Stable Matching

Applications: Assigning doctors to hospitals, students to schools

Problem:

Agents

\( V_1, V_2, V_3, \ldots \) and \( W_1, W_2, W_3, \ldots \)

\( \forall w \in W \) \( (V_3 \geq V_5 \geq \ldots) \)

\( \forall v \in V \) \( (V_3 \geq V_5 \geq \ldots) \)

\( \forall v, w \) \( (V_3 \geq V_5 \geq \ldots) \)

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Goal: Find a "stable" perfect matching

- Blocking pair

\( (V_2 > V_1) \) \( u_1 \) \( \rightarrow \) \( v_1 \) \( (u_1 > v_1) \)

\( (V_2 > V_1) \) \( u_2 \) \( \rightarrow \) \( v_2 \) \( (u_1 > v_2) \)

A matching is stable if no blocking pair

Gale-Shapley Alg (Differed Acceptance Alg)

\( (V_1, V_2, V_3) \) \( u_1 \)

\( (V_1, V_2, V_3) \) \( u_2 \)

\( (V_1, V_2, V_3) \) \( u_3 \)
Running time: $O(n^2)$
why is it perfect matching?

**Thm:** 65 are stable perfect matching.

**Pr:** $M: 65 op$

$(u,v) \in M$, $u \in U$, $v \in V$

**Case I:** $u$ never proposed to $v$.

Suppose $(u,v) \in M$

Then $v' \sim v$

**Case II:** $u$ proposed to $v$ but was rejected in favour of $v'$

Then $v' \sim v$

Finally, if $(u',v) \in M$ then $u'' > u' > u$

Either $(u > v)$ or $(u'' > v)$


Then: It is an optimal stable matching.

It meant stable matching for women.

"Men opt stable matching":

$$best-sm(u) = \arg\max \{v \mid (u,v) \in \text{some stable}\}$$

$$MOSM = \{u, best-sm(u)\} \forall u$$

**House Allocation**

A = set of agents, each with a house. Agent $i$ owns house $i$.

$ICA$ has a complete preference over $h_i, \ldots, h_n$

$(h_2 > h_3 > h_4 > h_5)$

$(h_4 > h_1 > h_3 > h_2)$
A Top Trading Cycle (TTC):

1. $A = n_1 \ldots n_n$, $H = h_1 \ldots h_n$

2. While $A \neq \emptyset$
   
   2.1 Each $i \in A$ points to her most preferred house in $H$, $\rightarrow G$
   
   2.2 $C$: All cycles in the graph $G$. (No sink in $G$ $\Rightarrow$ Not a DAG $\Rightarrow$ A cycle)
   
   2.3 Exchange houses along every cycle in $C$. $A = A \setminus$ agents in $C$, $H = H \setminus$ houses $\in C$

OBS1: A node can be part of at most one cycle in $G$.

Claim 1. Every agent only improves.

$G$: 

A Stronger Notion of Truthful News: Dominant Strategy Incentive Compatibility (DSIC)

An agent has "no" incentive to lie, no matter what others do.

Thm. TTC is DSIC.

Pf. $N_j = \{ i \}$ agent $i$ gets assigned in Round $i$.

OBS2. No agent in $N_1$ has any incentive to lie.

OBS3. By lying an agent can only change her outgoing edges in $G$ in any round, not the incoming edges.

OBS4: Let $i \in N_k \& h_i^1 > h_i^2 > \ldots > h_i^k$. Suppose $h_i^1 \ldots h_i^k$ are taken in Round 1 by agents $a_1 \ldots a_k \in N_i$, $e_i$ gets $h_i^k$.

$\Rightarrow a_1$ to $a_k$ are not pointing to $i$ in Round 1.
And no matter what preference order agent \( i \) reports she cannot change this fact.

\[ \Rightarrow i \text{ has no way to get any of } h_1, \ldots, h_k. \]

\[ \Rightarrow \text{Since, Out as } H \setminus \{ \text{Houses & N_i} \}, i \text{ is getting the most preferred house, she can not be & get better.} \]

**OBS 5:** let \( i \in N_j \) & \( h_1 \succ_i h_2 \succ_i \ldots \succ_i h_n \)

If the first available house in his order after \( 1, \ldots, (j-1) \) rounds is \( h_{(k+1)} \) then no matter what \( i \) reports she cannot get any of \( h_1, \ldots, h_k \).

The argument is the same as above. \( h_1, \ldots, h_k \) are assigned to some agents in \( N_1 \cup \ldots \cup N_{j-1} \Rightarrow \) None of them point to \( i \) in first \((j-1)\) rounds \( \Rightarrow \) No matter what \( i \) reports she can not get any of \( h_1, \ldots, h_k \).