CS 579. Computational Complexity
Problem Set 4

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due March 17, 2017

Collaboration Policy: The homework can be worked on in groups of up to 3 students each (2 would be optimal, but 1 and 3 are both accepted).

One submission per team is sufficient. Please write the solution for each of the problems on a separate sheet of paper. Write your team names and netids on each submission and please **staple** all the sheets together.

Submissions should be written in LaTeX, unless your handwriting is indistinguishable from LaTeX.

Homework is due before the end of class, March 17. Only one late homework per person will be allowed. If you submit more than one homework late, you will get no grade for the excess late homeworks.

Problem 1 (40 pts.)

Recall that the trace of a matrix A, denoted $\mathrm{tr}(A)$, is the sum of the entries along the diagonal.

1. Prove that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

2. Prove that if A is a random walk matrix of an n-vertex graph G and $k \geq 1$, then $\mathrm{tr}(A^k)$ is equal to n times the probability that if we select a vertex $v \in V(G)$ uniformly at random and take a k step random walk from vertex v, then we end up back at vertex v.

3. Prove that for every d-regular graph $G, k \in \mathbb{N}$ and vertex $v \in V(G)$ of G, the probability that a path of length k starting from v ends up back at v is at least as large as the corresponding probability in T_d, where T_d is the complete $(d-1)$-ary tree of depth k rooted at v (that is, every internal vertex has degree d, one parent and $d-1$ children).

4. (Extra Credit) Prove that for even k, the probability that a path of lengh k from the root v of T_d ends up back at v is at least $2^{k-k\log(d-1)/2+o(k)}$. (Hint: Use Catalan numbers and Stirling's approximation.)

Problem 2 (20 pts.)

Recall that if G is a d-regular graph with transition matrix M, then G^k is the d^k-regular multigraph with transition matrix M^k that has one edge for each path of length k in G (with repetitions).

1. Prove that if $h(G) \geq \epsilon$, then there is a $\kappa = \kappa(\epsilon)$ depending only on ϵ and not on the size of G such that $h(G^\kappa) \geq 1/10$.

2. Provide a counterexample to the following statement:

$$h(G^2) \geq \min(1/10, 1.01 \times h(G))$$

   Note: The statement may be true (it is an open question) if G^2 is replaced by G^3.

Problem 3 (40 pts.)

Let G be an undirected connected graph with n vertices and adjacency matrix A. Let d_i be the degree of vertex $i \in V(G)$. Define the normalized K-cut of a graph, into $K \geq 2$ disjoint sets of vertices $S_1, S_2, \cdots, S_K$, to be

$$\mathrm{NCUT}(S_1, S_2, \cdots, S_K) = \sum_{i=1}^{K} \frac{\mathrm{Cut}(S_i)}{\mathrm{Vol}(S_i)}$$

where $\mathrm{Cut}(S) = \sum_{i\in S, j\in V\setminus S} A(i,j)$ and $\mathrm{Vol}(S) = \sum_{i\in S} d_i$. Let h_i be an "indicator vector" for set S_i as follows:

$$h_i(v) = \begin{cases} \frac{1}{\sqrt{\mathrm{Vol}(S_i)}} & \text{if } v \in S_i \\ 0 & \text{otherwise} \end{cases}$$

Let H be an n by K matrix whose columns are the h_i, i.e. $H = [h_1, h_2, \cdots, h_K]$.

1. Show that $\mathrm{NCUT}(S_1, S_2, \cdots, S_K) = \mathrm{tr}(H^T L H)$, where L is the Laplacian of G, which is defined as $L = D - A$. D is a diagonal matrix with $D_{ii} = d_i$ for all $i \in V(G)$.

2. Show that $H^T D H = I_{K\times K}$

3. Show a spectral relaxation of minimum K-cut, generalizing Cheeger's rounding.