Fair Division: Max Nash Welfare

CS 580

7 September 2021

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Most slides are curtesy Prof. J. Garg

Allocation of Indivisible Items to Agents

- Set M of m indivisible items
- Set N of n agents
- Allocation $A = (A_1, \ldots, A_n)$ is a partition of items to agents where each item is assigned to at most one agent

Objectives

- Maximize the sum of valuations
  (**Utilitarian** Welfare):

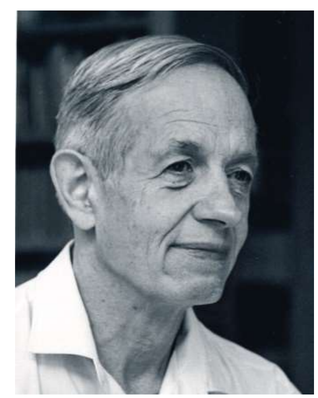
$$SW(A) = \sum_i v_i(A_i)$$

- Maximize the minimum of valuations
  (Max-Min-Fairness, **Egalitarian** Welfare):

$$SW(A) = \min_i v_i(A_i)$$

- Maximize the geometric mean of valuations
  (≈ **Efficiency** + **Fairness**, **Maximum Nash Welfare**):

$$\mathrm{N}W(A) = \left(\prod_{i \in A} v_i(A_i)\right)^{1/n}$$



Scale invariant

Maximum Nash Welfare (MNW)

- Maximum Nash welfare (MNW): An allocation A that maximizes the Nash welfare among all feasible allocations i.e.,

$$A^* = \arg\max_A (\prod_i v_i(A_i))^{1/n}$$

Additive Valuations $(v_i(A_i) = \sum_{j \in A_i} v_{ij})$:

- Divisible Items: MNW $\equiv$ CEEI $\Rightarrow$ Envy-free + Prop + PO + …
- Indivisible Items: MNW $\Rightarrow$ EF1 + PO + $\Omega(\frac{1}{\sqrt{n}})$-MMS [CKMPSW16]
  - Existence of EF1 + PO allocation

MNW: Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive Valuations

Additive $\subset$ GS $\subset$ Submodular $\subset$ XOS $\subset$ Subadditive

MNW: Generalizations

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The non-symmetric MNW Problem

- Non-symmetric MNW [HS72, K77]: un-equal agents' shares.
  - Agent i has a weight/share/clout/importance of w_i
- Allocation $A = (A_1, \ldots, A_n)$ is partition of items to agents

$$\mathrm{NW}(A) = \left(\prod_i v_i(A_i)^{w_i}\right)^{1/\sum_i w_i}$$

weighted geometric mean of agents' valuations

- A^*: allocation maximizing the NW
- ρ-approximate MNW allocation A satisfies:

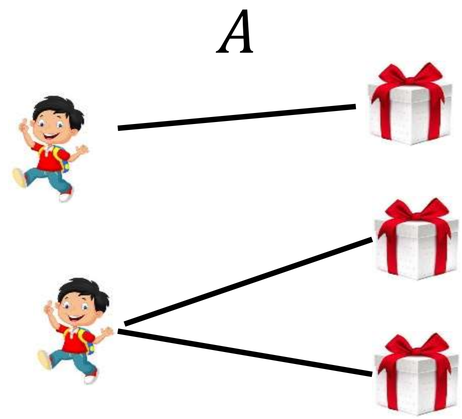
$$\rho.\,\mathrm{NW}(A) \geq \mathrm{NW}(A^*) = \mathrm{MNW}$$

Example (additive)

w_i

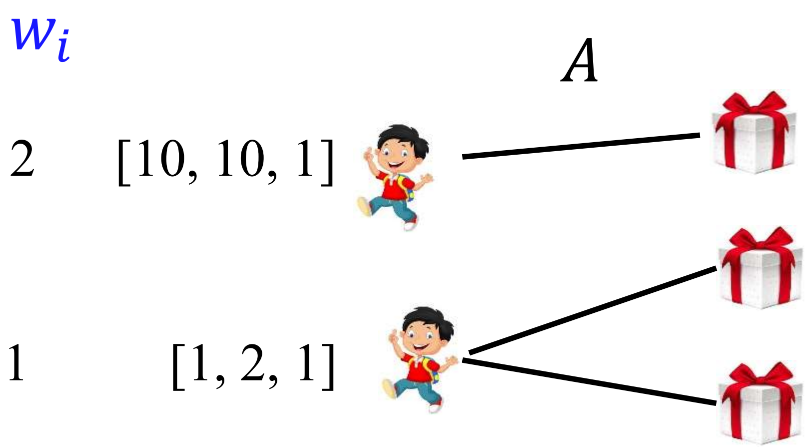
1 [10, 10, 1]

1 [1, 2, 1]



MNW=NW(A) = $(10^1 \cdot 3^1)^{1/2}$

Example (additive)



