PCP

Lecture 20
And Hardness of Approximation

Promise Problems

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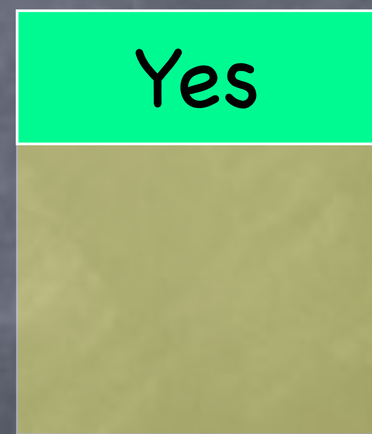

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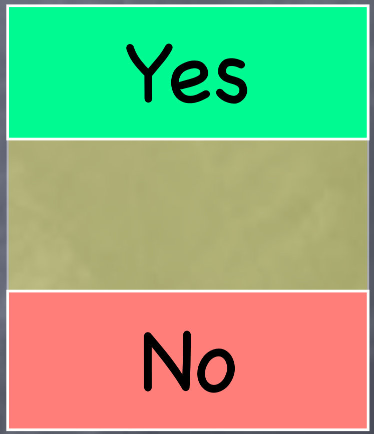

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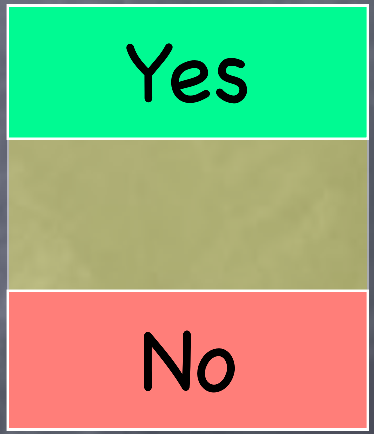

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- Specified by a Yes set and a No set, disjoint
  - A TM is said to decide a promise problem if it correctly answers Yes or No for inputs from these sets
  - For inputs outside the two, don't care
    - We're "promised" that such inputs are not given

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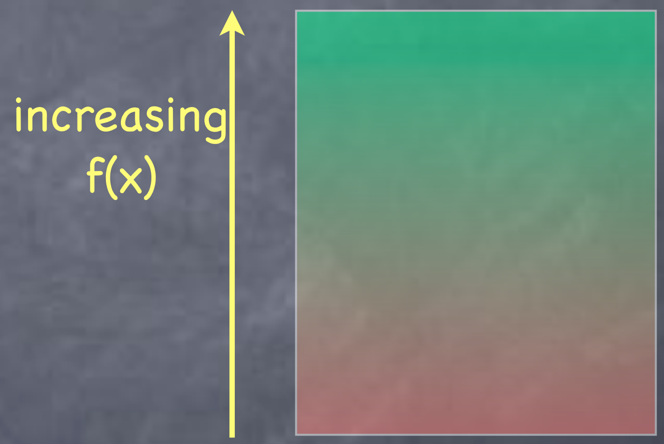

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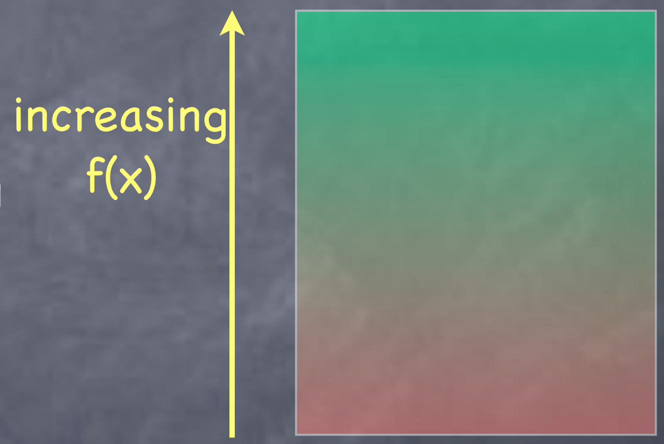

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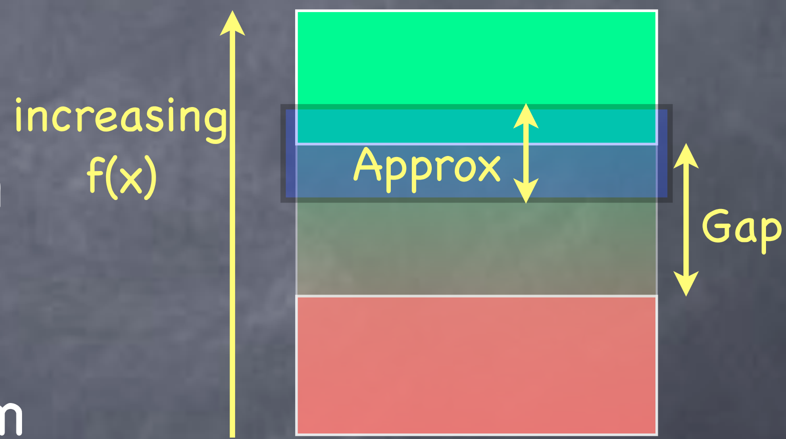

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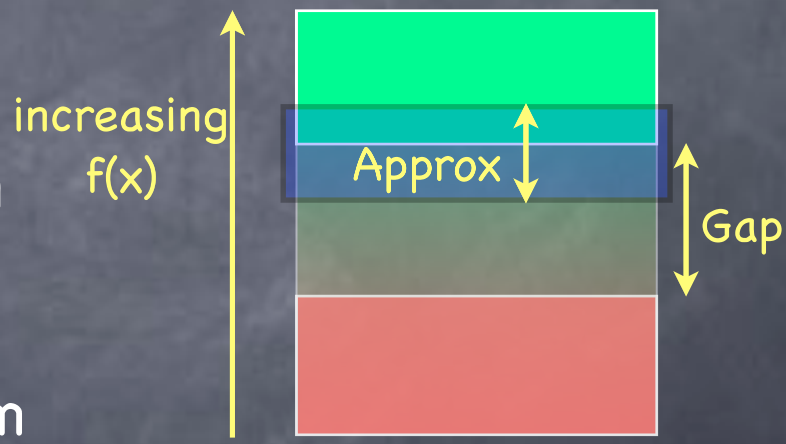

