Non-Uniform Computation & Circuits

Lecture 10

Wherein every language can be decided

Non-Uniform Computation

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- Non-uniform: A different "program" for each input size
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  - Not entirely realistic if the program family is uncomputable or very complex to compute

Non-uniform advice

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  - But $\{A_n\}$ can be uncomputable (even if just one bit long)
    - e.g. advice to decide undecidable unary languages

P/poly and P/log

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- P/log = $\cup_{c,k>0}$ DTIME(kn^c)/k log n

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  - e.g. unary undecidable languages
  - So P/log cannot be contained in any of the uniform complexity classes
- P/log contains P
  - Does P/log or P/poly contain NP?

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Recall

Search using Decision

- Suppose given "oracles" for deciding all NP languages, can we easily find certificates?
  - Yes! So, if decision easy (decision-oracles realizable), then search is easy too!
- Say, given x, need to find w s.t. $(x,w) \in L'$ (if such w exists)
  - consider L_1 in NP: $(x,y) \in L_1$ iff $\exists z$ s.t. $(x,yz) \in L'$. (i.e., can y be a prefix of a certificate for x).
  - Query L_1-oracle with (x,0) and (x,1). If $\exists w$, one of the two must be positive: say $(x,0) \in L_1$; then first bit of w be 0.
  - For next bit query L_1-oracle with (x,00) and (x,01)

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Use L_2 so that (x,z,pad) in L_2 iff (x,z) in L_1. Can query L_2 with same size instances

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- If no advice worked (one of them was correct), then input not in language

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    - $L = \{x | \exists z\ \forall w_1\ F(x, w_1, M(x,w_1; z)) \}$

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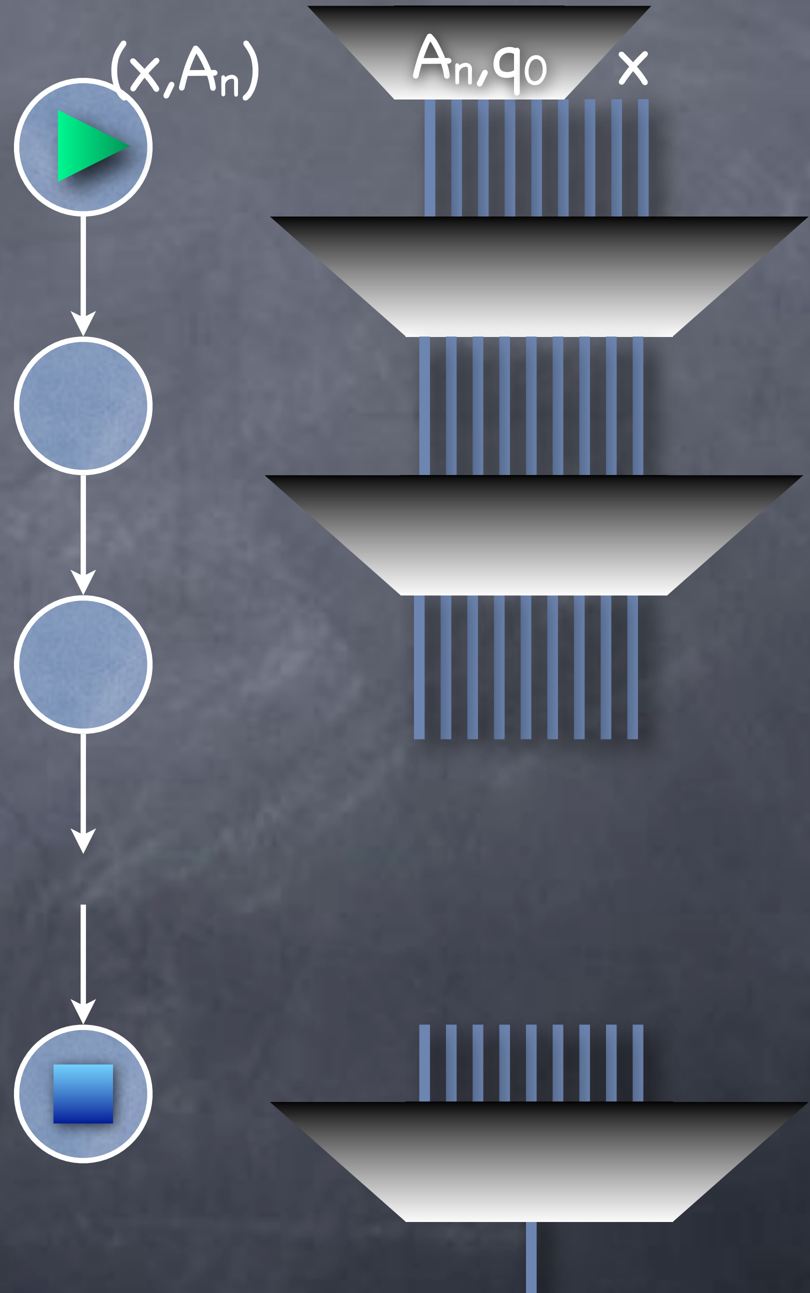

