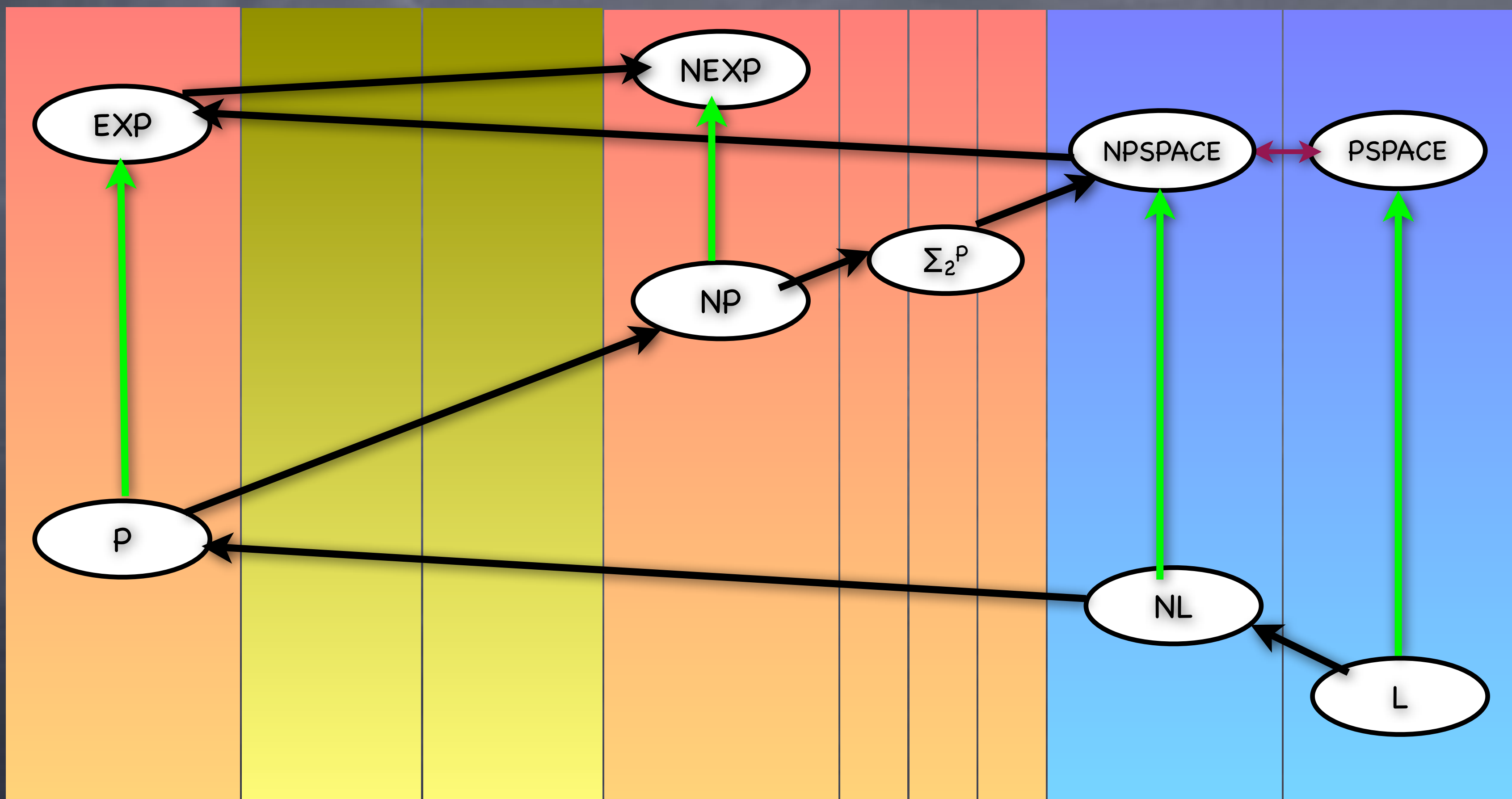
Probabilistic Computation

Lecture 14

BPP, ZPP

Zoo



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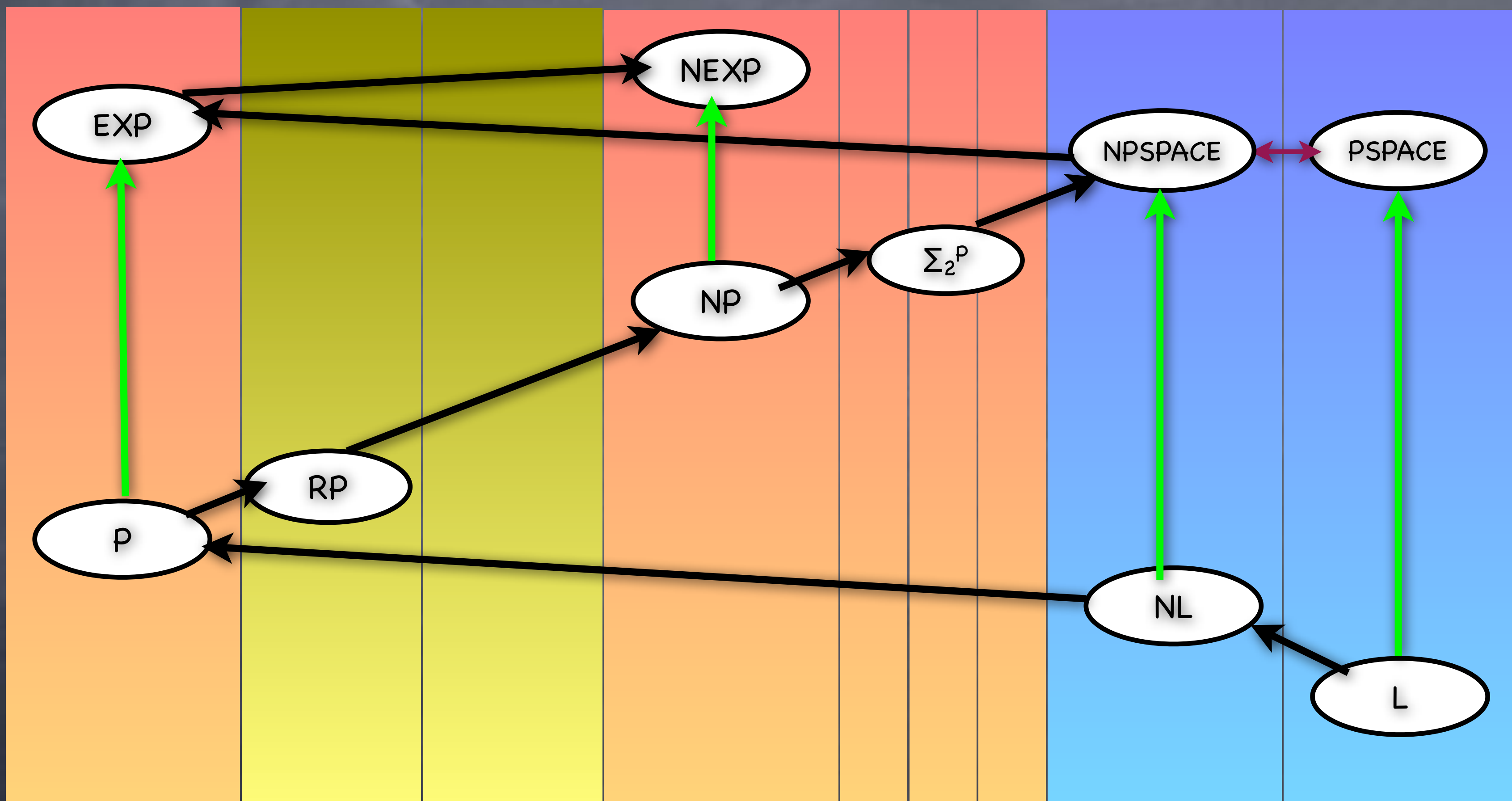

