Fair Division: Proportional, MMS

CS 580

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Most slides are curtesy Prof. J. Garg

Proportional (average)

- n agents
- M: set of m indivisible items (like cell phone, painting, etc.)
- Agent i has a valuation function $v_i : 2^m \to \mathbb{R}$ over subsets of items

Fairness:

Envy-free (EF)

Proportional (Prop):

Get value at least average of the grand-bundle

$$v_i(A_i) \geq \frac{1}{n} v_i(M)$$

	g_1	g_2	g_3	g_4
a_1	100	100	10	90
a_2	100	100	90	10

Sub-additive Valuations

Sub-additive:

$$v_i(A \cup B) \le v_i(A) + v_i(B), \qquad \forall A, B \in M$$

$\geq$ (super-additive)

Claim: $EF \Rightarrow Prop$ (sub-additive)

Proof: $(A_1, \ldots, A_n)$ is EF. $\Rightarrow$

$\forall i$: $V_i(A_i) \geq V_i(A_k)$ $\forall k$. $\Rightarrow$

$n V_i(A_i) \geq \sum_{k=1}^{n} V_i(A_k) \geq V_i(M) \Rightarrow$

$V_i(A_i) \geq \frac{1}{n} V_i(M)$ $\square$

Prop: May not always exist!

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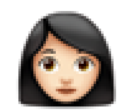


Proportional (Prop):

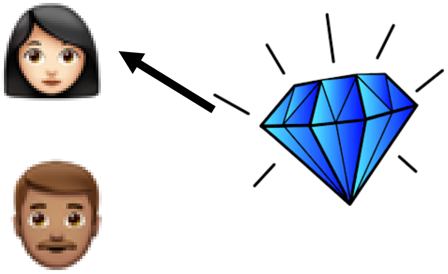
Get value at least average of the grand-bundle

$$v_i(A_i) \geq \frac{1}{n} v_i(M)$$

Proportionality up to One Item (Prop1)

- Prop1: A is proportional up to one item if each agent gets at least $1/n$ share of all items after adding one more item from outside:

$$v_i(A_i \cup \{g\}) \geq \frac{1}{n} v_i(M), \qquad \exists g \in M \setminus A_i, \forall i \in N$$



Prop1

Claim: EF1 implies Prop1 for subadditive valuations

$\Longrightarrow$ Envy-cycle procedure outputs a Prop1 allocation

Proof:

Prop1

- EF1 implies Prop1 for subadditive valuations
  $\Longrightarrow$ Envy-cycle procedure outputs a Prop1 allocation

- +PO: Additive Valuations
  - EF1 + PO allocation exists but no polynomial-time algorithm is known!
  - Prop1 + PO? Algorithm based on competitive equilibrium.

Proportionality

- A set N of n agents, a set M of m indivisible items

- Proportionality: Allocation $A = (A_1, \dots, A_n)$ is proportional if each agent gets at least $1/n$ share of all items:

$$v_i(A_i) \geq \frac{v_i(M)}{n}, \qquad \forall i \in N$$

Cut-and-choose?

Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent i to propose a partition of items into n bundles with the condition that i will choose at the end
- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

Maximin Share (MMS) [B11]

Cut-and-choose.

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- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

- $\Pi \coloneqq$ Set of all partitions of items into n bundles
- $\mu_i \coloneqq \max_{A \in \Pi} \min_{A_k \in A} v_i(A_k)$

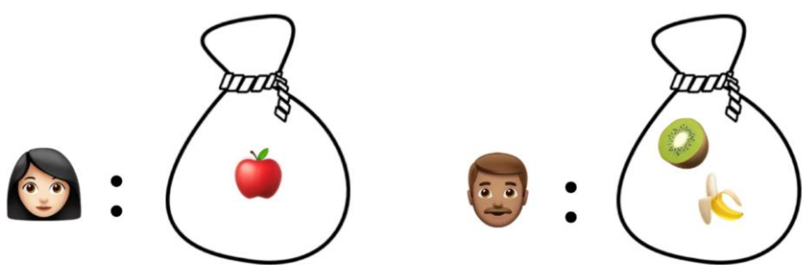
- MMS Allocation: A is called MMS if $v_i(A_i) \geq \mu_i$, $\forall i$
- Additive valuations: $v_i(A_i) = \sum_{j \in A_i} v_{ij}$