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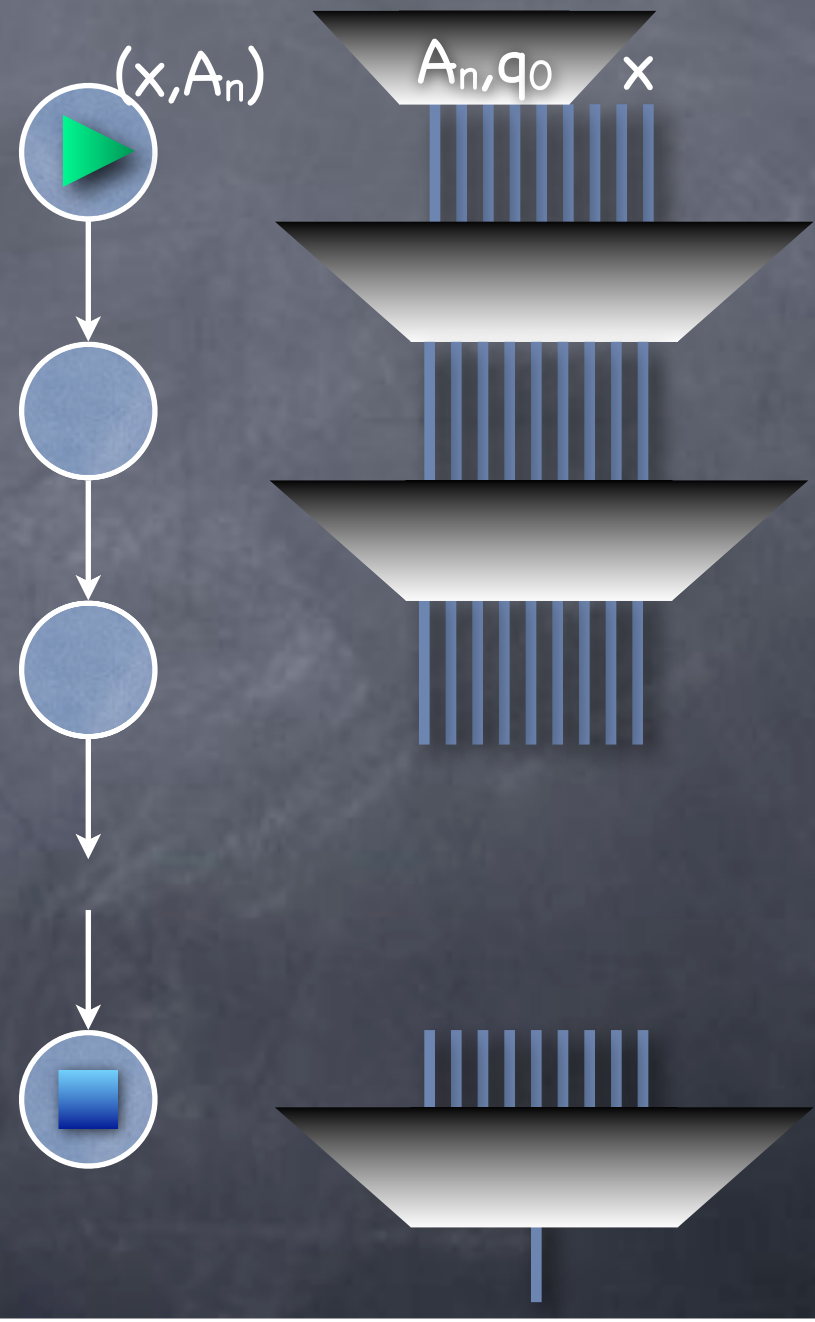

