CS 580RM: Algorithmic Game Theory, Fall 2021
HW 2 (due on Wednesday, 13th Oct at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (*Nash equilibrium: existence and computation*)
   (a) (1 points) Does every zero-sum game have a Pure Nash equilibrium, where every player plays a single move with probability 1? Give a proof or a counter example to support your answer.
   (b) (2 point) Is the problem of computing an ϵ-approximate Nash equilibrium for a given $\epsilon > 0$ scale invariant? That is, if (x, y) is an ϵ-NE of game (A, B), then for $\alpha, \beta \geq 0$ and $a, b \in \mathbb{R}$ is it also an ϵ-NE of game $(\alpha A + a, \beta B + b)$? Justify your answer.
   (c) (3 points) Compute all Nash equilibria of the game whose payoff bimatrix is given in Table 1.

5,3	9,0	-8,3
0,30	5,3	10,3
5,-10	5,10	5,3

Table 1: Payoff bimatrix of a 3X3 game

(d) (4 points) Game (A, B) is said to be symmetric if $B = A^T$. First note that, both players have the same number of moves in a symmetric game. Prove that a symmetric game always has a symmetric NE, i.e., NE (x, y) such that $y = x$.

[Hint: Modify Nash's proof]

2. (*TFNP classes*)

(a) (1 point) Prove that every 1-D Sperner problem instance has an odd number of solutions.

(b) (3 points) Prove that PPAD $\subseteq$ PPP. That is show that the canonical problem of PPAD reduces to the canonical problem of PPP (see lecture slides).

(c) (6 points) The colorful Caratheodory theorem (CCT) is as follows. In d-dimensions, we are given points colored with one of the $\{1, \ldots, (d+1)\}$ colors. Furthermore, let S_i be the set of points with color i, then $|S_i| = (d+1)$ and $\mathbf{0}$ is in the convex hull of S_i. We say that set S is colorful if it has exactly one point from each S_i. CCT proves that there exists a colorful set S whose convex hull contains $\mathbf{0}$.

Look up the proof of CCT and using it show that finding such a colorful set S is in PLS. Note: You do not need to prove CCT as part of your answer.

3. (*Other game and equilibrium notions*)

(a) (2 points) Compute all Nash equilibria of the game shown in Table 2.

-1.4	1,-8	3,2	3,-2
4,-5	5,-2	5,-6	5,-8
-3,-2	3,-5	8,-1	7,-5
3,1	-3,1	3,0	7,-2

Table 2: Payoff bimatrix of a 4X4 game

[Hint: Apply iterated dominance.]

(b) (3 points) Alice and Bob are playing a game (A, B) in rounds where in t^{th} round they update their strategies as follows, starting at $x(0)$ and $y(0)$ that are uniform strategies of Alice and Bob respectively.

$$\begin{aligned} \forall i, \quad x_i(t) &= x_i(t-1)\frac{(Ay)_i}{x^T Ay} \\ \forall j, \quad y_j(t) &= y_j(t-1)\frac{(x^T B)_j}{x^T By} \end{aligned}$$

Show that $(x(t), y(t)) = (x(t-1), y(t-1))$ if and only if $(x(t), y(t))$ is a Nash equilibrium. Note: Assume both A and B are non-negative and non-zero.

(c) (3 points) Given a game (A, B), show that each of it's correlated equilibrium is also a coarse correlated equilibrium.

(d) (2 points) Write the normal form representation of the extensive form game shown in fig 1. The game is zero-sum, and Alice's payoffs are shown in the figure.

