Fair Division: MMS (Continued) and Max Nash Welfare

CS 580

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Maximin Share (MMS)
(Last time)

Normalized valuations

Scale free ⇒ wlog $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$

Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

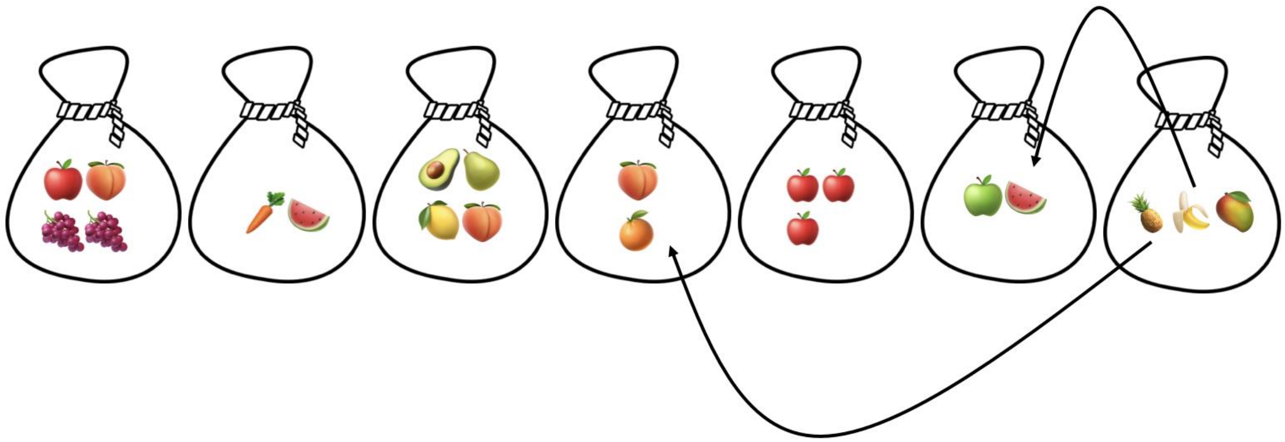
Valid Reduction (α-MMS): If there exists $S \subseteq M$ and $i^* \in N$

$v_{i^*}(S) \geq \alpha . \mu_{i^*}^n(M)$ - bag S makes i happy

$\mu_i^{n-1}(M \setminus S) \geq \mu_i^n(M), \forall i \neq i^*$ - remaining agents don't mind (their MMS value over remaining items and agents is not lower)

