player 2 follows the suggestion. Similarly, for any $u \in S_2$ for player 2 summing up the second inequalities in (2) for all j while $j' = u$, we get,

$$\beta(p) \geq \sum_{j \in S_2} \sum_{i \in S_1} B(i,u)p(i,j) = \sum_{i \in S_1} B(i,u) \sum_{j \in S_2} p(i,j) = \sum_{i \in S_1} B(i,u)\sigma_i \quad (4)$$

Again, r.h.s. is the payoff of player 2 if she always plays u, while player 1 follows the suggestion. Conditions (3) and (4) together with Claim 2 proves that every CE is a CCE.

(d) Player 1 has 3 strategies available, while Player 2 has 9 strategies available, as Player 2 cannot distinguish between Player 1 playing strategy R or P. Let Player 1 be the row player, and Player 2 be the column player. Since the game is zero-sum, the normal-form representation of the game is

	RR	RP	RS	PR	PP	PS	SR	SP	SS
R	0, 0	0, 0	0, 0	-1,1	-1,1	-1,1	1,-1	1,-1	1,-1
P	1,-1	1,-1	1,-1	0,0	0,0	0,0	-1,1	-1,1	-1,1
S	-1,1	1,-1	0,0	-1,1	1,-1	0,0	-1,1	1,-1	0,0

4. (*Stackelberg strategies*) (10 points) The employees of a company called Galaxity all live in the same suburb far from their office, thus have to travel from a common point central in the suburb to the office everyday. To discourage too many employees choosing the same (shortest) route everyday and congesting it, the company sets up a toll booth on some street $i \in [n]$ everyday that the employees can not know before hand. If street i is chosen on a particular day and an employee traverses this street, then the company gets a reward r_i while the employee gets ζ_i cost, otherwise the company pays a cost c_i and the employee's reward is ρ_i.

   Design a polynomial time algorithm to compute the Stackelberg strategy of the company.
   Solution.

   Consider the graph corresponding to the street layout of Galaxity, i.e., edges of the graph correspond to the streets, and vertices are ends of streets. Let s be the common starting point, t be the office, E be the set of all $s - t$ paths in Galaxity, and let each path in E be denoted by $e = [e_1, \dots, e_n]$, where $e_i = 1$ if street i is on the path, and $e_i = 0$ otherwise.

   Suppose the strategy of the employees is $p = [p_e], i \in E$, i.e., the employees choose to traverse path $e \in E$ with probability p_e, and suppose that setting up a toll booth on street i is the company's best response to p. We will compute the p that maximizes the employees' utility for which i is the company's best response. The Stackelberg strategy of the employees is the p that maximizes their utility over all the strategies corresponding to each $i \in [n]$.

   The following program computes the p for a given i. For this, let the probability that the employees traverse street i be denoted by x_i. x_i is the sum of all probabilities that they traverse some path containing i, i.e., $\sum_{e:i \in e} p_e$.

$$
\begin{array}{lll}
\max & x_i \cdot r_i + (1 - x_i) \cdot c_i & \\
\text{s.t.} & x_i \zeta_i + (1 - x_i)\rho_i \geq x_k \cdot \zeta_k + (1 - x_k)\rho_k, & \forall k \in [n] \setminus \{i\} \\
& x_i = \sum_{e:i \in e} p_e, & \forall i \in [n] \\
& \sum_e p_e = 1, & \forall e \in E \\
& p_e \geq 0, & \forall e \in E
\end{array}
$$

   But this program has an exponential number of variables ($|E|$). Consider the dual. Let y_k, a_i, b be the variables corresponding respectively to the first, second and third set of constraints.

$$
\begin{array}{lll}
\min & \sum_{k \neq i} (\rho_k - \rho_i) y_k - b & \\
\text{s.t.} & b \geq e_i \Big((r_i - c_i) - \sum_{k \neq i} y_k(\rho_i - \zeta_i) \Big) + \sum_k e_k \cdot y_k(\rho_k - \zeta_k), & \forall e = [e_1, \dots, e_n] \in E \\
& y_k \geq 0, & \forall k \in [n] \setminus \{i\}
\end{array}
$$

   Define $w_i = (r_i - c_i) - \sum_{k \neq i} y_k(\rho_i - \zeta_i)$, and $w_k = y_k(\rho_k - \zeta_k)$, for all $k \neq i$. As the first set of constraints must be satisfied for all paths, they must be satisfied for the max weight $s - t$ path as well. Finding the weight of the max weight $s - t$ path can be done in polynomial time via max-flow. Hence, checking if given b, y_k's are feasible can be done in polynomial time, by checking that b is greater than the weight of the max weight $s - t$ path according to the weights w_i defined above. In other words, there is a polynomial time separation oracle for the above LP.