MMS value/partition/allocation

Agent\Items	🍎	🍌	🥝
👩	3	1	2
👨	4	4	5

👩	🍎	🍌 🥝
Value	3	3
MMS Value	3	

👨	🍎 🍌	🥝
Value	8	5
MMS Value	5	



MMS value/partition/allocation

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MMS Value	3	

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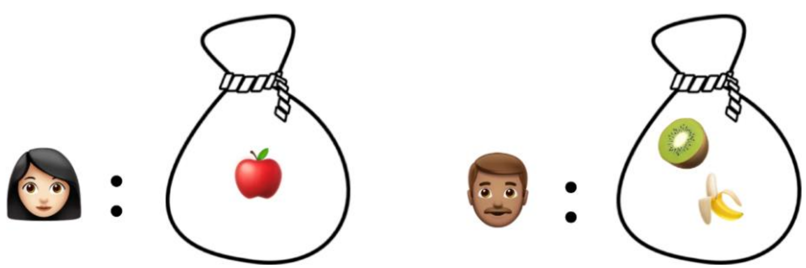

Finding MMS value is NP-hard!

What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$: yes 

  $\Longrightarrow$ A PTAS to find $(1 - \epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$: NO [PW14]

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- $n \geq 3$: NO [PW14]

- α-MMS allocation: $v_i(A_i) \geq \alpha . \mu_i$ $\alpha \in (0,1)$
  - 2/3-MMS exists [PW14, AMNS17, BK17, KPW18, GMT18]
  - 3/4-MMS exists [GHSSY18]
  - $(3/4 + 1/(12n))$-MMS exists [GT20]

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$

suppose not. Let $(A_1, \ldots, A_n)$ be the MMS partition of agent i

$$v_i(A_1) \geq \ldots \geq v_i(A_n) > 1$$

$$v_i(M) = \sum_j v_{ij} = \sum_{k=1}^{n} \underbrace{v_i(A_k)}_{1} > n$$

$n\mu_i$

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- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

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	🍎	🍌	🍐	🍍	🥥
👩	3	1	2	5	4
👨	4	4	5	3	2

	1	**2**	**3**	**4**	**5**
👩	5	4	3	2	1
👨	5	4	4	3	2

Challenge

- Allocation of high-value items!
- If for all $i \in N$
  - $v_i(M) = n \Rightarrow \mu_i \leq 1$
  - $v_{ij} \leq \epsilon, \forall i, j$

$\epsilon = \max_{i,j} v_{ij}$

$v_{ij} \le \epsilon, \forall i, j$

$\cancel{g_1}$ $\cancel{g_2}$ - - - - g_m

Claim: After round k, if i remains then v_i(remaining goods) $\ge n - k$.

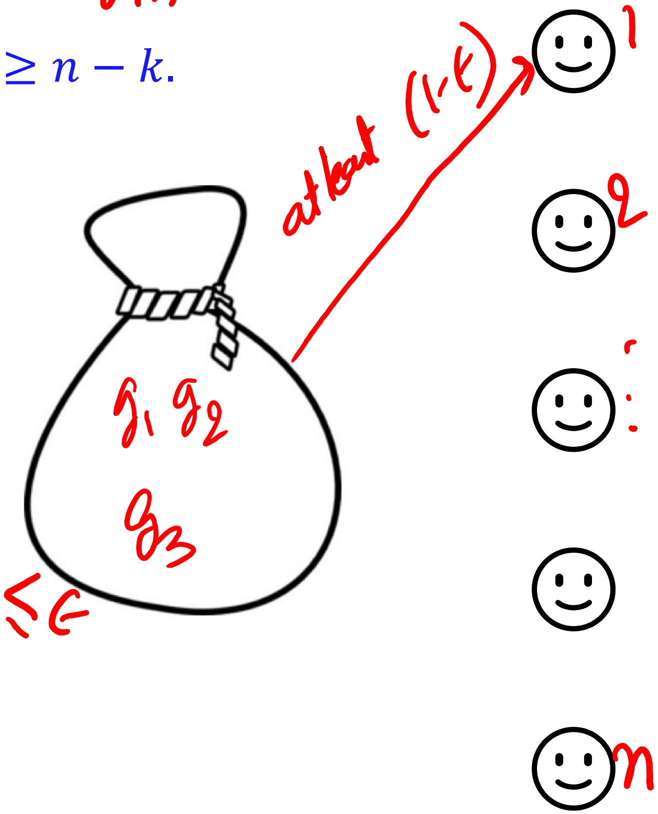
$\{g_1, g_2, g_3\} \Rightarrow A_1$

if n doesn't shout out then

$v_n(A_1) < 1-\epsilon$

o.w. $v_n(A_1 \setminus g_3) < 1-\epsilon$, $v_n(g_3) \le \epsilon$

$\Rightarrow v_n(A_1) < 1-\epsilon+\epsilon = 1$



Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag B
- Keep adding items to B until some agent i values it $\ge (1-\epsilon)$
- Assign B to i and remove them