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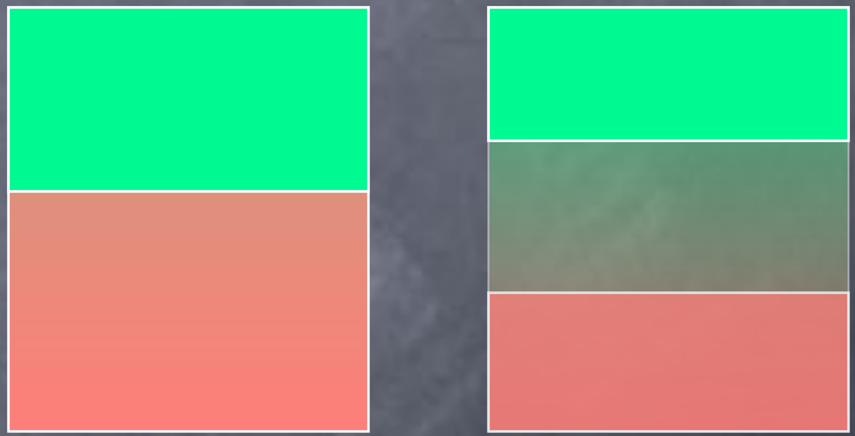
  - PCP(log m,q): q-query (non-adaptive) verifier, tosses at most log m coins

Decision Problem to Gap Problem



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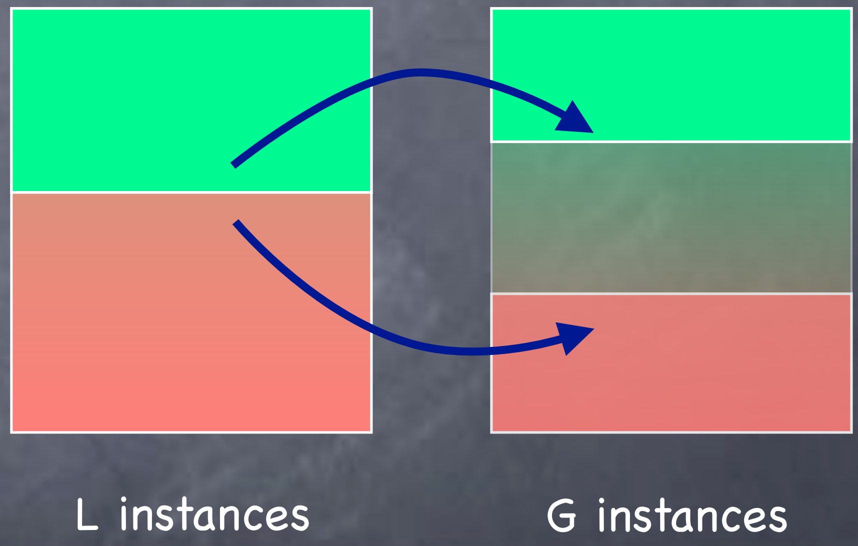

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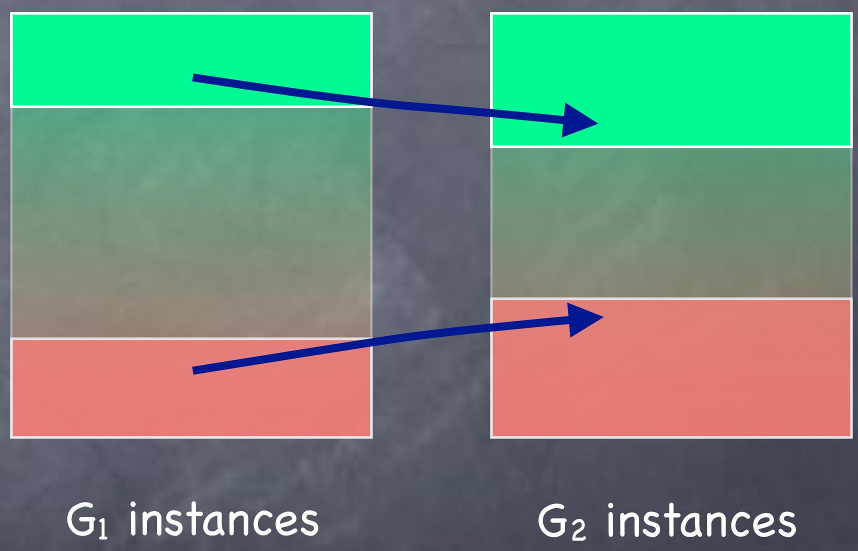

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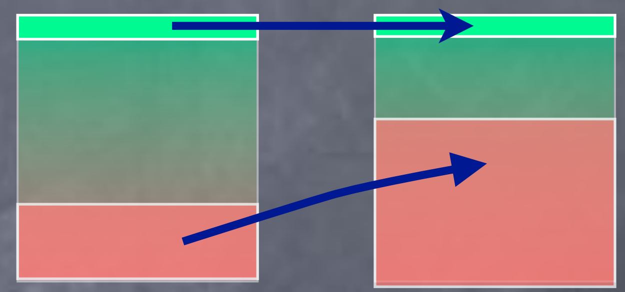
Max-qCSP to Max-3SAT



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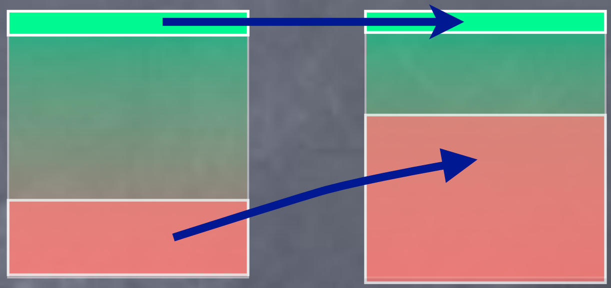

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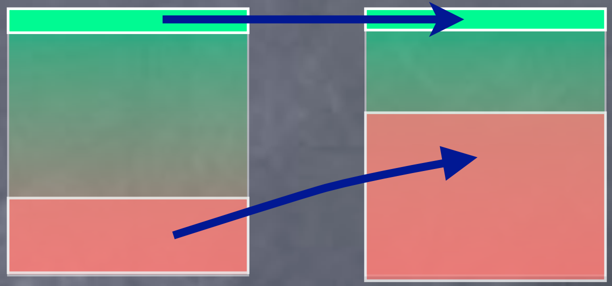

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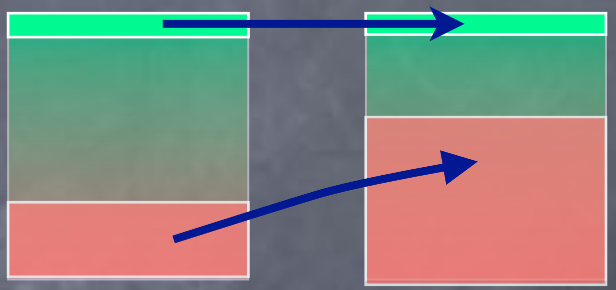

