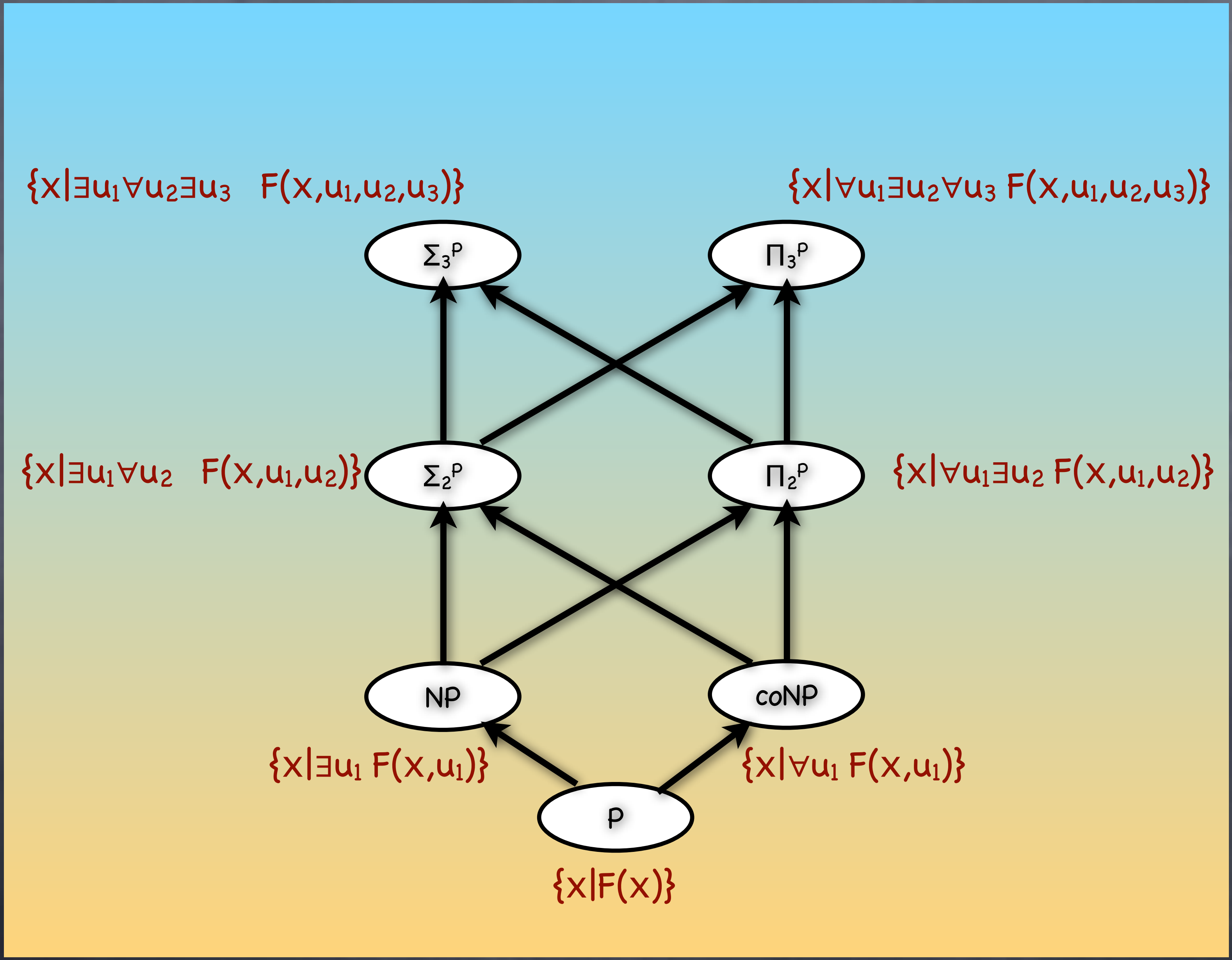
Computational Complexity

Lecture 8

More of the Polynomial Hierarchy

Oracle-based Definition

Recall PH



{x|∃u1∀u2∃u3 F(x,u1,u2,u3)}
{x|∀u1∃u2∀u3 F(x,u1,u2,u3)}
Σ3P
Π3P
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Σ2P
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{x|∀u1∃u2 F(x,u1,u2)}
NP
coNP
{x|∃u1 F(x,u1)}
{x|∀u1 F(x,u1)}
P
{x|F(x)}

Oracle Machines

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  - Can run an oracle machine with any oracle
  - Oracle fully specified by the input-output behavior
  - Language oracle: answer is a single bit
    - This is what we consider

Oracle Machines (ctd.)