Valid Reductions

For any singleton set S, condition 2 of valid reductions always true

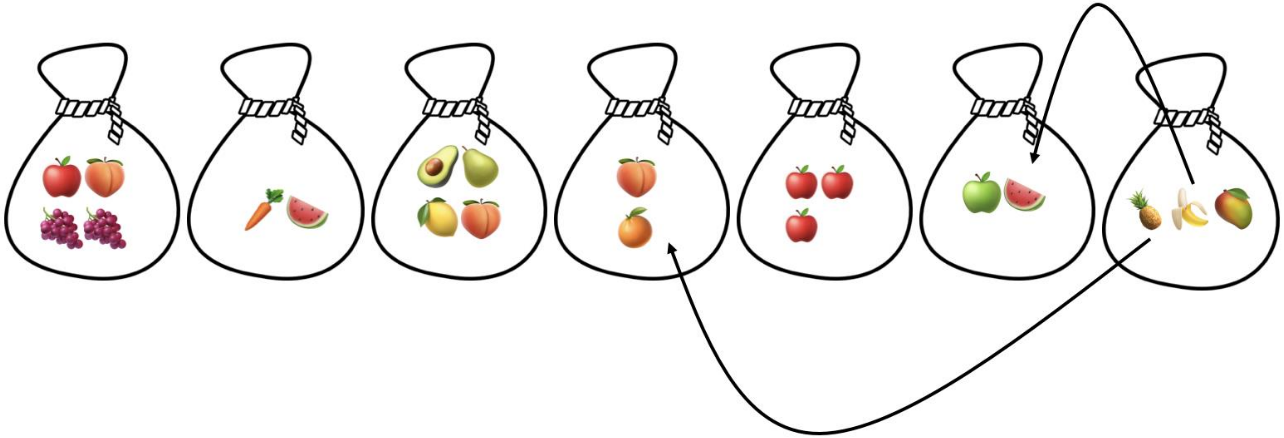


1/2-MMS Allocation

Step 1: Valid Reductions

- If $v_{i1} \geq 1/2$ then assign item 1 to i

Step 2: Bag Filling



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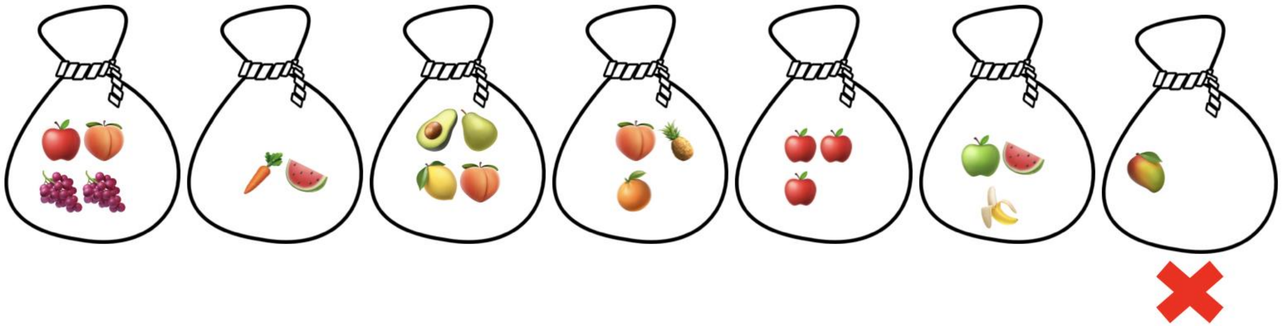

1/2-MMS Allocation

- Re-normalization

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i1} \geq 1/2$ then assign item 1 to i
- After every valid reduction, normalize valuations

Step 2: Bag Filling

2/3-MMS Allocation [GMT19]

- If all $v_{ij} \leq 1/3$ then?

Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i

 $\longrightarrow \quad \exists i : \quad v_i(B) \geq 2/3$

$v_i(\square) > 1/3$ — ①

$v_i(S_{High}) + v_i(S_{Low}) = n$

$v_i(S_{High}) \leq \frac{2}{3} \cdot n$

$v_i(S_{Low}) \geq n - v_i(S_{high}) \geq n - \frac{2}{3} n \geq \frac{1}{3} \cdot n$

2/3-MMS Allocation [GMT19]