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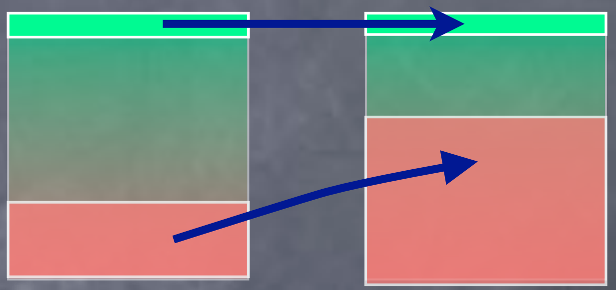

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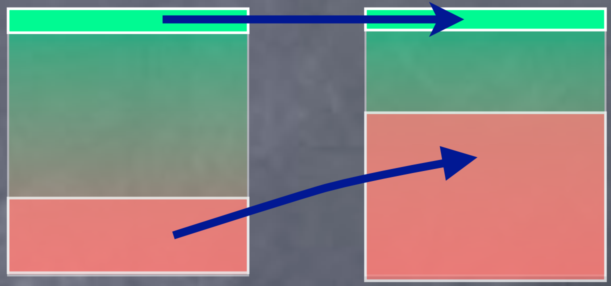

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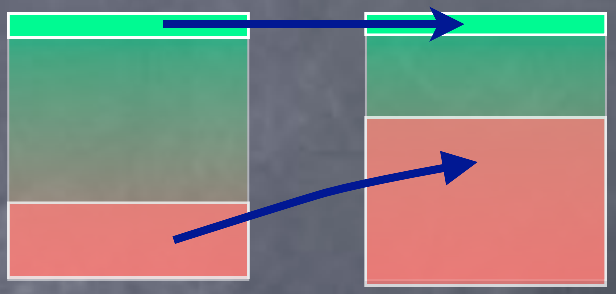

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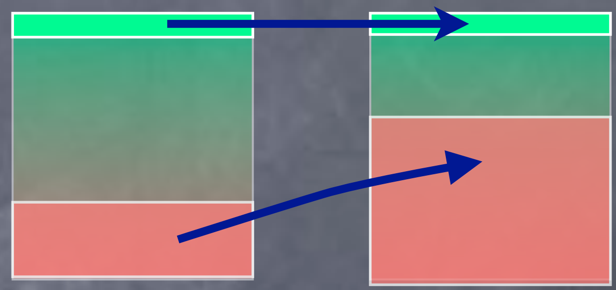

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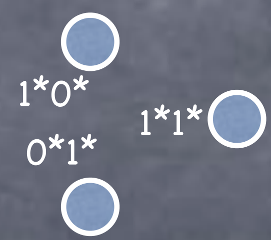

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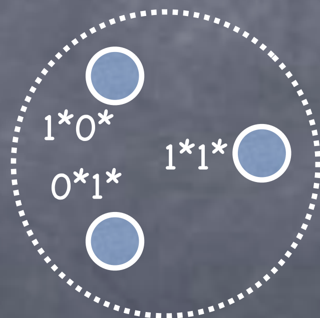

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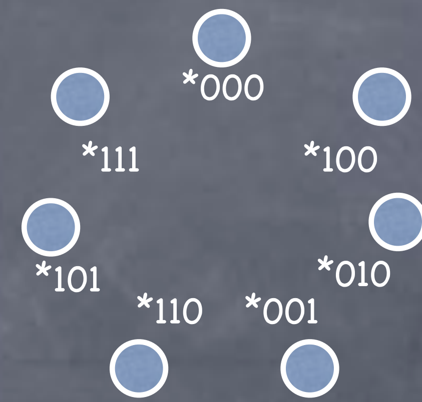

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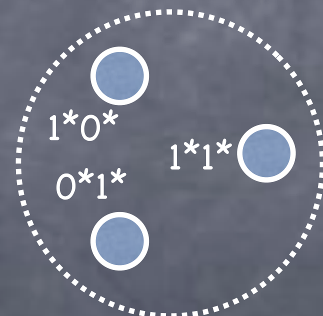

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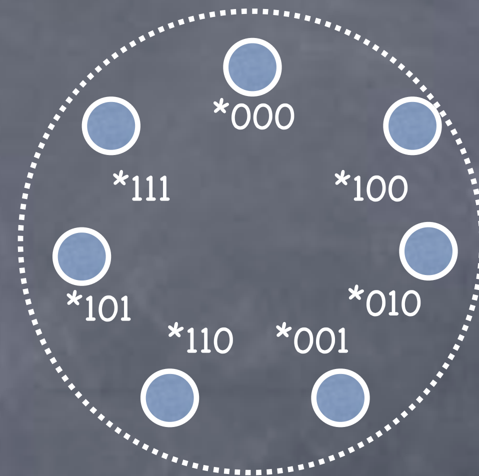

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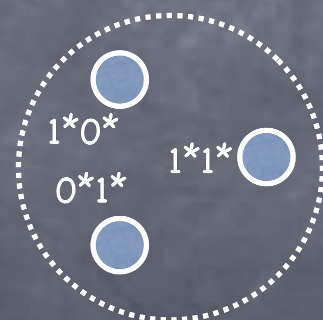

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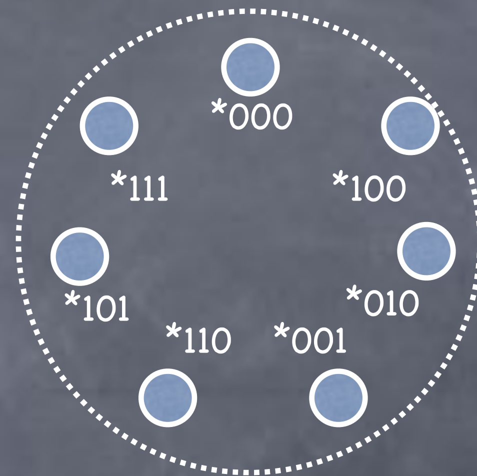

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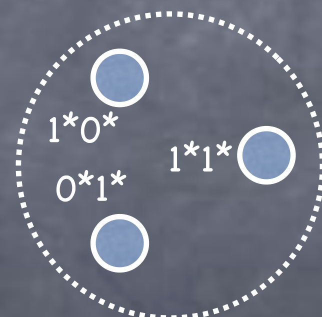