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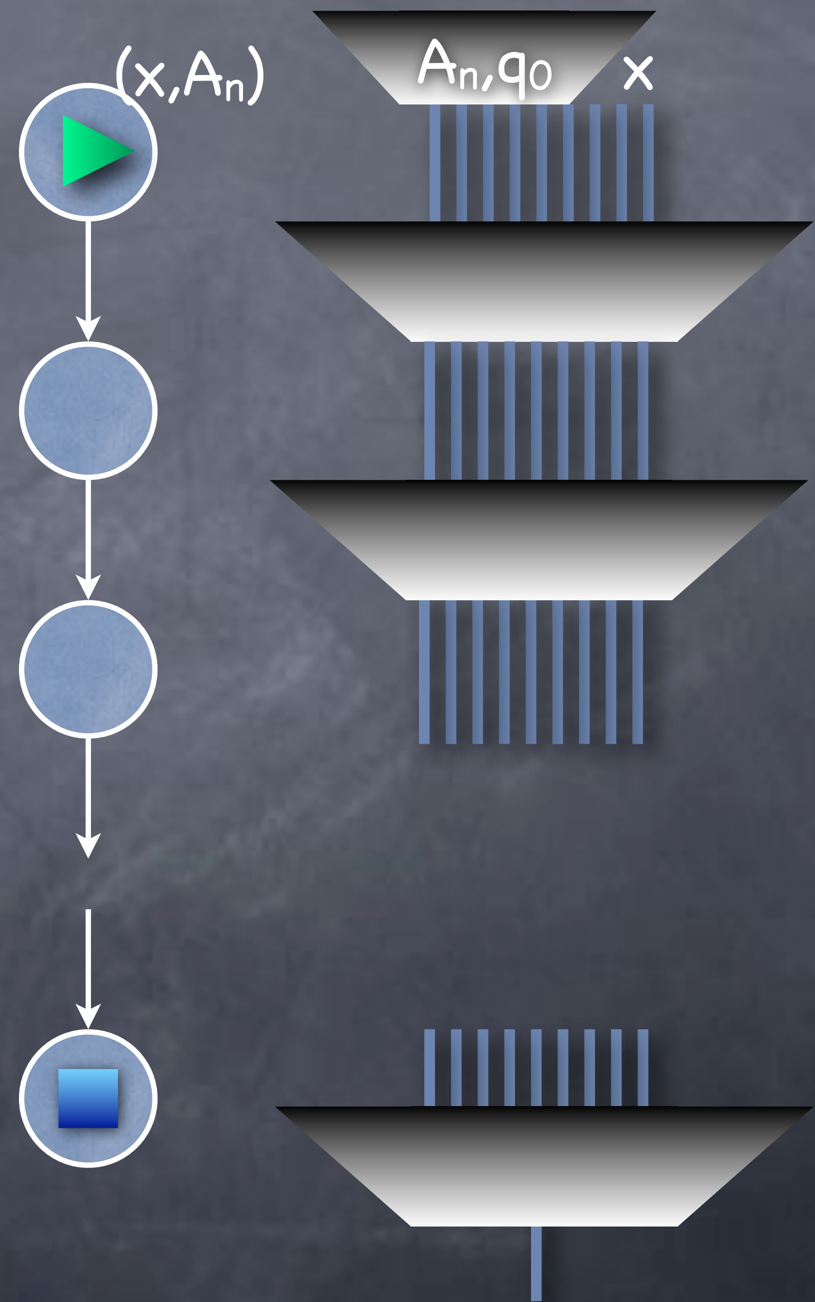