   Using the separation oracle, we solve the dual and find the optimal objective value, the expected optimal utility of the employees, for each pure strategy of the company. We return the strategy corresponding to the highest utility among these.

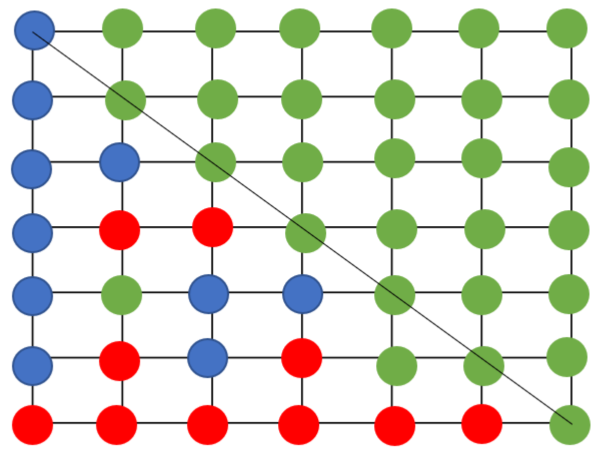

Figure 2: Illustrating the special case of Sperner

5. (*Bonus problem*)

   (a) Prove that finding NE in game (A, B) reduces to finding a symmetric NE in a symmetric game.

   (b) Prove that checking if 1-D Sperner has more than one solution is NP-complete.

   (c) Consider the special case of the 2-D Sperner problem on a square that adds the following further restrictions to the legal colorings of the boundary vertices. Every vertex on the upper side of the diagonal connecting the top-left and bottom right vertices, including all except the top-left vertex on the diagonal vertices must have the same color. Further, each vertex on the left boundary must have the same color as that of the top-left vertex, and each vertex on the bottom boundary must have the color assigned to the bottom-right vertex.

   Figure 2 illustrates this special case. For clarity, most diagonal edges forming the triangles are not shown, except those on the main diagonal. Given that the 2-D Sperner problem we discussed in the class is PPAD-hard, prove that this special case of the problem is also PPAD-hard. That is, given an arbitrary 2-D Sperner instance reduce it to this special case of 2-D Sperner.

Solution.

(a) Given a game (A, B), we create the followiung symmetric game (C, C^T):

$$C = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}.$$

We show that a symmetric Nash equilibrium of (C, C^T) gives a Nash equilibrium of (A, B). Suppose $z = (x|y)$ is a symmetric Nash equilibrium of the game (C, C^T), where x corresponds to the first m slots of z and y corresponds to the remaining n slots. We begin by showing that we must have $x > 0$ and $y > 0$, i.e. both x and y cannot be the zero vector.

For the sake of contradiction, suppose that $y = 0$. Then for $i \leq m$, we have

$$(Cz)_i = \sum_j C_{ij} z_j = \sum_{j=1}^{m} 0 \cdot z_j + \sum_{j=m+1}^{m+n} C_{ij} \cdot 0 = 0.$$

In particular, since $y = 0$, we know that if $z_i > 0$, then we must have $i \leq m$, and consequently $(Cz)_i = 0$. However, if we take $i > m$, then we have

$$(Cz)_i = \sum_j C_{ij} z_j = \sum_{j=1}^{m} B^T_{i-m,j} x_j + \sum_{j=m+1}^{m+n} 0 \cdot 0 = (B^T x)_{i-m} > 0.$$

Note that $(B^T x)_{i-m} > 0$ follows from the fact that all entries of B are strictly positive and there is some strictly positive entry in x (otherwise $\sum_i z_i = 0$, which contradicts z being a probability distribution), hence the sum $\sum_{j=1}^{m} B^T_{i-m,j} x_j$ contains terms which are all non-negative and at least one term which is strictly positive. Thus, there is some i such that $(Cz)_i > 0$. Taking this together with the previous analysis yields the implication

$$z_i > 0 \implies (Cz)_i = 0 < \max_k (Cz)_k,$$

which contradicts the fact that z is a symmetric Nash equilibrium. Thus, it must be the case that $y > 0$, i.e. y has some strictly positive entry. By an analogous argument, we also have $x > 0$.