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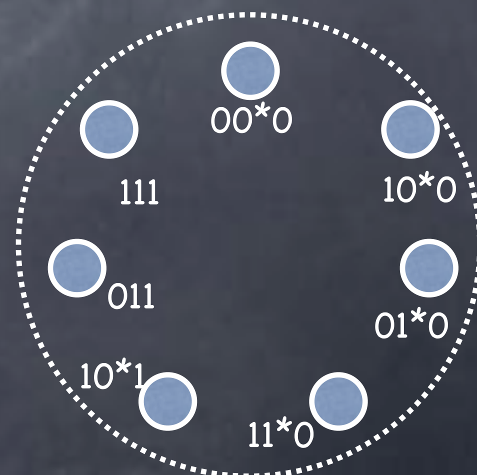

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  - vertices: each clause's sat assignments (for its variables)
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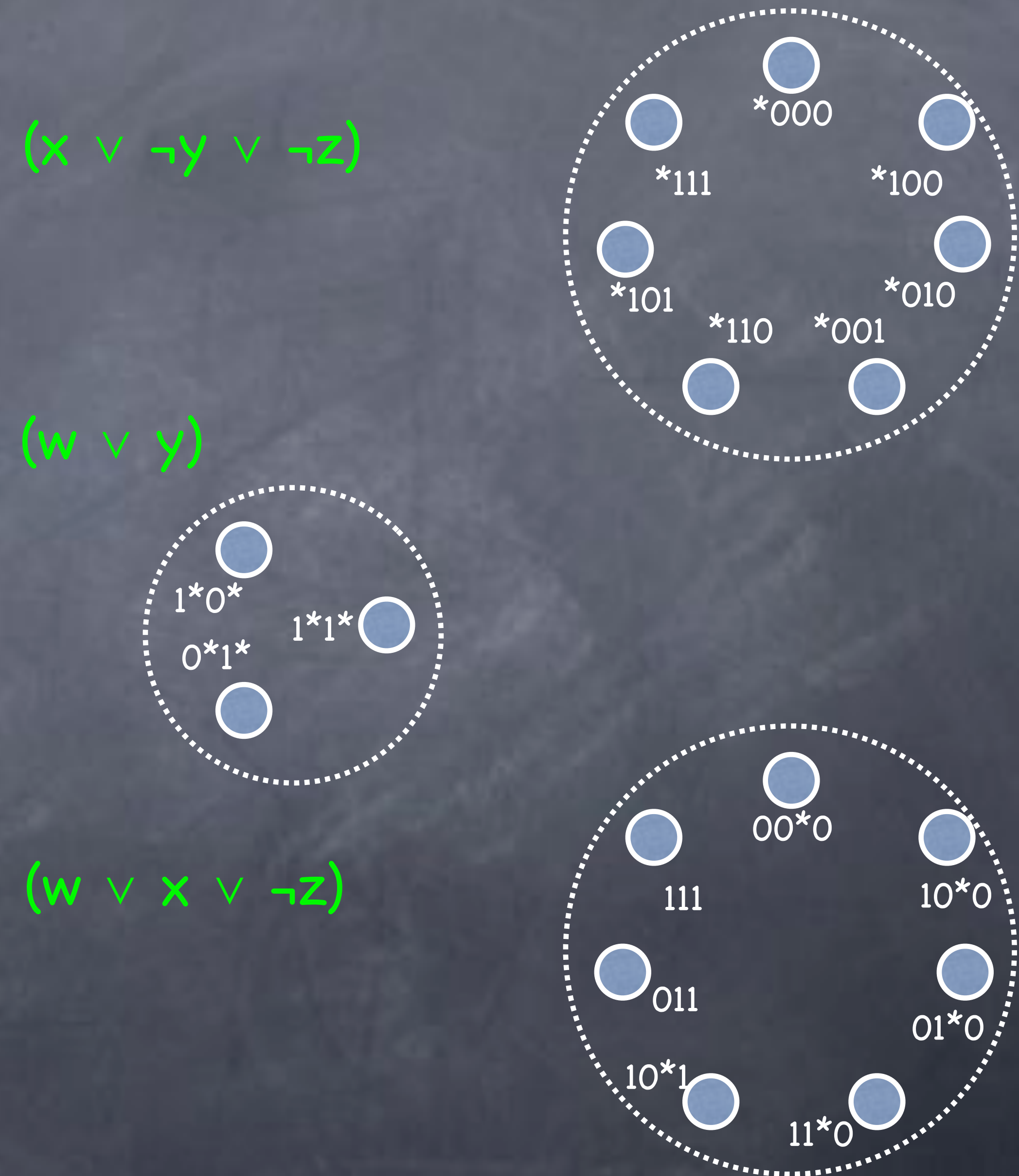

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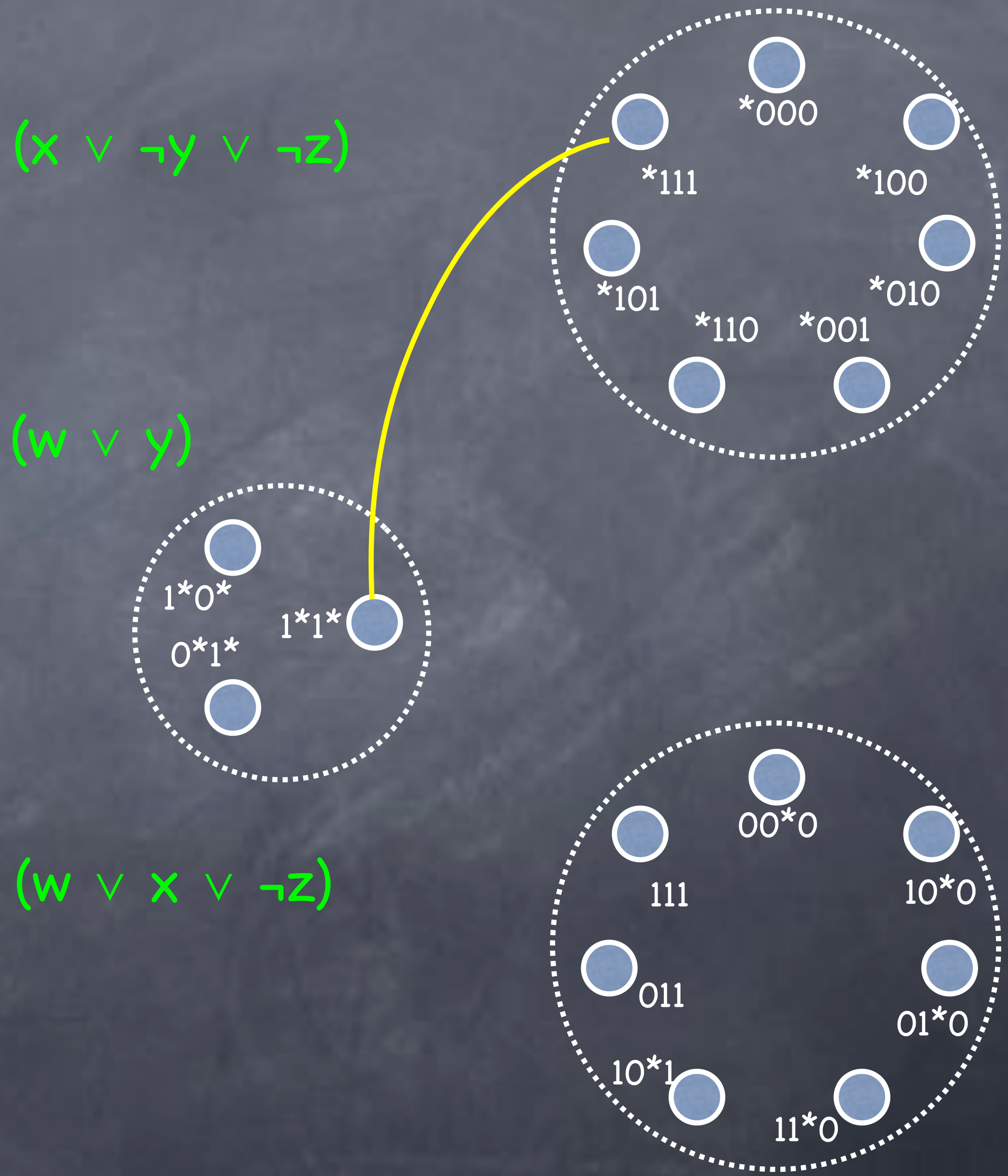