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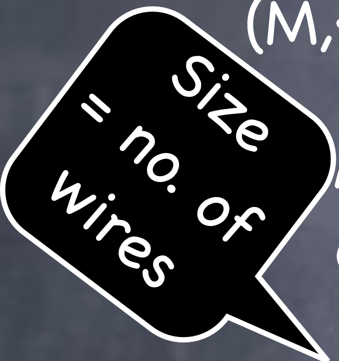

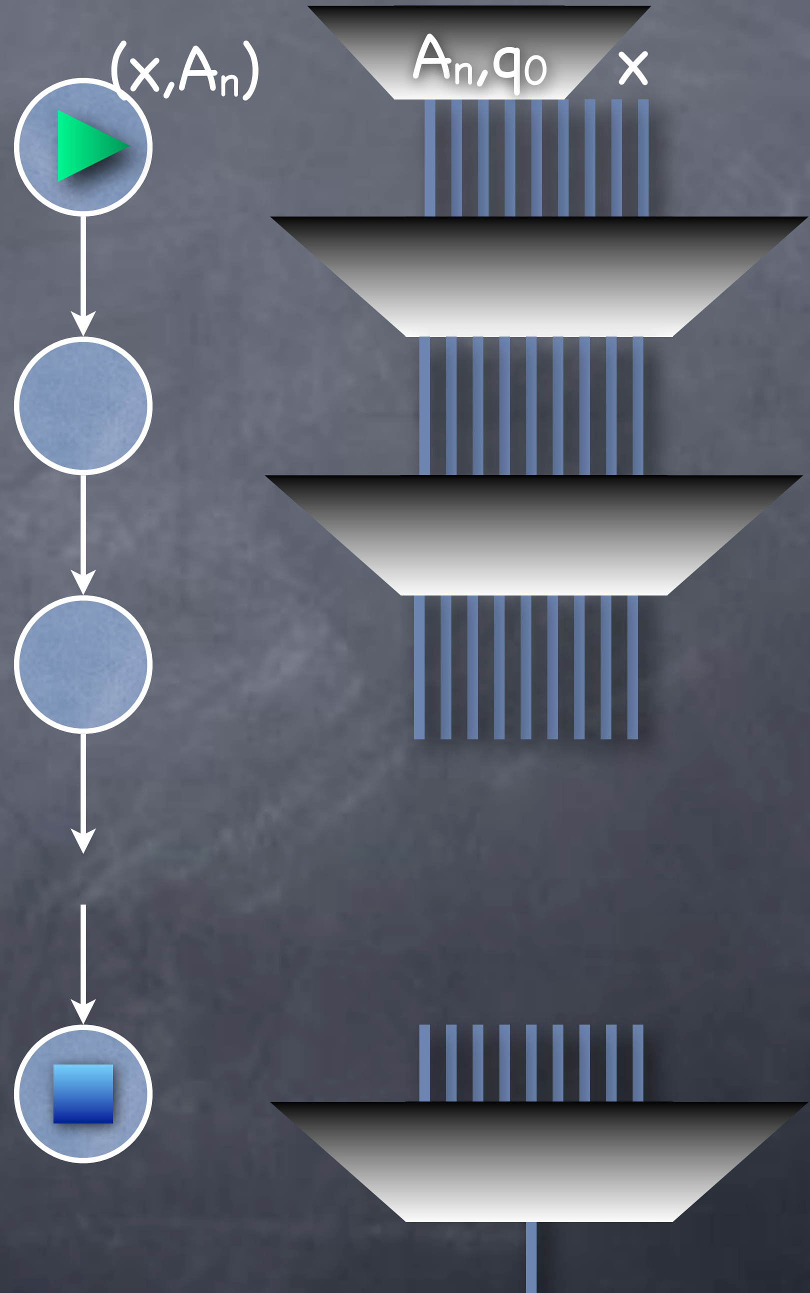

