CS 580RM: Algorithmic Game Theory, Fall 2021
HW 1 (due on Wednesday, 22nd Sept at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (*Envy-freeness and Proportionality*)

   (a) (2 points) Give an example with monotone valuations where a Prop1 allocation is not EF1.

   (b) (3 points) Prove that EF1 $\Rightarrow$ Prop1 for instances with additive valuations.

   (c) (5 points) Prove that when agents have (non-identical) additive valuations but the same ordered preferences, an EFX allocation can be found in polynomial time.
   [*Hint: Try to see if any of the algorithms we covered in class works*]

2. (*MMS*)

   (a) (2 points) For additive valuation functions, we showed $MMS_i \leq \frac{v_i(M)}{n}$ for all agents i. Give an example with sub-additive valuation functions where this is not true.

   (b) (1 point) Prove that if an α-MMS allocation exists for an instance, then an α-MMS+PO allocations also exists.

   (c) (1 point) When agents have identical valuation functions, what is the highest value of $\alpha \in (0, 1]$ for which an α-MMS allocation always exists?

(d) (6 points) For the case with additive valuation functions when $v_{ij} \leq \epsilon$ for all i, j, prove that an EF1+$(1-\epsilon)$-MMS allocation exists and can be computed in polynomial time.

3. (*Max Nash welfare*) Suppose there are m indivisible goods, and n agents with additive valuations.

   (a) (2 points) Is an MNW allocation always PO? Give a short explanation for your answer.

   (b) (8 points) Let $A = (A_1, \ldots, A_n)$ be an MNW allocation. Suppose for every agent i you are given good $g_i \in A_i$ that gives her the maximum value, i.e., $g_i \in \text{argmax}_{g \in A_i} v_{ig}$. Prove that you can now find a (cn)-MNW allocation, where c is a constant, in polynomial time. [Hint: Max weight matchings]

4. (*Stable Matchings*)

   (a) (2 points) Give an example of a stable matching instance with more than one stable matching.

   (b) (2 points) Suppose we consider the variant of finding a stable matching of roommates as follows. There is one set of n people, each person has a preference list for the remaining people, with the highest ranked person in the list most preferred as a roommate. Give an example where a stable matching of roommates does not exist.

   (c) (6 points) We discussed in the class that the Differed Acceptance algorithm where men propose gives the worst choice to women among all stable matchings. This indicates the Gale-Shapley algorithm is not truthful. Give an example where a woman can get a better choice by misreporting her preferences.

5. (*Bonus*)

   (a) Prove that when agents have binary additive valuation functions (that is, $v_{ij} \in \{0, 1\}$ for all i, j) then an EF1+PO allocation can be found in polynomial time.

   (b) An α-EFX allocation is where for all pairs of agents i, j, and every good $g \in A_j$, we have $v_i(A_i) \geq \frac{1}{2} v_i(A_j \backslash \{g\})$. Prove that when the valuation functions of agents are subadditive, $\frac{1}{2}$-EFX $\Rightarrow$ α-MMS for $\alpha = 1/(cn)$, for some constant c where n is the number of agents.

   (c) Prove that an MNW allocation is also EF1+PO.

   (d) Which of the Arrow's axioms does the instant run-off voting system violate? Give an example to illustrate this. The instant runoff system is as follows: Repeatedly eliminate candidates that have the least number of first-place preferences among all voters, until only one candidate (arbitrarily choose among the last remaining if they all will be eliminated in the next round) remains.

   (e) Prove that the stable matching algorithm matches every woman with her worst feasible choice.