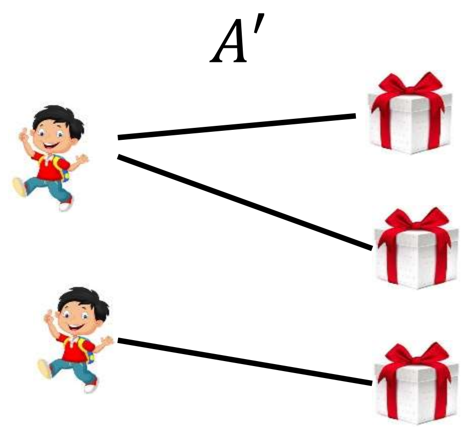
$$\mathrm{NW}(A) = (10^2 \cdot 3^1)^{1/3}$$

Example (additive)

w_i

2 [10, 10, 1]

1 [1, 2, 1]

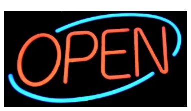


$$\mathrm{NW}(A) = (10^2 \cdot 3^1)^{1/3} < (20^2 \cdot 1^1)^{1/3} = \mathrm{NW}(A') = \mathrm{MNW}$$

MNW Approximations: Additive

	Lower bound	**Upper Bound**
Symmetric	1.069	1.45
Non-symmetric	1.069	$O(n)$

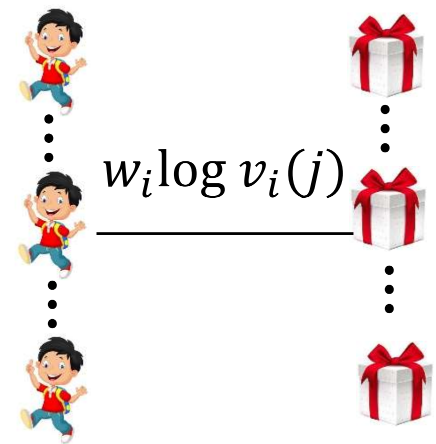
n: # of agents



Constant factor? sublinear?

$m = n$: Matching

$$\mathrm{NW}(A) = \left(\prod_i v_i(A_i)^{w_i}\right)^{1/\sum_i w_i}$$



$m = n$: Matching

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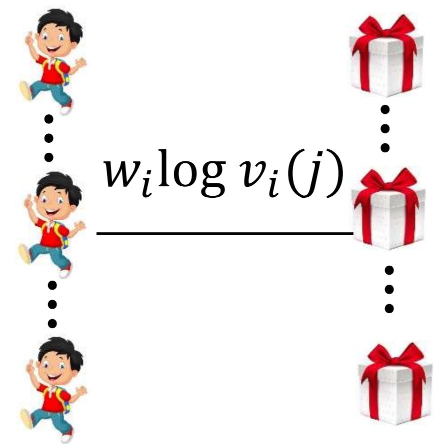
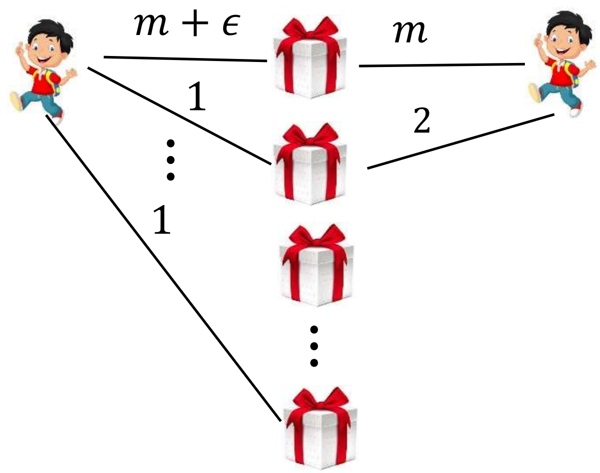


$$\text{MNW Alloc.} = \underset{A}{\operatorname{argmax}}\ \mathrm{NW}(A)$$
$$= \underset{A}{\operatorname{argmax}} \log \mathrm{NW}(A)$$
$$= \underset{A}{\operatorname{argmax}} \sum_i w_i \log v_i(A_i)$$

Claim: If $m = n$, then max-weight matching outputs MNW

$m > n$

- How good is max-weight matching?



$$\mathrm{NW}(A^*) \simeq m$$

$$\mathrm{NW}(A) \simeq \sqrt{2m}$$

Issue: Allocation of high-value items!

Approach:

1. Handle highest-valued items separately
2. Round Robin

Round Robin Procedure

A_i^R: Agent i's bundle. Guarantee (per agent) ?

- $H_i = n$ highest-valued items of agent i. $u_i = v_i(M \setminus H_i)$

Claim. $v_i\left(A_i^R\right) \geq \frac{u_i}{n}$

$O(n)$-MNW + EF1 [GKK20]

- $H_i = 2n$ highest-valued items for agent i. $H = \cup_i H_i$
- $u_i = v_i(M \setminus H_i)$
- Allocate as per max-weight matching from H with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate $M \setminus (\cup_i y_i^*)$ using round-robin procedure

Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

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Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

Approximation Analysis: A General Approach

- A_i^* = What agent i gets at the Max NW allocation (OPT)
- A_i = What your algorithm gives to agent i.