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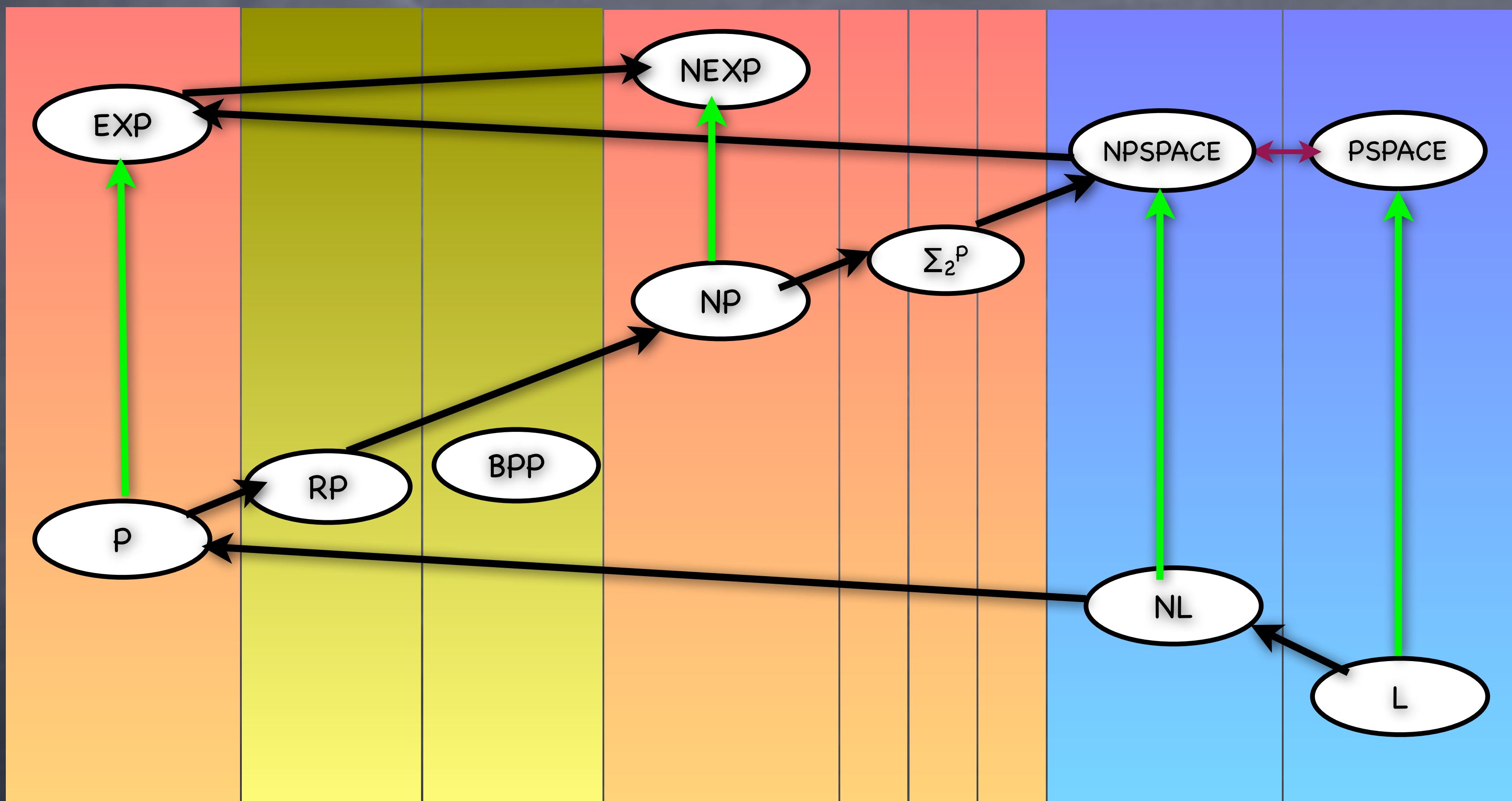