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  - languages of the form $\{x \mid \forall w\ (x,w) \in L\}$, where L in P^A

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- If A in NP?
  - Oracle for A is an oracle for A^c too! $NP^A = NP^{A^c}$
  - $NP \cup co\text{-}NP \subseteq NP^{SAT}$
    - Can we better characterize NP^{SAT}?

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- $NP^{\Sigma_k} = \bigcup_{A \in \Sigma_k} NP^A = NP^{\Sigma_k SAT}$
- Will show $NP^{\Sigma_k} = \Sigma_{k+1}^P$ (alt. definition for Σ_{k+1}^P)

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- $NP^{\Sigma_k} = \cup_{A \in \Sigma_k} NP^A = NP^{\Sigma_k SAT}$
- Will show $NP^{\Sigma_k} = \Sigma_{k+1}^P$ (alt. definition for Σ_{k+1}^P)
  - In particular, $NP^{NP} = \Sigma_2^P$

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- Consider $L \in \Sigma_{k+1}^P$

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  - $L = \{ x | \exists w \ (x,w) \in L' \}$, where L' in Π_k^P

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- Consider $L \in \Sigma_{k+1}{}^P$
  - $L = \{ x | \exists w\ (x,w) \in L' \}$, where L' in $\Pi_k{}^P$
  - So L in $NP^{L'}$ where L' in $\Pi_k{}^P$

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    - So $NP^{L'} \subseteq NP^{\Pi_k} = NP^{\Sigma_k}$
- So $\Sigma_{k+1}{}^P \subseteq NP^{\Sigma_k}$

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  - $L = \{ x | \exists w\ (x,w) \in L' \}$, where L' in Π_k^P
  - So L in $NP^{L'}$ where L' in Π_k^P
    - So $NP^{L'} \subseteq NP^{\Pi_k} = NP^{\Sigma_k}$
- So $\Sigma_{k+1}^P \subseteq NP^{\Sigma_k}$
- Now to show $NP^{\Sigma_k} \subseteq \Sigma_{k+1}^P$

$$NP^{\Sigma_k} \subseteq \Sigma_{k+1}$$

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- To show $NP^A \subseteq \Sigma_{k+1}^P$ if A in Σ_k^P

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  - For B $\in NP^A$ poly-time TM M s.t. B = $\{ x | \exists w\ M^A(x,w)=1\}$

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- To show $NP^A \subseteq \Sigma_{k+1}{}^P$ if A in $\Sigma_k{}^P$
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  - i.e., $B = \{ x | \exists w\ \exists ans\ M^{\langle ans \rangle}(x,w)=1$ and "ans correct"$\}$

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  - To show $C = \{(x,w,ans) \mid M^{<ans>}(x,w)=1$ and "ans correct"$\}$ in Σ_{k+1}^P

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  - i.e., B = { x| $\exists$w $\exists$ans $M^{\text{<ans>}}(x,w)=1$ and "ans correct"}
  - To show C = {(x,w,ans) | $M^{\text{<ans>}}(x,w)=1$ and "ans correct"} in Σ_{k+1}^P
    - Then B also in Σ_{k+1}^P

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  - C={(x,w,ans) | $M^{<ans>}(x,w)=1$ $\wedge$ [(ans=1 $\wedge$ $\exists u_1 \forall u_2 \ldots Q_k u_k$ F(z,u_1,...)=1) or (ans=0 $\wedge$ $\forall v_1 \exists v_2 \ldots Q'_k v_k$ F(z,v_1,...)=0)] }

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    in Σ_{k+1}^P
  - Changes for t queries: $z=Z(x,w) \rightarrow (z^{(1)} \ldots z^{(t)}) = Z(x,w,ans)$, $u_i \rightarrow u_i^{(1)} \ldots u_i^{(t)}$, $v_i \rightarrow v_i^{(1)} \ldots v_i^{(t)}$, and use conjunction of t checks (for $j=1,\ldots,t$) of the form $[\ (ans^{(j)}=1 \wedge F(z^{(j)},u_1^{(j)},\ldots)=1)$ or $(ans^{(j)}=0 \wedge F(z^{(j)},v_1^{(j)},\ldots)=0)\ ]$