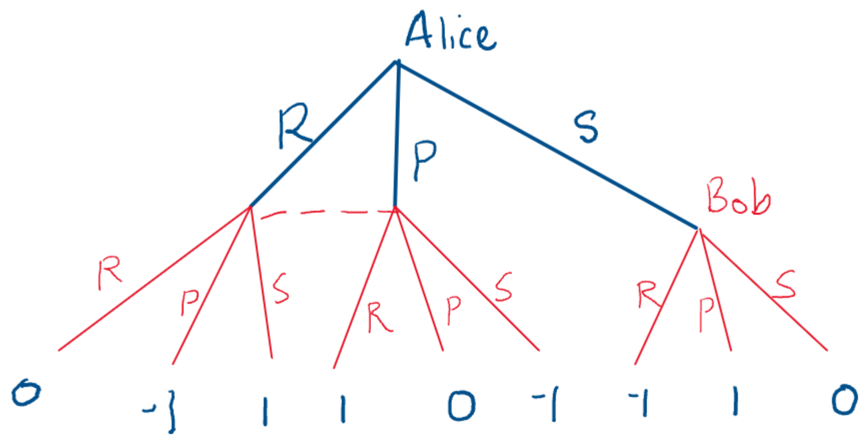

Figure 1: Extensive Form Game

4. (*Stackelberg strategies*) (10 points) The employees of a company called Galaxity all live in the same suburb far from their office, thus have to travel from a common point central in the suburb to the office everyday. To discourage too many employees choosing the same (shortest) route everyday and congesting it, the company sets up a toll booth on some street $i \in [n]$ everyday that the employees can not know before hand. If street i is chosen on a particular day and an employee traverses this street, then the company gets a reward r_i while the employee gets ζ_i cost, otherwise the company pays a cost c_i and the employee's reward is ρ_i.

   Design a polynomial time algorithm to compute the Stackelberg strategy of the company.

   Note: The road network of the town can be represented by a graph where all the employees live at a node s, and the office is at node t. An employee's (attacker) strategy is to choose a path from s to t, while the company's (defender) strategy is to choose an edge to set up a toll booth.

5. (*Bonus problem*)

   (a) Prove that finding NE in game (A, B) reduces to finding a symmetric NE in a symmetric game.

   (b) Prove that checking if 1-D Sperner has more than one solution is NP-complete.

   Note: Imagine a 1-dimensional grid with grid points (nodes) numbered 0 to $2^n - 1$, that are colored with either 0 or 1 color and we know that the color of node 0 is 0, and the color of node $2^n - 1$ is 1. The color function is represented succinctly by a boolean circuit, say $C : \{0, 1\}^n \rightarrow \{0, 1\}$. C takes the boolean representation of a node number and outputs a color 0 or 1, such that C(0)=0 and C(1)=1. So the input size is $n + (\#gates\ in\ C)$, and not the number of grid points, which is 2^n.

   (c) Consider the special case of the 2-D Sperner problem on a square that adds the following further restrictions to the legal colorings of the boundary vertices. Every vertex on the upper side of the diagonal connecting the top-left and bottom right vertices, including

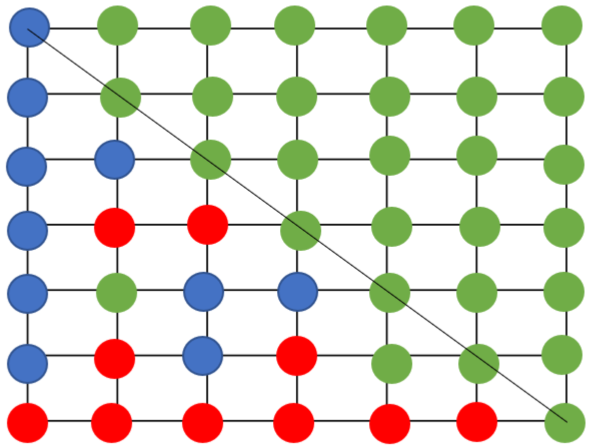

Figure 2: Illustrating the special case of Sperner

all except the top-left vertex on the diagonal vertices must have the same color. Further, each vertex on the left boundary must have the same color as that of the top-left vertex, and each vertex on the bottom boundary must have the color assigned to the bottom-right vertex.

Figure 2 illustrates this special case. For clarity, most diagonal edges forming the triangles are not shown, except those on the main diagonal. Given that the 2-D Sperner problem we discussed in the class is PPAD-hard, prove that this special case of the problem is also PPAD-hard. That is, given an arbitrary 2-D Sperner instance reduce it to this special case of 2-D Sperner.