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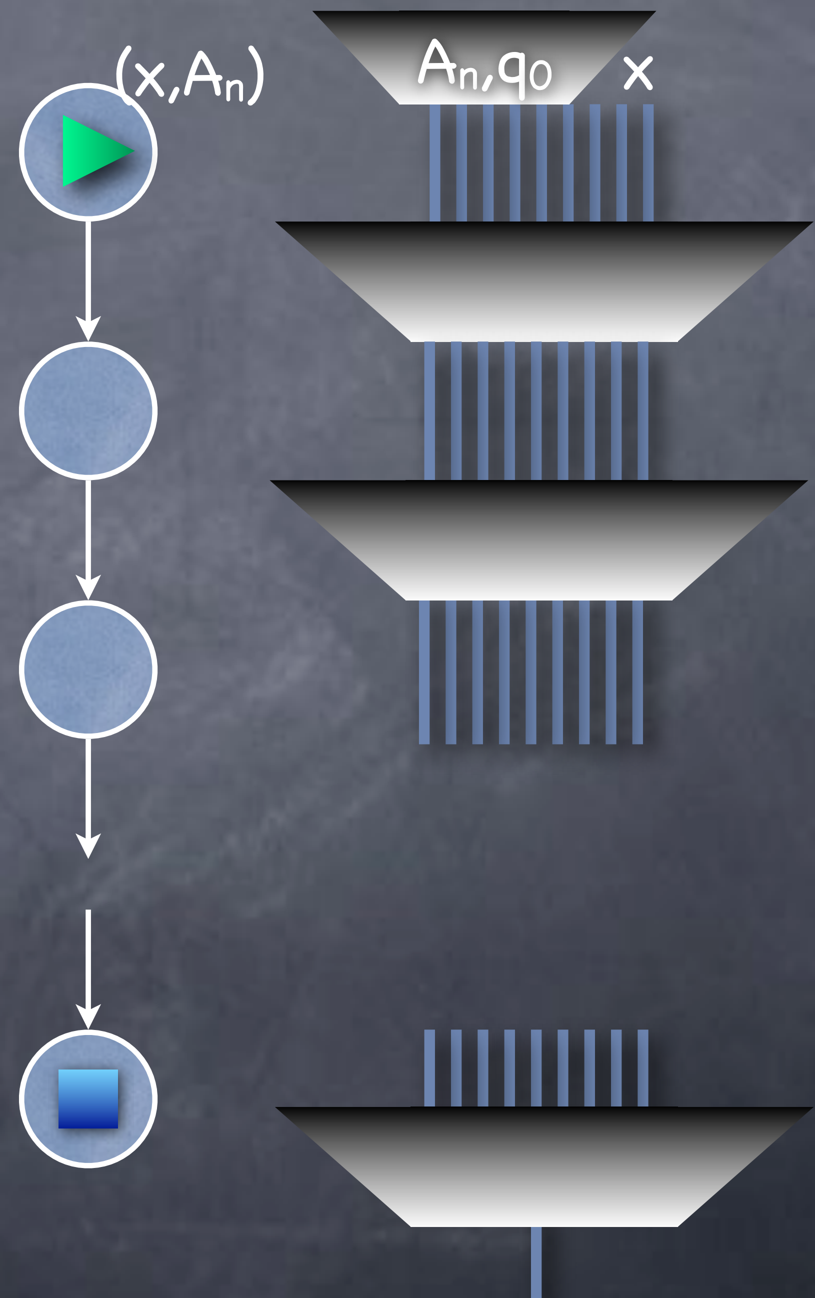