Oracle Version

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- **$\Sigma_{k+1}^P = NP^{\Sigma_k}$** (with $\Sigma_0^P = P$)

Oracle Version

- $\Sigma_{k+1}{}^{P}$ = NP^{Σ_k} (with $\Sigma_0{}^{P}$ = P)
- $\Pi_{k+1}{}^{P}$ = co-NP^{Π_k} (with $\Pi_0{}^{P}$ = P)

Oracle Version

- **Σ_{k+1}^P = NP^{Σ_k}** (with $\Sigma_0^P = P$)
- **Π_{k+1}^P = co-NP^{Π_k}** (with $\Pi_0^P = P$)
  - Π_{k+1}^P = co-(NP^{Σ_k}) = co-NP^{Σ_k} = co-NP^{Π_k}

Δ_k^P

Δ_k^P

- $\Delta_{k+1}^P = P^{\Sigma_k} = P^{\Pi_k}$

Δ_k^P

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$\Delta_k{}^P$

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- $\Delta_{k+1}^P \subseteq \Sigma_{k+1}^P \cap \Pi_{k+1}^P$ (why?)

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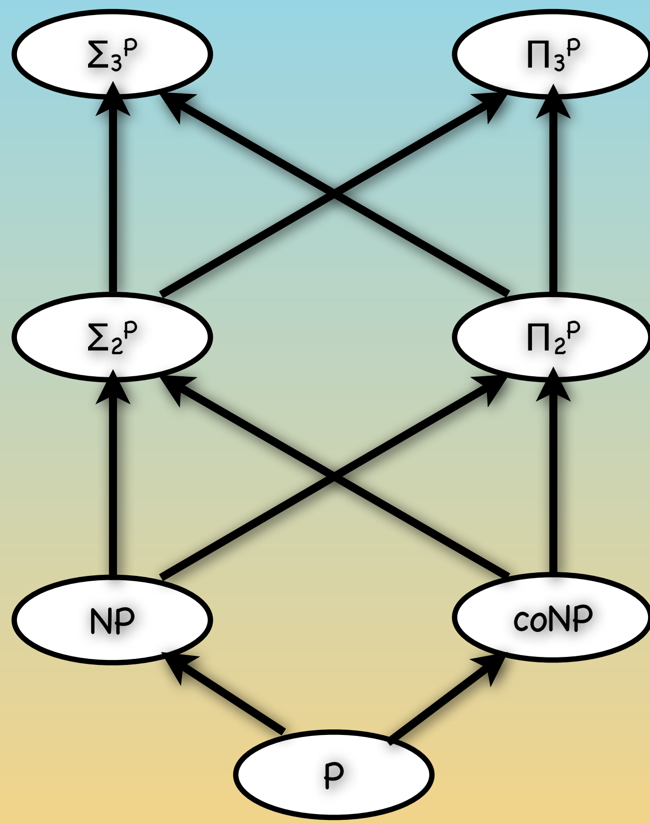
- $\Delta_{k+1}^P \subseteq \Sigma_{k+1}^P \cap \Pi_{k+1}^P$ (why?)
  - $P^{\Sigma_k} \subseteq NP^{\Sigma_k} \cap coNP^{\Sigma_k}$

PH



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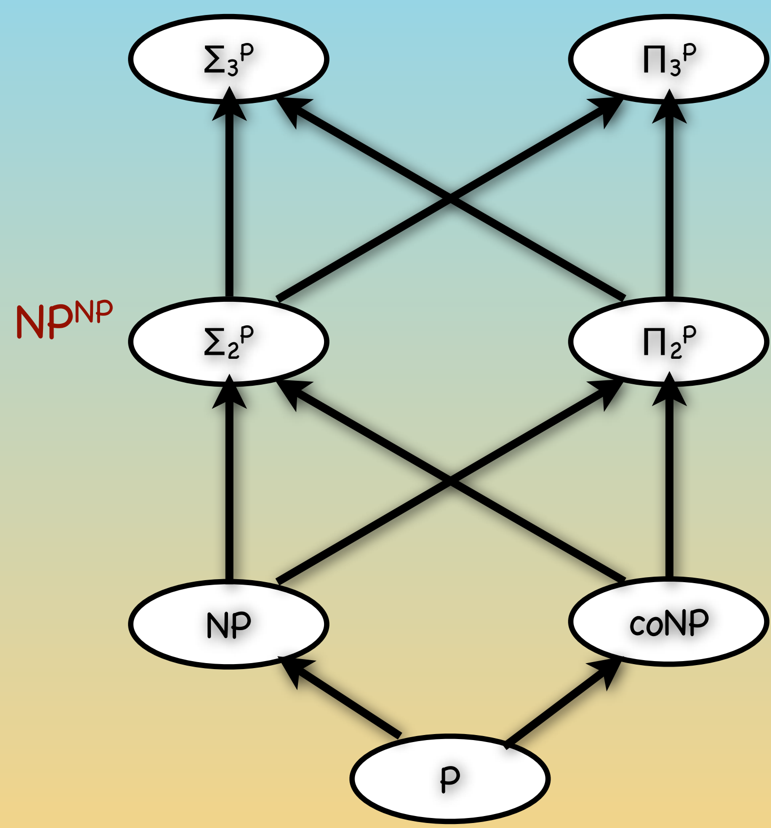