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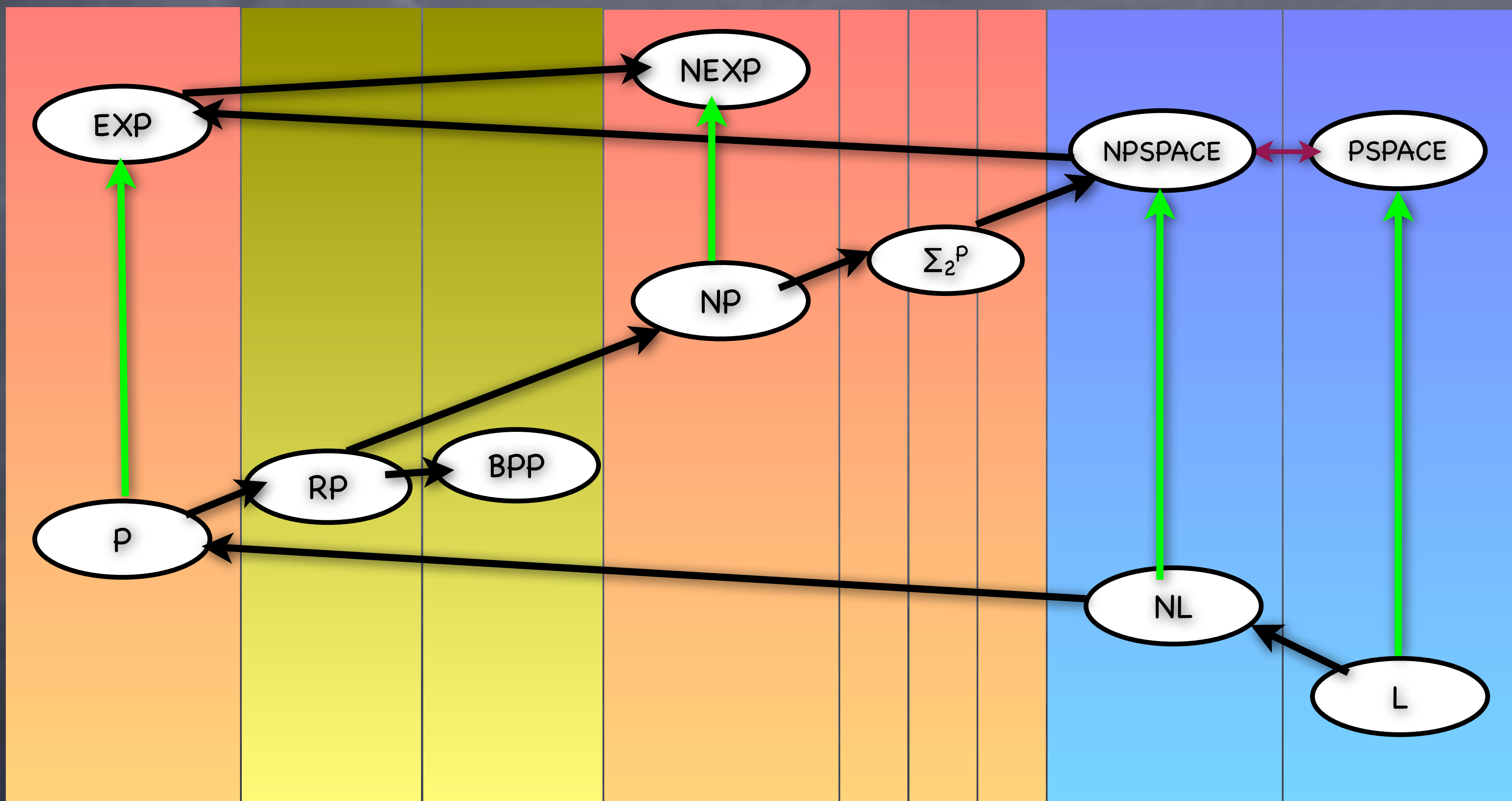

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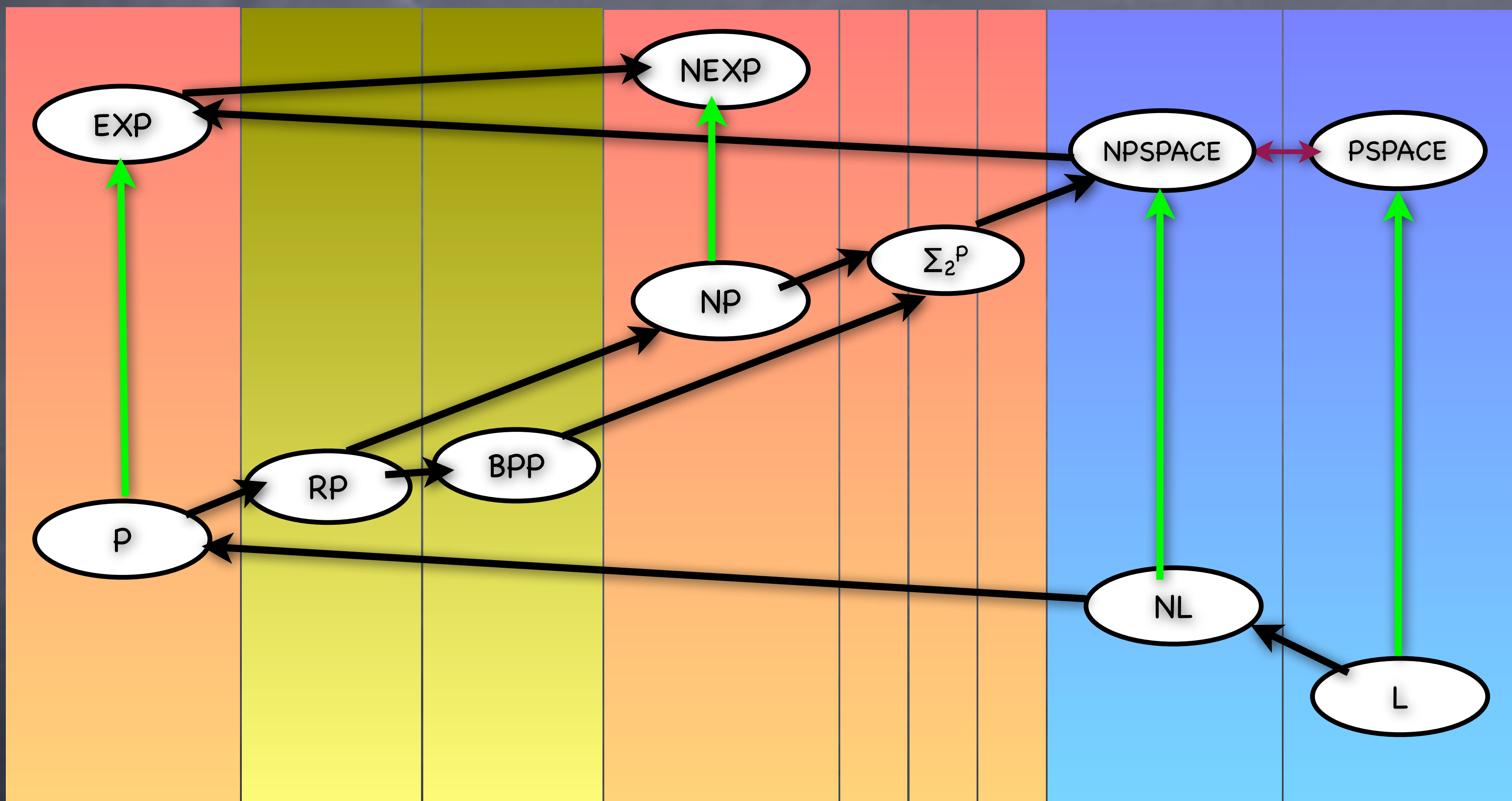

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    - Just run M(x) for t steps and accept if it accepts?
      - If $(M,x,1^t)$ in L, we will indeed accept with prob. > 2/3
      - But M may not have a bounded gap. Then, if $(M,x,1^t)$ not in L, we may accept with prob. very close to 2/3.