ρ-approximation: To show that $NW(A) \geq \frac{1}{\rho} NW(A^*)$

- $v_i(A_i^*) \leq \rho$ [an intermediate bound]
- [an intermediate bound] $\leq v_i(A_i)$

Upper bounding the optimum $v_i(A_i^*)$

- $H_i = 2n$ highest-valued items of agent i. $u_i = v_i(M \setminus H_i)$
- g_i^*: highest-valued item in MNW allocation A_i^*

$$v_i(A_i^*) = v_i(A_i^* \cap H_i) + v_i\big(A_i^* \cap (M \setminus \mathrm{H}_i)\big)$$

Upper bounding the optimum $v_i(A_i^*)$

- $H_i = 2n$ highest-valued items of agent i. $u_i = v_i(M \setminus H_i)$
- g_i^*: highest-valued item in MNW allocation A_i^*

$$v_i(A_i^*) = v_i(A_i^* \cap H_i) + v_i\big(A_i^* \cap (M \setminus \mathrm{H}_i)\big)$$
$$\leq 2n v_i(g_i^*) + u_i = 2n\left(v_i(g_i^*) + \frac{u_i}{n}\right)$$

- If at A, $v_i(A_i) \geq v_i(g_i^*) + \frac{u_i}{n}$, then A is $(2n)$-approx!

$O(n)$-MNW + EF1 [GKK20]

- $H_i = 2n$ highest-valued items for agent i. $H = \cup_i H_i$
- $u_i = v_i(M \setminus H_i)$
- Allocate as per max-weight matching from H with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate $M \setminus (\cup_i y_i^*)$ using round-robin procedure

Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

- $H_i = 2n$ highest-valued items for agent i
- $u_i = v_i(M \setminus H_i)$
- Allocate one item to each agent using max-weight matching with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate remaining items using round-robin procedure

Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

Claim. $v_i(A_i^*) \leq 2n(v_i(g_i^*) + \frac{u_i}{n})$, where g_i^* is the highest-valued item in A_i^*

$$\Rightarrow NW(A) \geq \left(\Pi_i \left(v_i(y_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}} \geq \left(\Pi_i \left(v_i(g_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

$$\geq \frac{1}{2n} \left(\Pi_i \left(v_i(A_i^*)\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

Additive valuations are restrictive

100

Additive valuations are restrictive

100

100

Additive valuations are restrictive

100

+

100

$125 \neq 100 + 100$

Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive

Additive $\subset$ GS $\subset$ Submodular $\subset$ XOS $\subset$ Subadditive

non-negative monotone: $v(S) \leq v(T), \quad S \subseteq T$

Subadditive: $v(A \cup B) \leq v(A) + v(B), \quad \forall A, B$

MNW Approximations: Symmetric Agents

Additive ⊂ (SC ⊂ OXS ⊂ GS / Budget additive) ⊂ Submodular ⊂ XOS ⊂ Subadditive

Valuation	**Lower bound**	**Upper Bound**
Additive Budget additive Separable concave	1.069	1.45
OXS Gross-Substitutes	1.069	$O(1)$
Submodular XOS Subadditive	1.58	$O(n)$

n: # of agents

MNW Approximations: Non-symmetric Agents

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Envy-free (EF) Allocation

Claim: An EF allocation A is $O(n)$-approximation

½-EFX:
Max-matching + Envy-cycle procedure

½-EFX Allocation

- ½-EFX allocation A: $v_i(A_i) \geq \frac{1}{2} v_i(A_j \setminus \mathrm{g}), \forall g \in A_j, \forall i, j$

Claim: If $|A_i| \geq 2, \forall i,$ then A is $O(n)$-approximation

$O(n)$ Algorithm [CGM.20]

- H_i : n highest-valued items for agent i. $H = \cup_i H_i$
- $u_i = v_i(M \setminus \mathrm{H}_i)$
- Allocate as per max-weight matching from H with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $Y = \cup_i y_i^*$
- $A \leftarrow$ Allocate $M \setminus \mathrm{Y}$ using ½-EFX algorithm

Theorem. A is $O(n)$-MNW and ½-EFX allocation

$O(n)$ Algorithm

Claim: A is $O(n)$-MNW

Proof (sketch):

- g_i^*: highest-valued item in MNW allocation A_i^*
- $v_i(A_i^*) \leq \; n v_i(g_i^*) + v_i(M \setminus H_i) = n\left(v_i(g_i^*) + \frac{v_i(M \setminus H_i)}{n}\right)$
- $v_i(A_i) \geq v_i(y_i^*)$
- $v_i(A_i) \geq \frac{v_i(M \setminus Y)}{4n} \geq \frac{v_i(M \setminus H_i) - n v_i(y_i^*)}{4n}$

- $v_i(A_i) \geq \frac{1}{8}\left(v_i(y_i^*) + \frac{v_i(M \setminus H_i)}{n}\right) \geq \frac{1}{8}\left(v_i(g_i^*) + \frac{v_i(M \setminus H_i)}{n}\right)$

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