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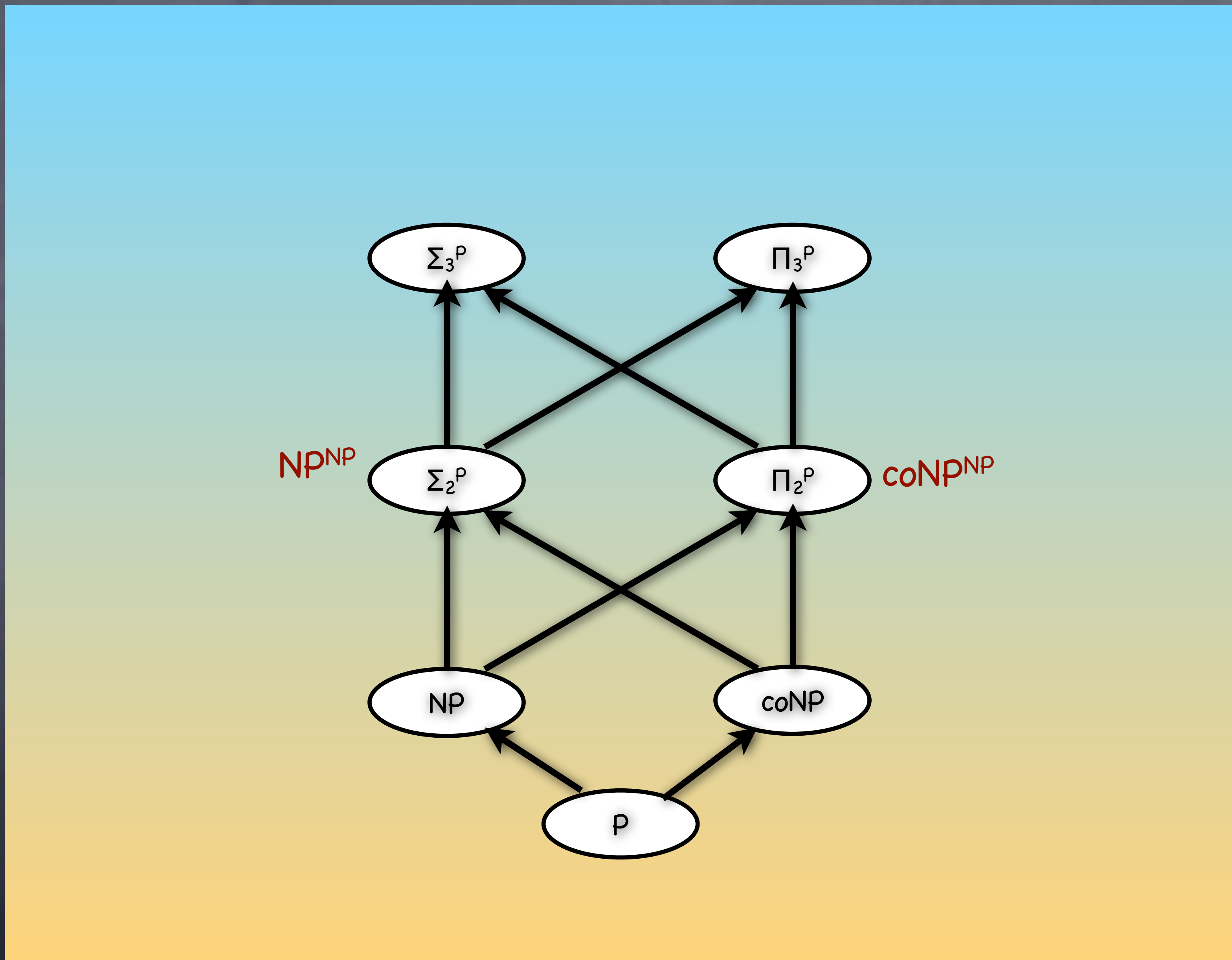