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  - After t steps: $p^{(t)} = M^t p$

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  - Accept if $(M^t\, p^{start})_{accept} > 2/3$ where p^{start} is the probability distribution with all the weight on the start configuration

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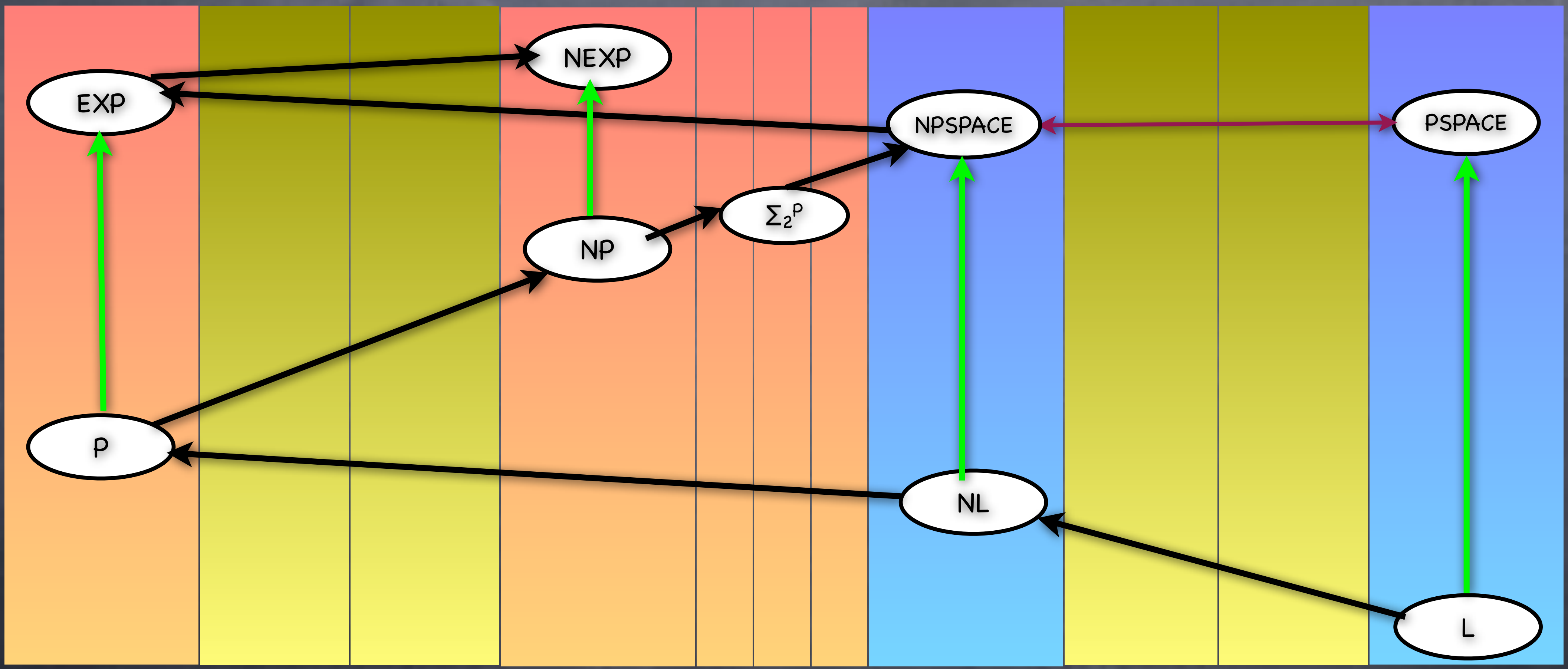

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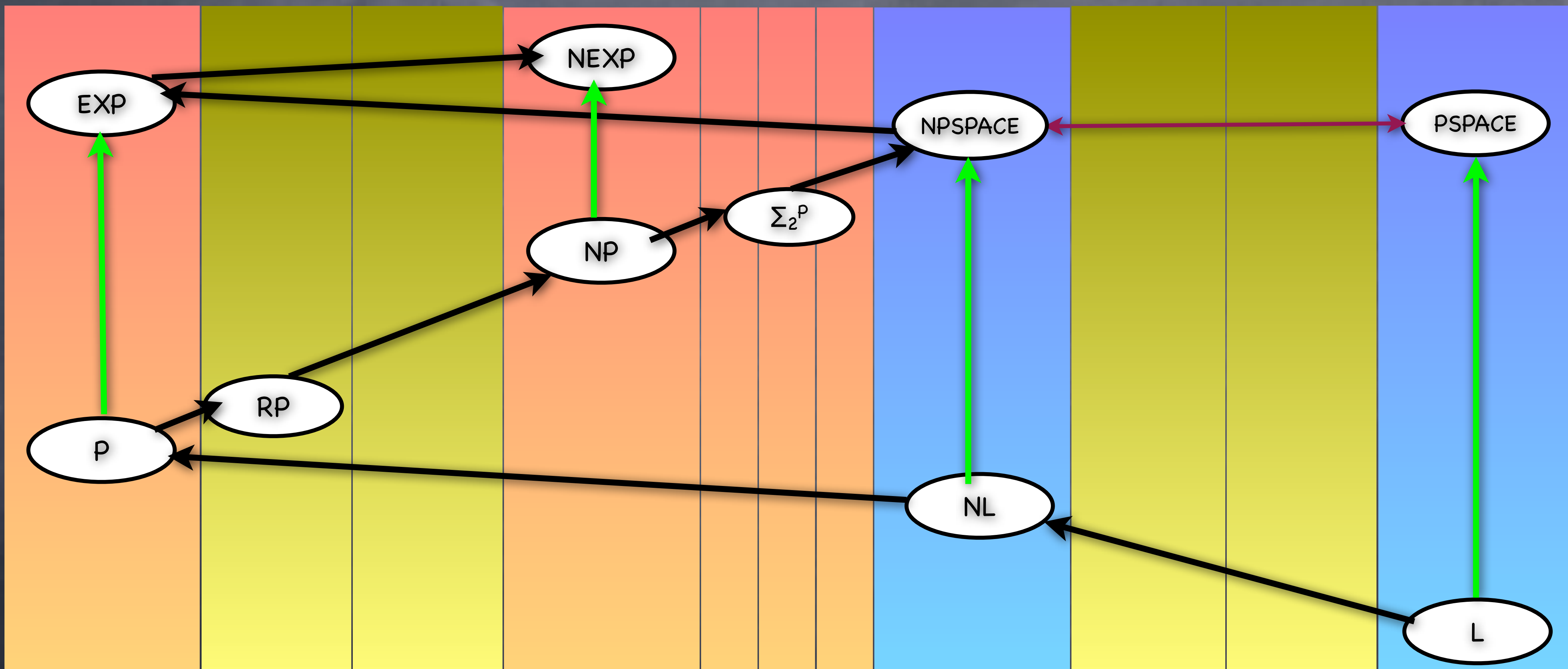