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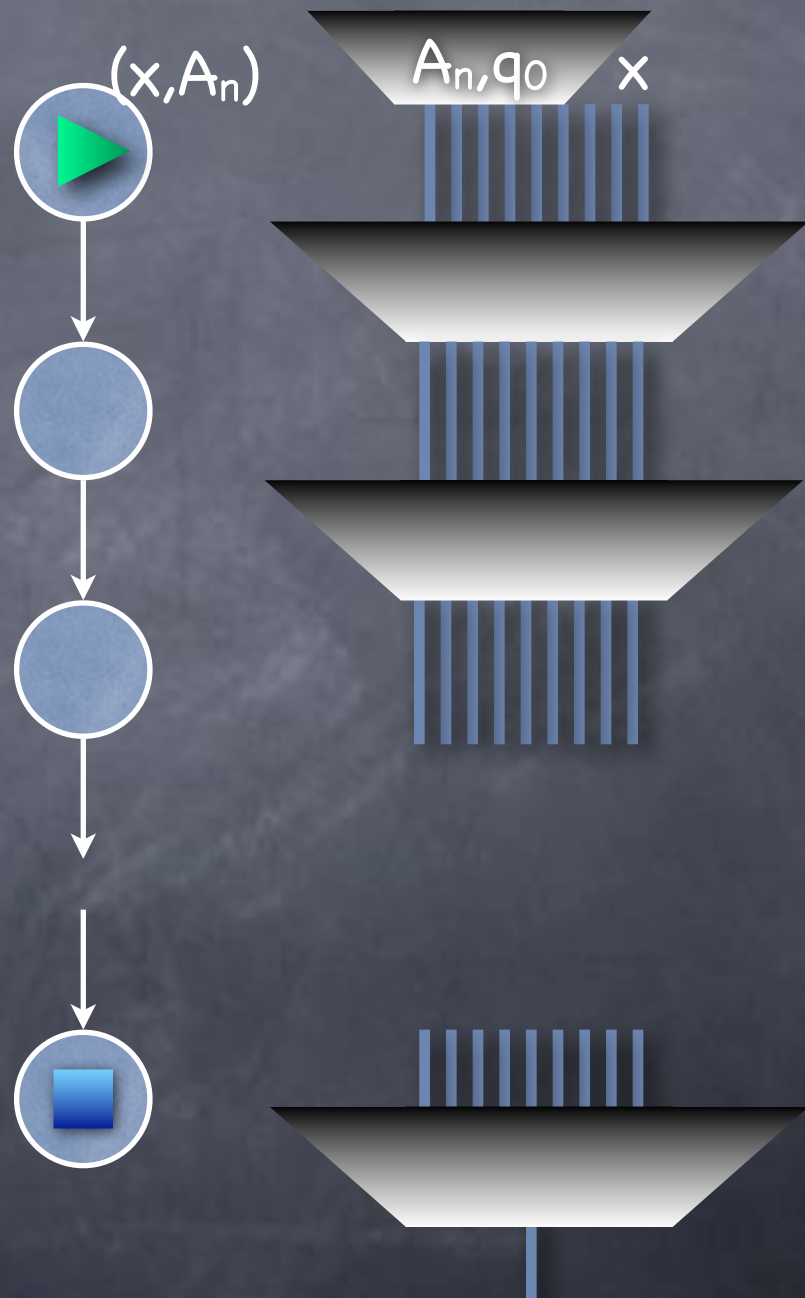
  - Conversely, given $\{C_n\}$, can use description of C_n as advice A_n for a "universal" TM
    - $|A_n|$ comparable to size of circuit C_n



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  - P/poly $\subseteq$ SIZE(poly): Transformation from Cook's theorem, with advice string hardwired into circuit

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    - All of them in SIZE(T), most not in SIZE(T′)

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  - $NC^0 \subsetneq AC^0$: Consider L = {1,11,111,...}

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- So NC = AC

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- Open problem: Is NC = P?

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  - Total "work" is size of the circuit

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