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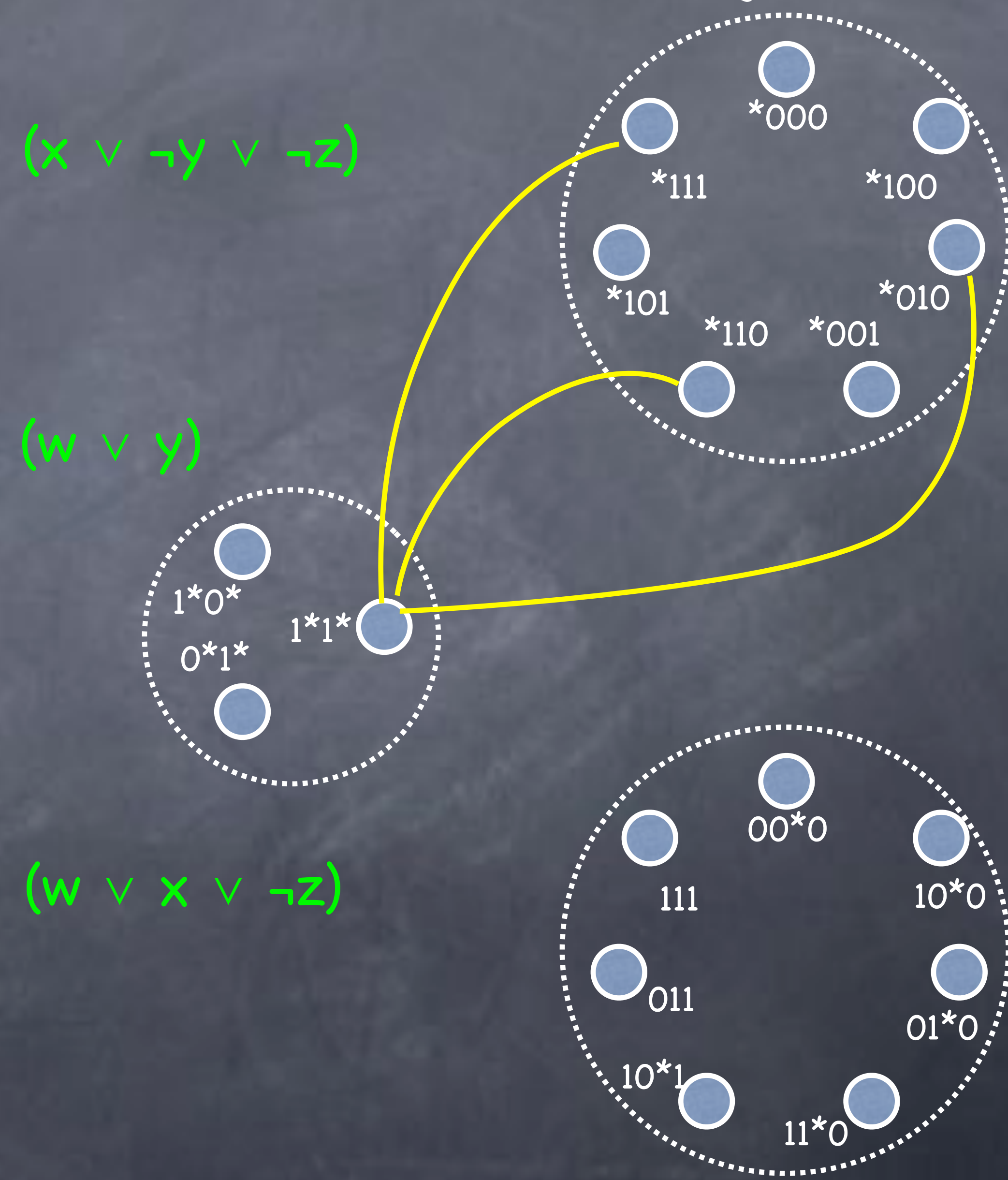