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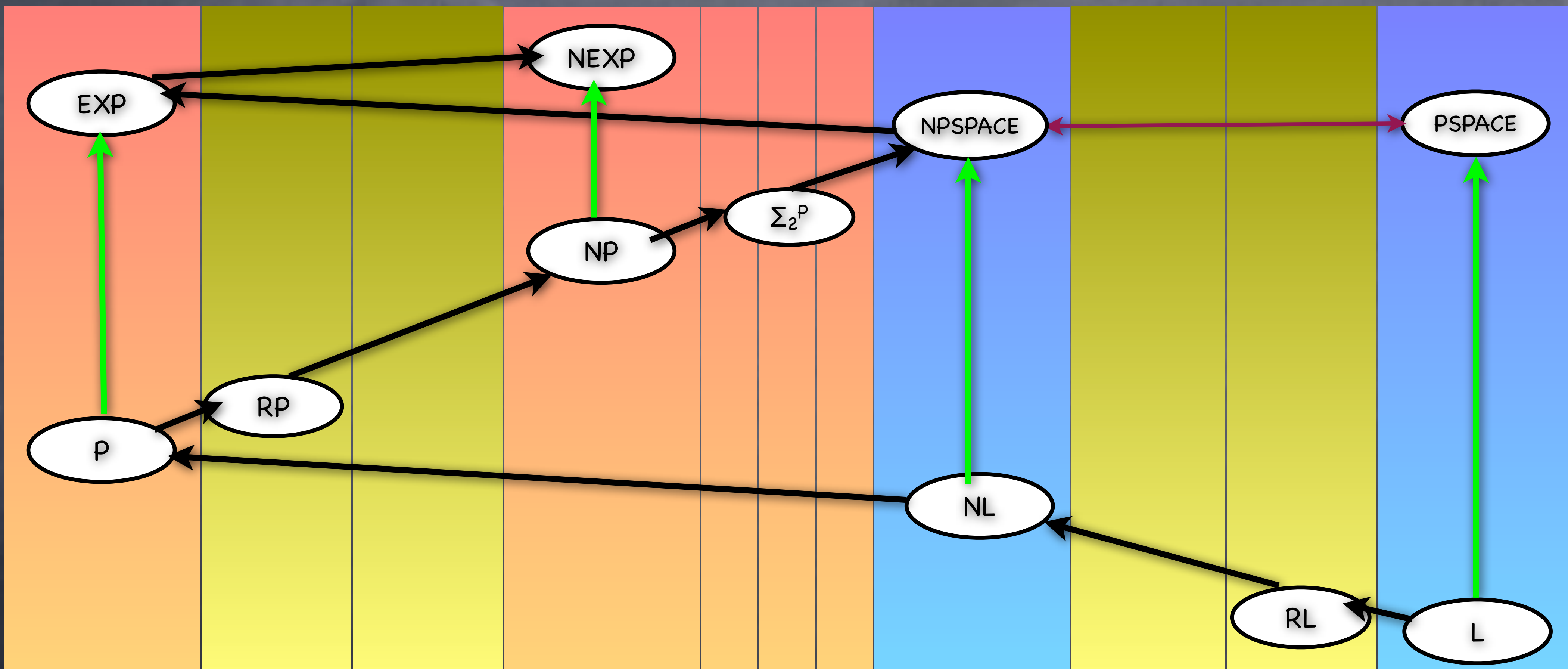

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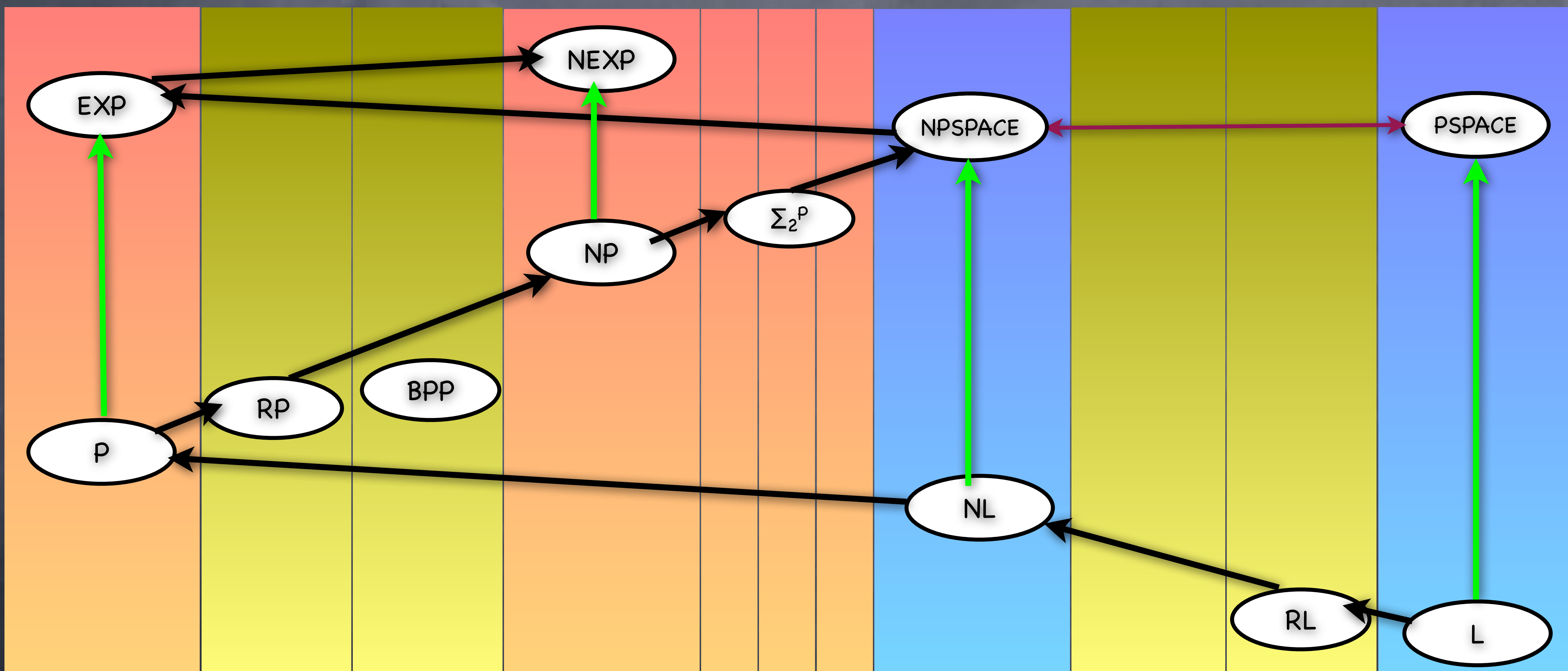