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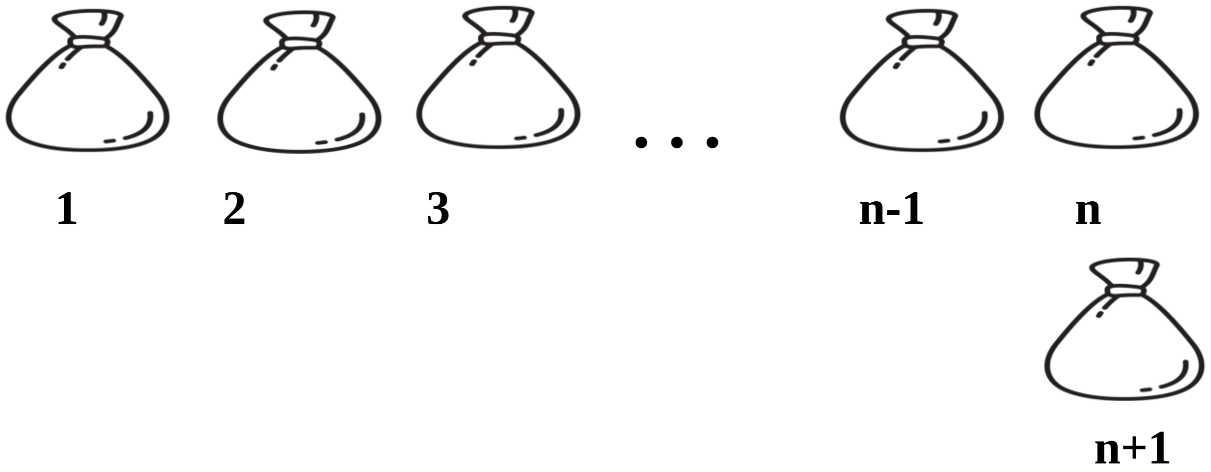
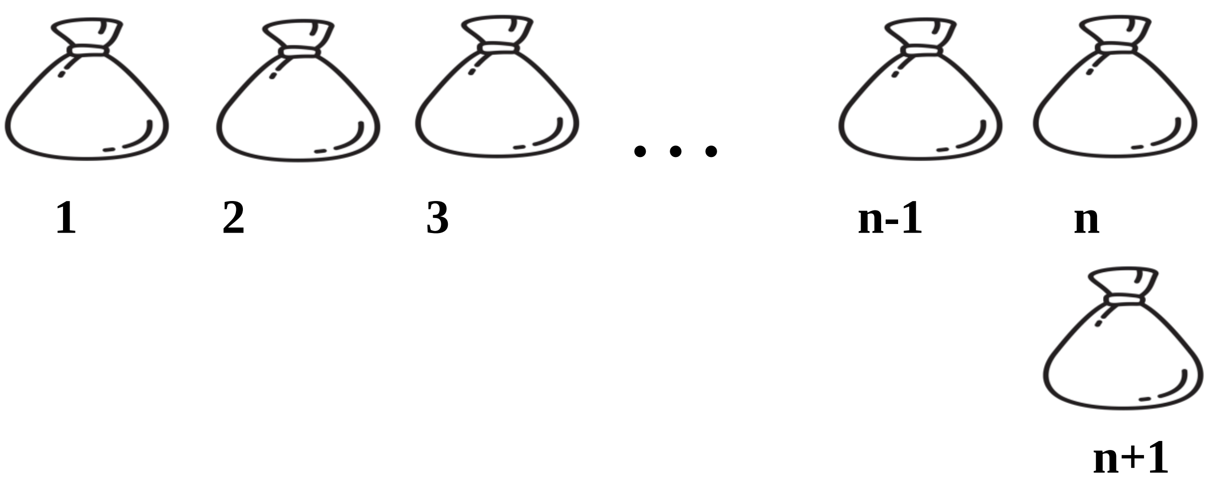


2/3-MMS Allocation [GMT19]

Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i
- If $v_{in} + v_{i(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i



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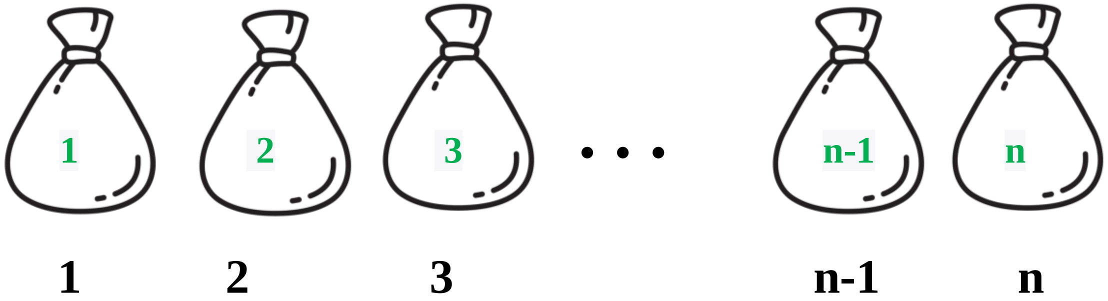
Step 2: Generalized Bag Filling