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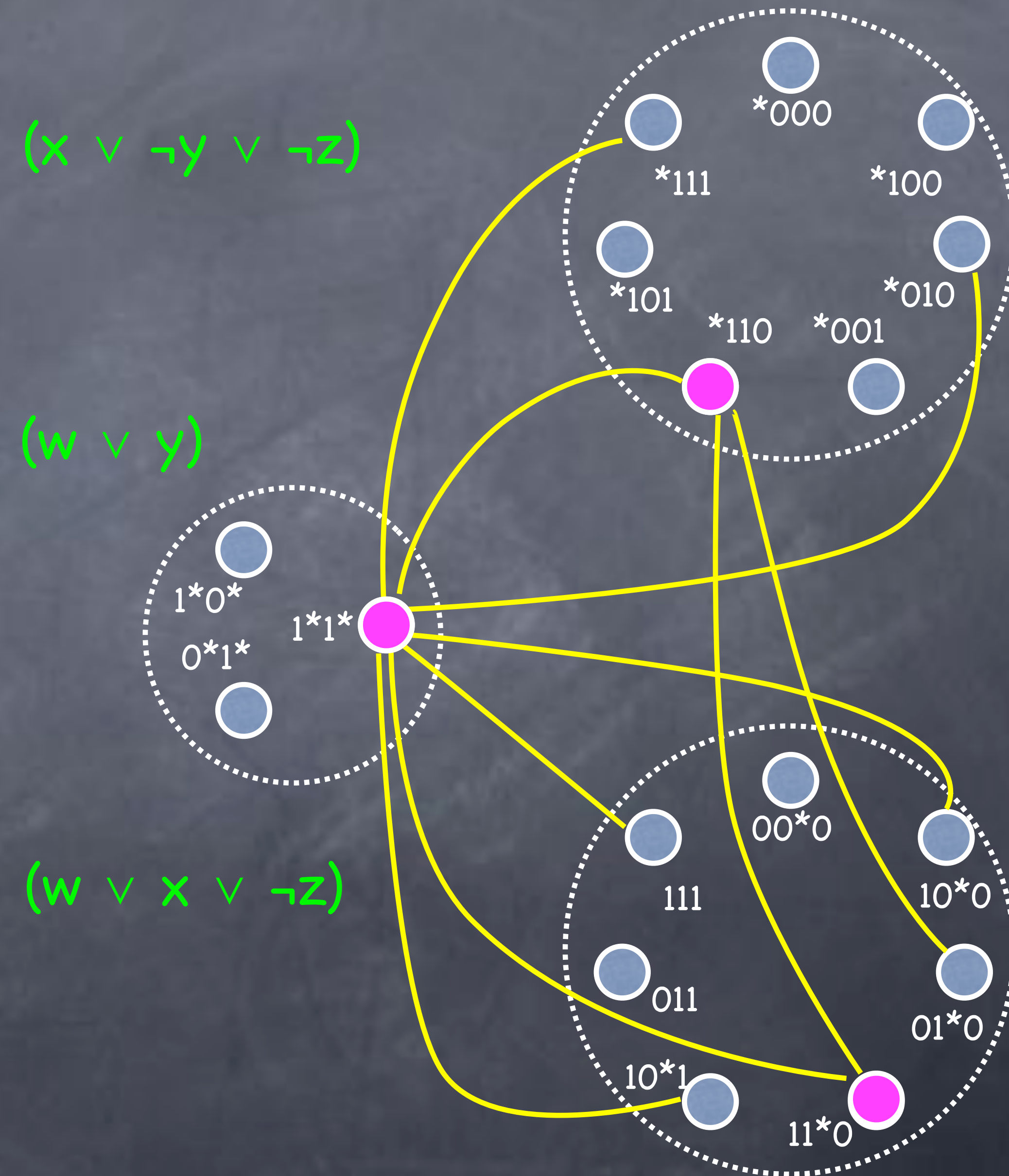

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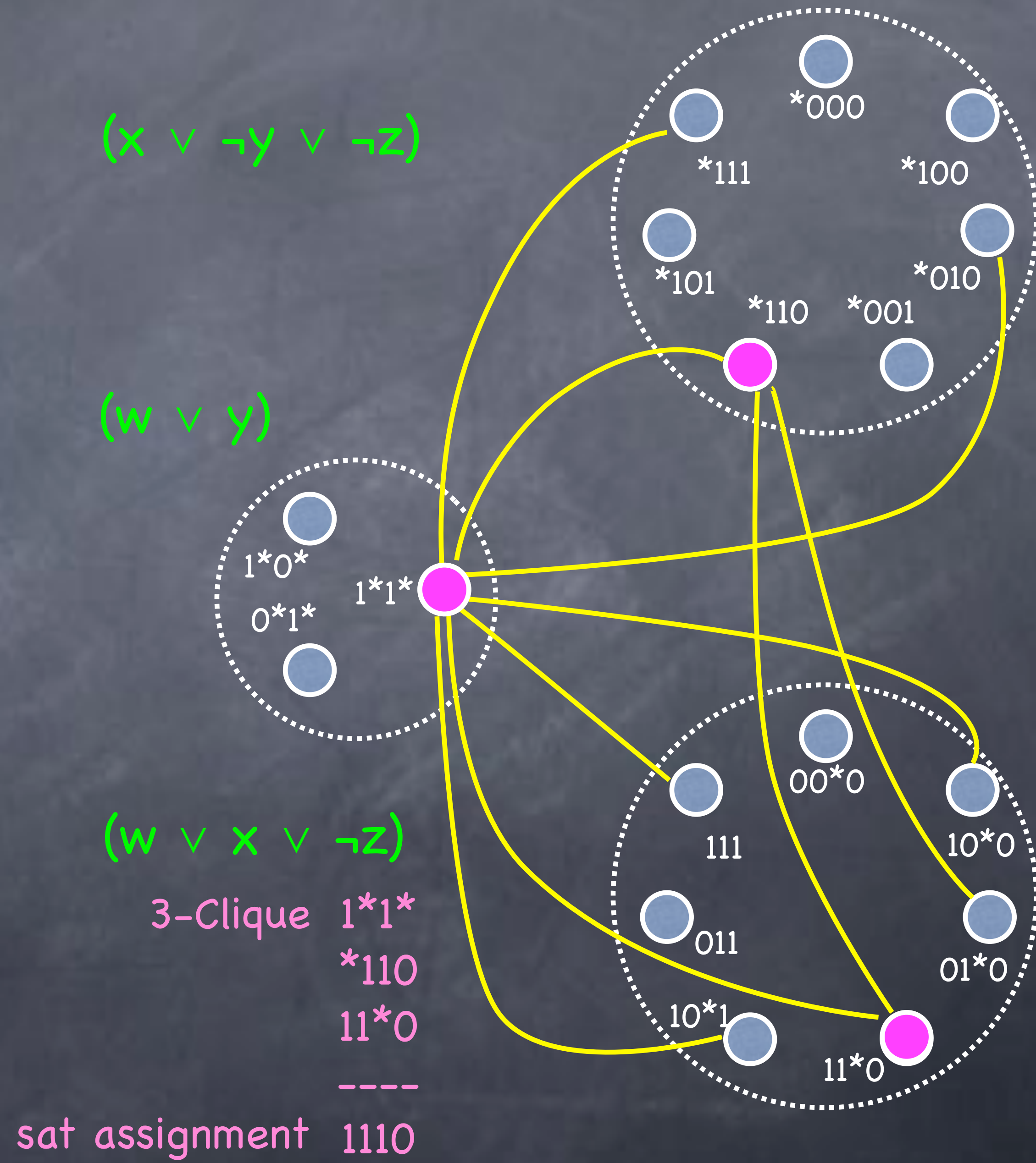