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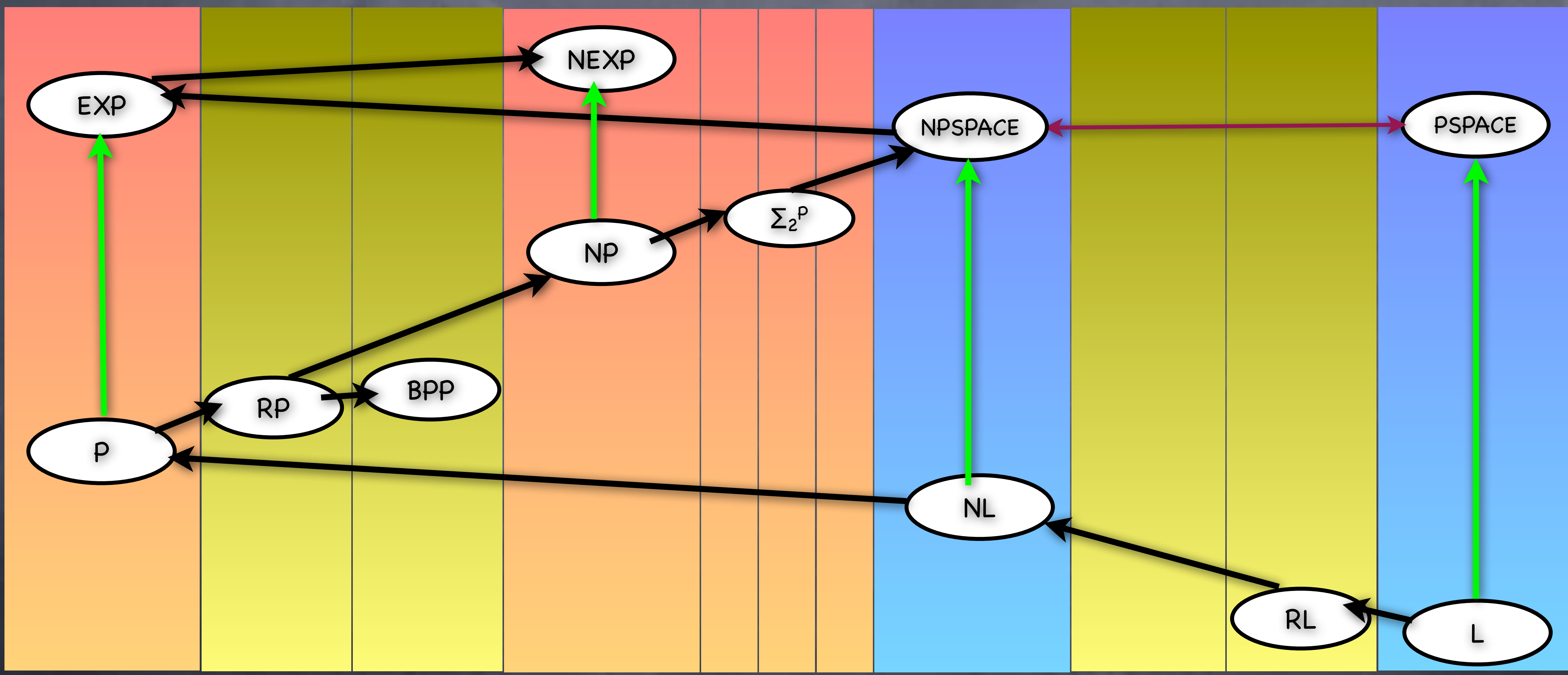