$v_{ij} \leq \epsilon, \forall i, j$

$g_1 \; g_2 \; \ldots \; g_m$

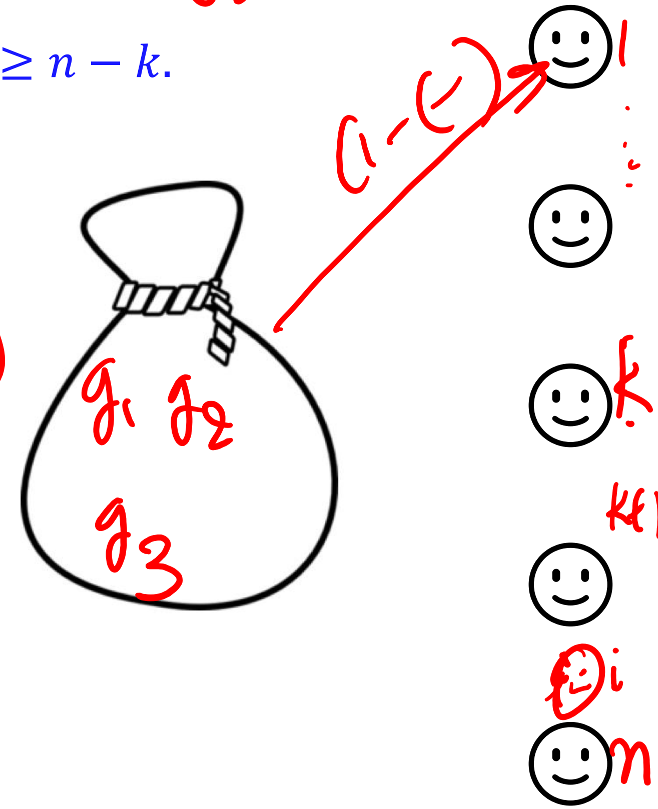
Claim: After round k, if i remains then v_i(remaining goods) $\geq n - k$.

$k = 1$

$v_i(A_1) < 1$

$\Rightarrow v_i(M \setminus A_1) = v_i(M) - v_i(A)$

$> n - 1$



Bag Filling Algorithm:

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- Keep adding items to B until some agent i values it $\geq (1 - \epsilon)$
- Assign B to i and remove them

$v_{ij} \leq \epsilon, \forall i, j$

Thm: Every agent gets at least $(1-\epsilon)$.

$\Rightarrow (1-\epsilon)$-MMS

pf:

$$v_i(A_i) \geq (1-\epsilon) = (1-\epsilon)\cdot 1 \geq \underbrace{(1-\epsilon)}_{=\,\alpha}\cdot \mu_i$$

Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag B
- Keep adding items to B until some agent i values it $\geq (1-\epsilon)$
- Assign B to i and remove the

Warm Up: 1/2-MMS Allocation

- If all $v_{ij} \leq 1/2$ then?
  - Done, using bag filling.

- What if some $v_{ij} > \frac{1}{2}$?

Valid Reductions

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \le 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \ge v_{i2} \ge \cdots v_{im}, \forall i \in N$
- Valid Reduction (α-MMS): If there exists $S \subseteq M$ and $i^* \in N$
  - $v_{i^*}(S) \ge \alpha.\mu_{i^*}^n(M)$
  - $\mu_i^{n-1}(M \setminus S) \ge \mu_i^n(M), \forall i \neq i^*$

  $\Rightarrow$ We can reduce the instance size!

$i^* = n$

Claim: $(A_1, \ldots, A_{n-1})$ is an α-MMS allocation of $M \setminus S$ to $\{1, \ldots, n-1\}$ agents then $(A_1, \ldots, A_{n-1}, S)$ is α-MMS allocation in the original instance.

Pf: $i < n$, $v_i(A_i) \ge \alpha \mu_i^{n-1}(M \setminus S) \ge \alpha \mu_i^n(M)$

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