- Initialize n bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$



2/3-MMS Allocation [GMT19]

- (Re)normalization

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Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i
- If $v_{in} + v_{i(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i
- After every valid reduction, normalize valuations

Step 2: Generalized Bag Filling

- Initialize a bag B with the last high value item
- Fill low value items in B until some i has $v_i(B) \geq 2/3$

2/3-MMS Allocation [GMT19]

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Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i
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Step 2: Generalized Bag Filling

- Initialize a bag B with the last high value item
- Fill low value items until some i has $v_i(B) \geq 2/3$
- Remove i and B
- Re-compute the set of high value items

Summary

Covered

- Additive Valuations:
  - Prop1 (poly-time algorithm)
  - 2/3-MMS allocation (polynomial-time algorithm)

Not Covered

- Prop1+PO
- $\left(\frac{3}{4}+\right)$-MMS allocation [GT20]
- More general valuations
  - MMS [GHSSY18]
- Groupwise-MMS [BBKN18]
- Chores: 11/9-MMS [HL19]

Major Open Questions (additive)

- c-MMS + PO: polynomial-time algorithm for a constant $c > 0$
- Existence of 4/5-MMS allocation? For 5 agents?

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Nash social welfare

Allocation of Indivisible Items to Agents

- Set M of m indivisible items
- Set N of n agents
- Allocation $A = (A_1, \ldots, A_n)$ is a partition of items to agents where each item is assigned to at most one agent

Objectives

- Maximize the sum of valuations (**Utilitarian** Welfare):

$$SW(A) = \sum_i v_i(A_i)$$

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  (**Utilitarian** Welfare):

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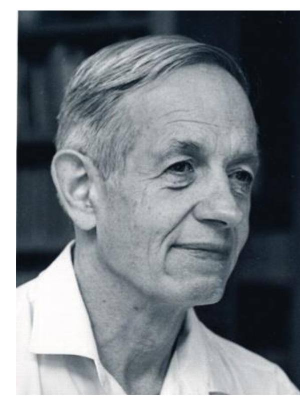
- Maximize the minimum of valuations
  (Max-Min-Fairness, **Egalitarian** Welfare):

$$SW(A) = \min_i v_i(A_i)$$

- Maximize the geometric mean of valuations
  (≈ **Efficiency + Fairness**, **Maximum Nash Welfare**):

$$\mathrm{N}W(A) = \left(\prod_{i \in A} v_i(A_i)\right)^{1/n}$$



Scale invariant

Maximum Nash Welfare (MNW)

- Maximum Nash welfare (MNW): An allocation A that maximizes the Nash welfare among all feasible allocations i.e.,

$$A^* = \arg\max_A (\prod_i v_i(A_i))^{1/n}$$

Additive Valuations ($v_i(A_i) = \sum_{j \in A_i} v_{ij}$):

- Divisible Items: MNW ≡ CEEI ⇒ Envy-free + Prop + PO + …
- Indivisible Items: MNW ⇒ EF1 + PO + $\Omega(\frac{1}{\sqrt{n}})$-MMS [CKMPSW16]
  - Existence of EF1 + PO allocation

MNW (additive)

- APX-hard [Lee17]; 1.069-hardness [G.HM18]

Approximation:

- ρ-approximate MNW allocation A satisfies: $\rho . \mathrm{NW}(A) \geq MNW$
  - 2 [CG15, CDGJMVY17], e [AMOV18]
  - 1.45 [BKV18] (pEF1 approach)

Close the gaps!

- Fairness Guarantees
  - Prop1 + PO + $\frac{1}{2n}$-MMS + 2-MNW [GM19]

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