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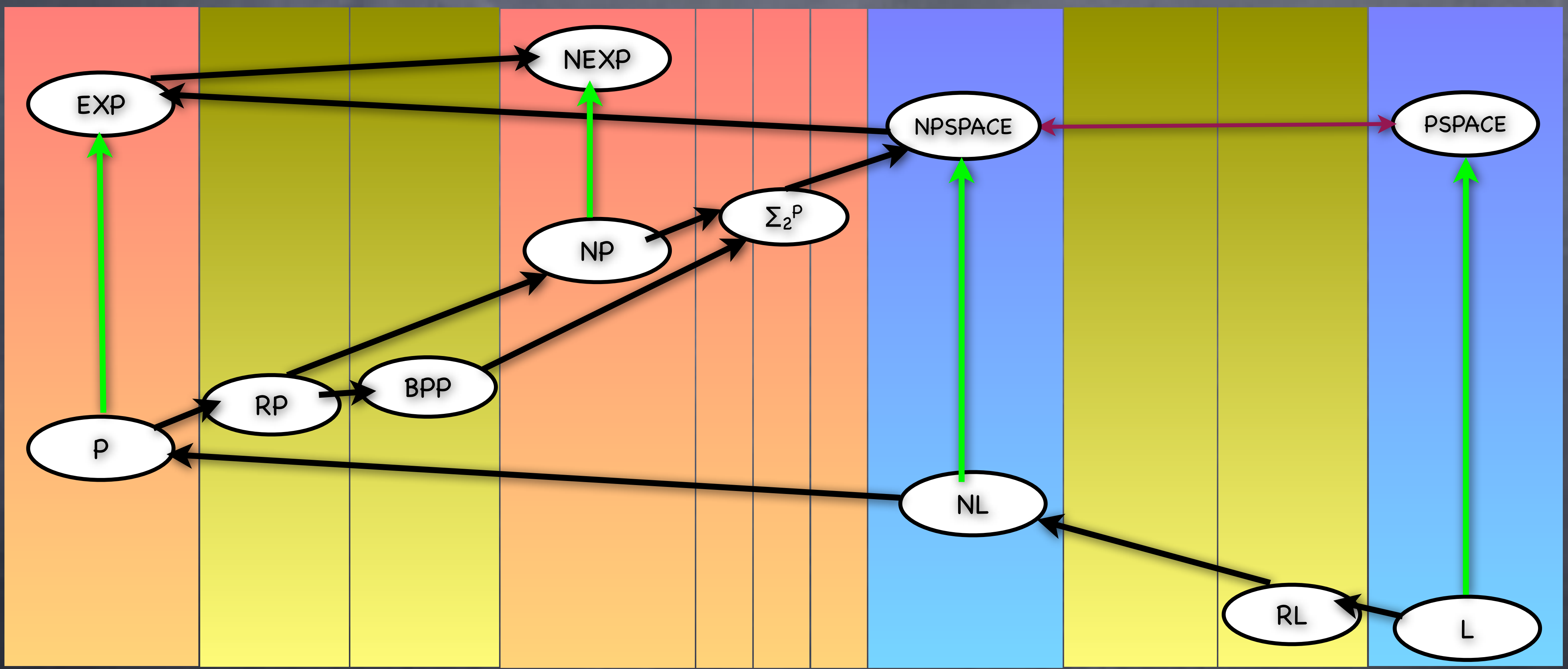

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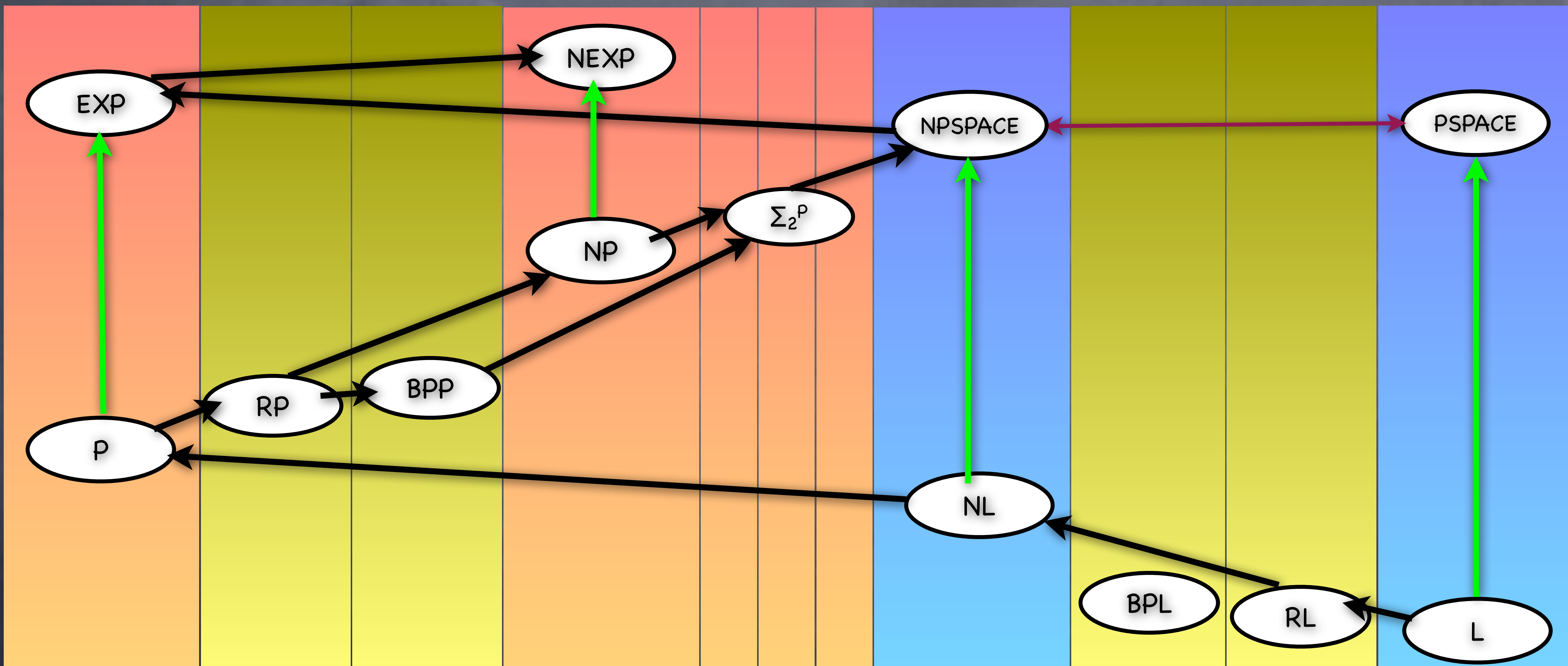