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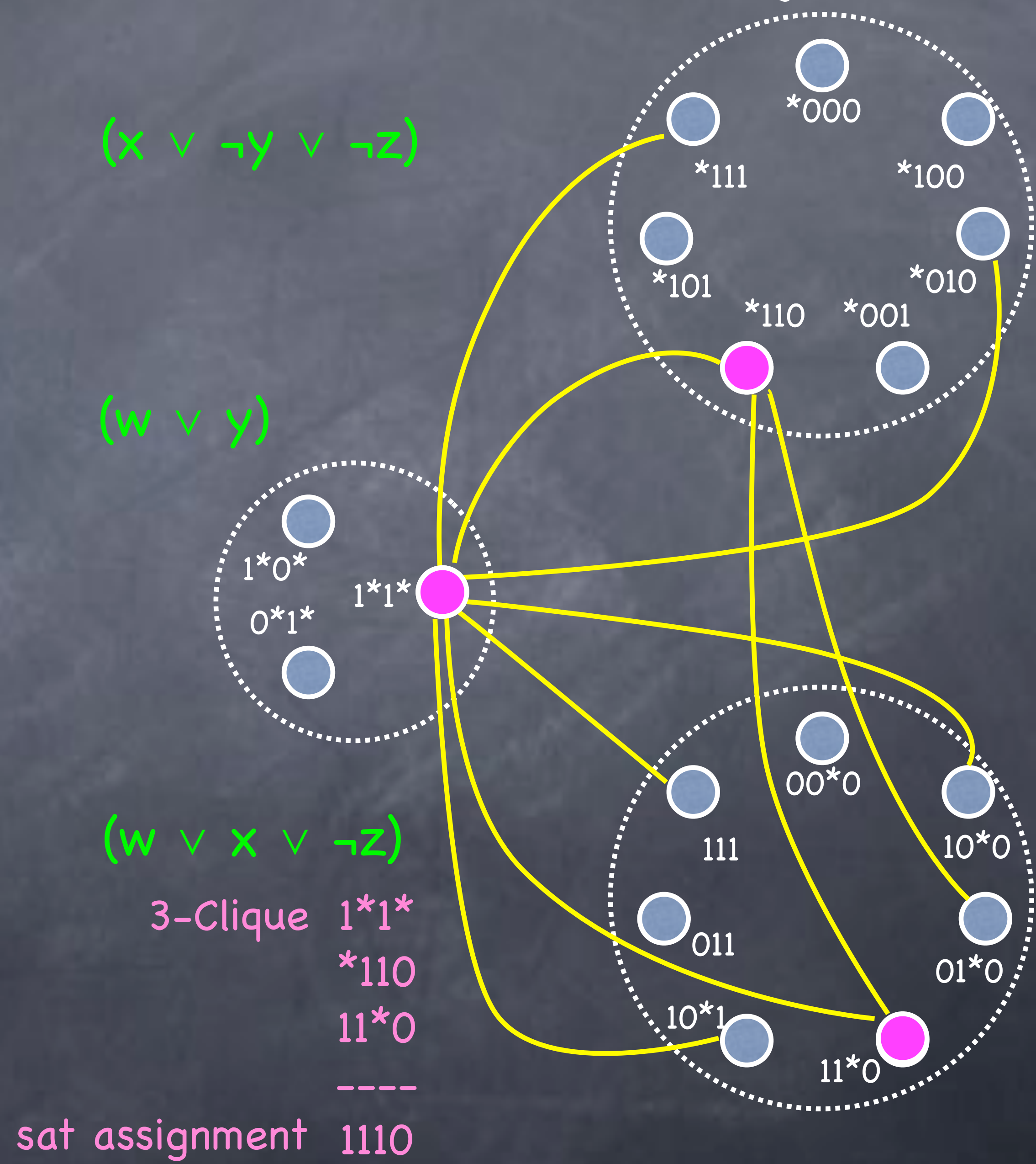

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- Need to check that the encoded vector is the evaluation of the clauses on an assignment, and that encoding is valid

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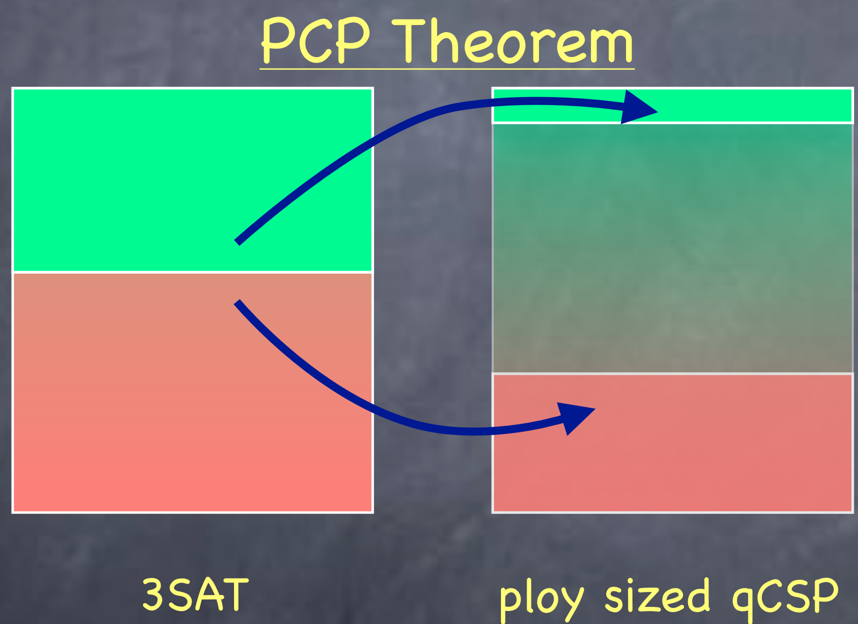