We claim that $\left(\frac{x}{\sum_i x_i}, \frac{y}{\sum_i y_i}\right)$ gives a Nash equilibrium of the game (A, B). Since both $x > 0$ and $y > 0$, we avoid the potential issue of division by zero, i.e. our proposed equilibrium is well-defined. Note that for $z = (x|y)$, the definition of C immediately gives us

$$(Cz)_i = \begin{cases} (Ay)_i & i \leq m \\ (B^t x)_{i-m} & i > m \end{cases}$$

Suppose that for some $i \in S_1$, we have $x_i > 0$. Then by construction, we must have $z_i > 0$. Since z is a symmetric Nash equilibrium of (C, C^T), this implies

$$(Ay)_i = (Cz)_i = \max_k \, (Cz)_k = \max_k \, (Ay)_k.$$

Similarly, suppose that for some $j \in S_2$, we have $y_j > 0$. Then $z_{m+j} > 0$, so it follows from z being a symmetric Nash equilibrium that

$$(x^T B)_j = (Cz)_{m+j} = \max_k \, (Cz)_k = \max_k \, (x^T B)_k.$$

The two results above immediately imply that (x, y) is a Nash equilibrium of the game (A, B).

(b) Reduction from 3-SAT: Given a 3-SAT instance $\phi(x)$ with n variables $(x_1, \ldots, x_n)$ and m clauses, we will construct a Sperner instance on $[0, 2^{(n+1)} - 1]$ grid.

In what follows, by $\phi(k)$ for an integer k we mean $\phi(x)$ where $x = (x_1, \ldots, x_n)$ is the binary representation of k with x_1 being highest significant bit (hsb) and x_n being the lowest significant bit (lsb). By $int(x)$ we mean the integer represented by binary string x considering x_1 as hsb and x_n as lsb.
If $\phi(0) = 1$, then return ϕ is satisfiable. If $\phi(1) = 1$, then return ϕ is satisfiable.

Otherwise, construct a Sperner instance where Color function is implemented as follows: Let Color(0)=red and Color(1)=blue. For $k \in [1, 2^n - 1]$, Color(k)=red if $\phi(k) = 1$, otherwise Color(k)=blue. For $k \in [2^n, 2^{n+1} - 1]$ let Color(k)=blue.

Correctness. Clearly, if ϕ is satisfiable at binary representation of 0 or 1, we correctly return that it is satisfiable. If not, then we create a Sperner instance with the above implementation of Color. It creates a dummy solution at $k = 0$.
If there is another solution k' then surely $1 < k' < 2^n$ and $\phi(k') = 1$. On the other hand, if ϕ is satisfiable, then let $k' = \max\{int(x) \mid \phi(x) = 1, x \in 0, 1^n\}$. Clearly, $1 < k' < 2^n$, Color(k')=red and Color($k' + 1$)=blue. This implies that the Sperner has at least two solutions, one at $k = 0$ and another at k'.

(c) Notice that in the version of 2-D Sperner discussed in class, two of the sides of the rectangle's boundary had to have disallowed the same color. Without loss of generality, we assume these two be the top and right boundaries. The first step is to notice that such an instance is topologically equivalent to a 2-D Sperner instance on a triangle, where we let the traingle's sides be the bottom side, the left side and the top/right sides combined of the rectangle. Therefore, we can assume that our 2-D Sperner instance is on a triangle instead, and without loss of generality, we can display such a triangle as a right triangle witht he top/right side as the hypotenuse. The next step is to notice that we can add one more boundary "layer" around our triangle instance, where the vertices of the new boundary that are on the same side have the same color, save for the corner vertices. We can color the corner vertices with the only color not disallowed on both adjacent sides, as in the standard 2-D Sperner instance. This is valid, since we do not violate any premises of the 2-D Sperner instance and we also do not create any panchromatic triangles this way.

Now we have a 2-D Sperner instance that is a right triangle, with each vertices on the same boundary side having the same color, except for the corner vertices. Now, we can "stretch" the hypotenuse and extend the triangle instance into a rectangle, and color any new vertices with the same color as the hypotenuse. The result is a 2-D Sperner instance exactly like the one in the problem statement. This reduces standard 2-D Sperner to the 2-D Sperner variant described in the problem statement. Since standard 2-D Sperner is PPAD-Hard, the 2-D Sperner variant described in the problem statement is also PPAD-Hard.