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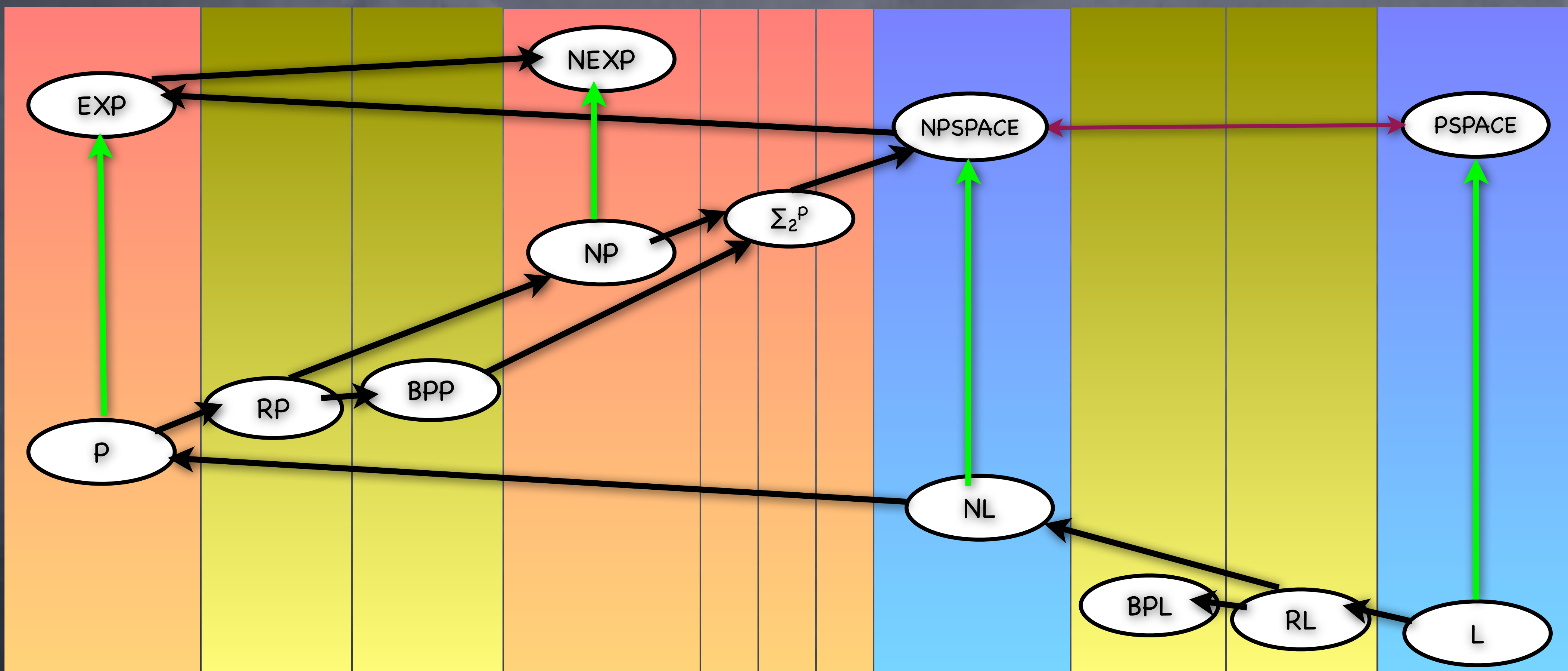

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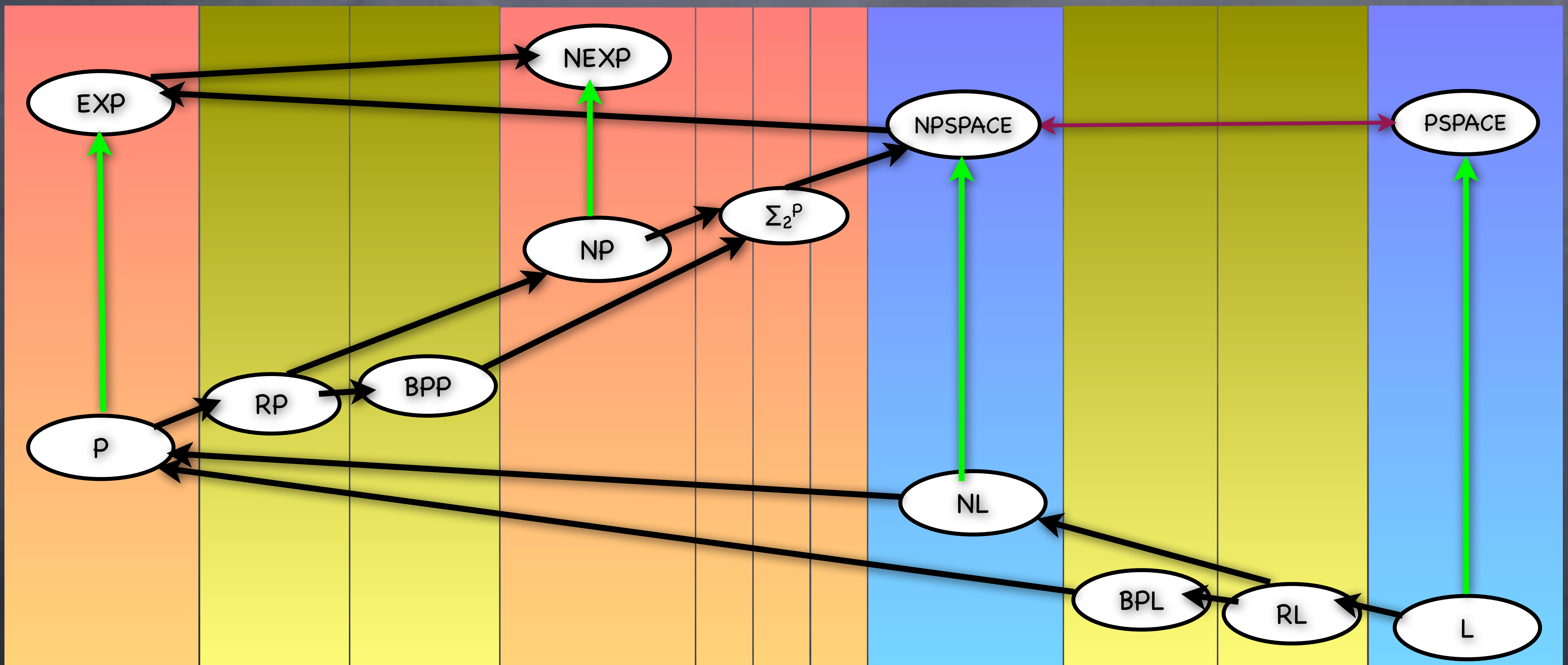

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  - $T(n) \leq cn + 3cn$ as required

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- Pr[error] at most 1/a if truncated after a times expected running time

$RP \cap co\text{-}RP \subseteq ZPP$

RP $\cap$ co-RP $\subseteq$ ZPP

- If $L \in$ RP $\cap$ co-RP, then a ZPP algorithm for L:
  - Run both RP and coRP algorithms
  - If former says yes or latter says no, output that answer
  - Else, i.e., if former says no and latter yes, repeat
    - Expected number of repeats = O(1)

Today

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- Zoo
  - BPL $\subseteq$ P
- Expected running time
- Zero-Error probabilistic computation
- ZPP = RP $\cap$ co-RP