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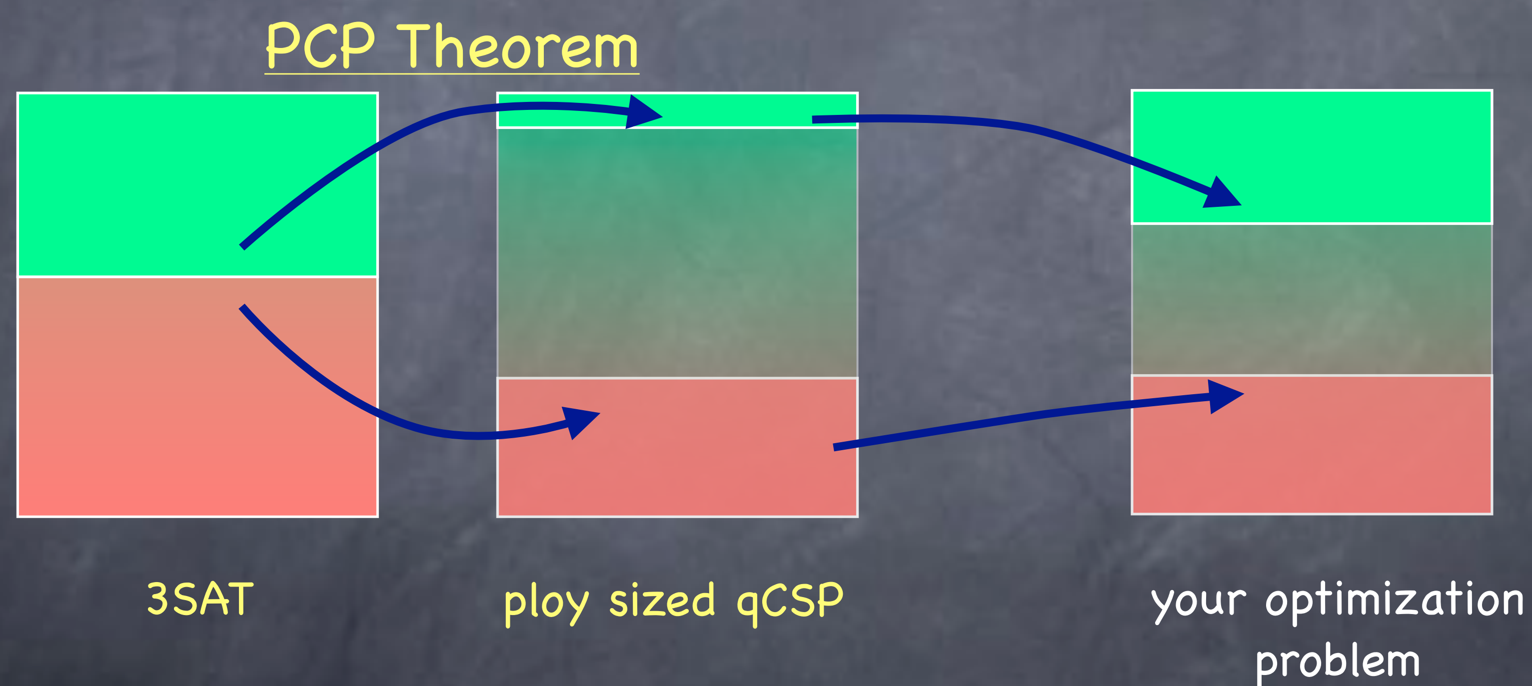

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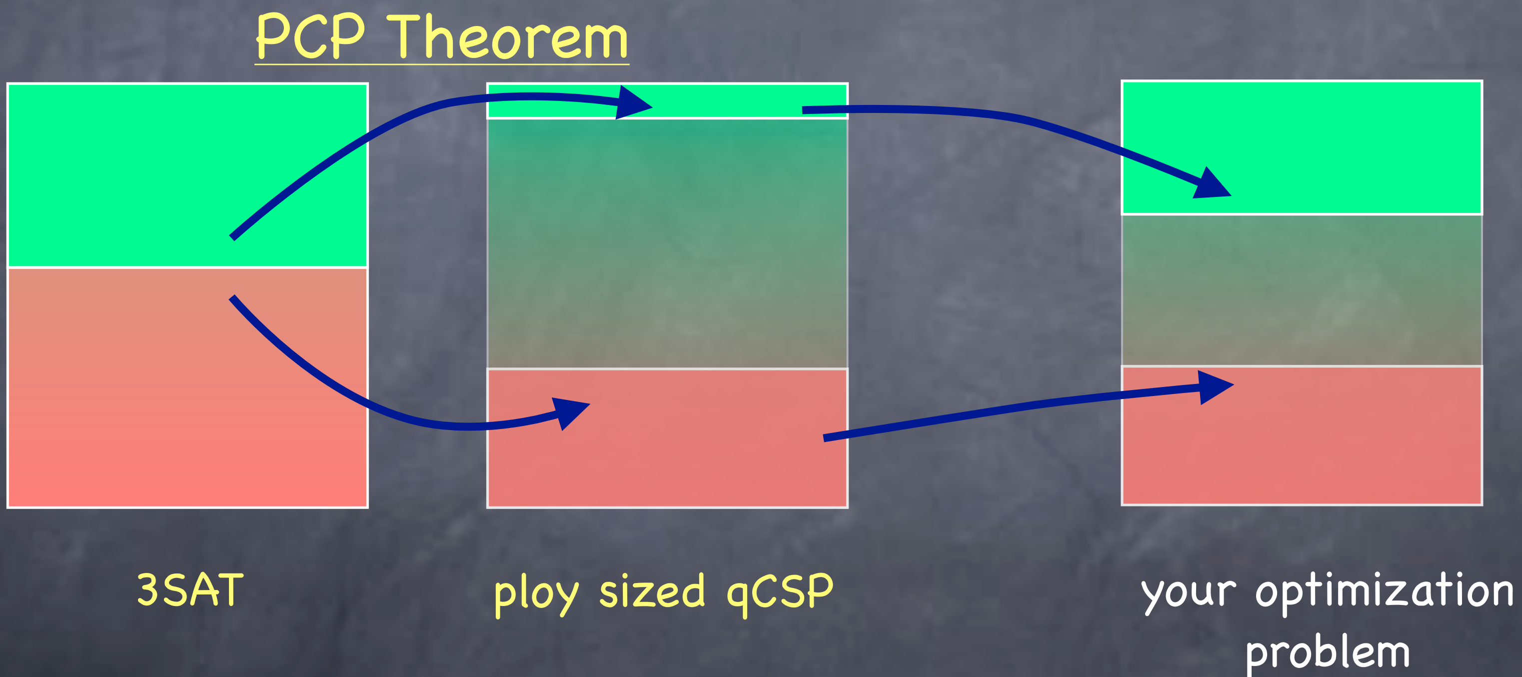

- Variants of these reductions to get different hardness results for different approximations