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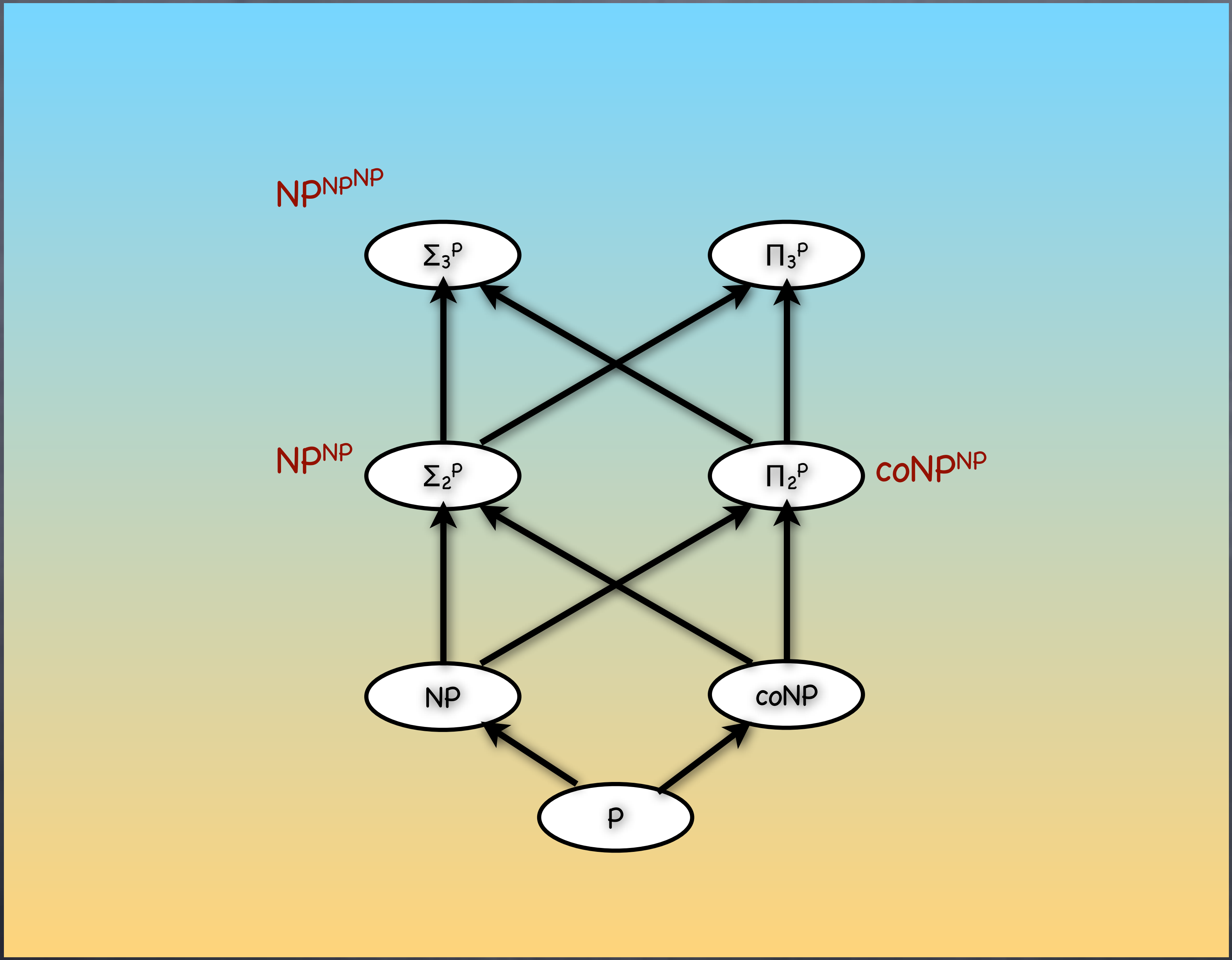

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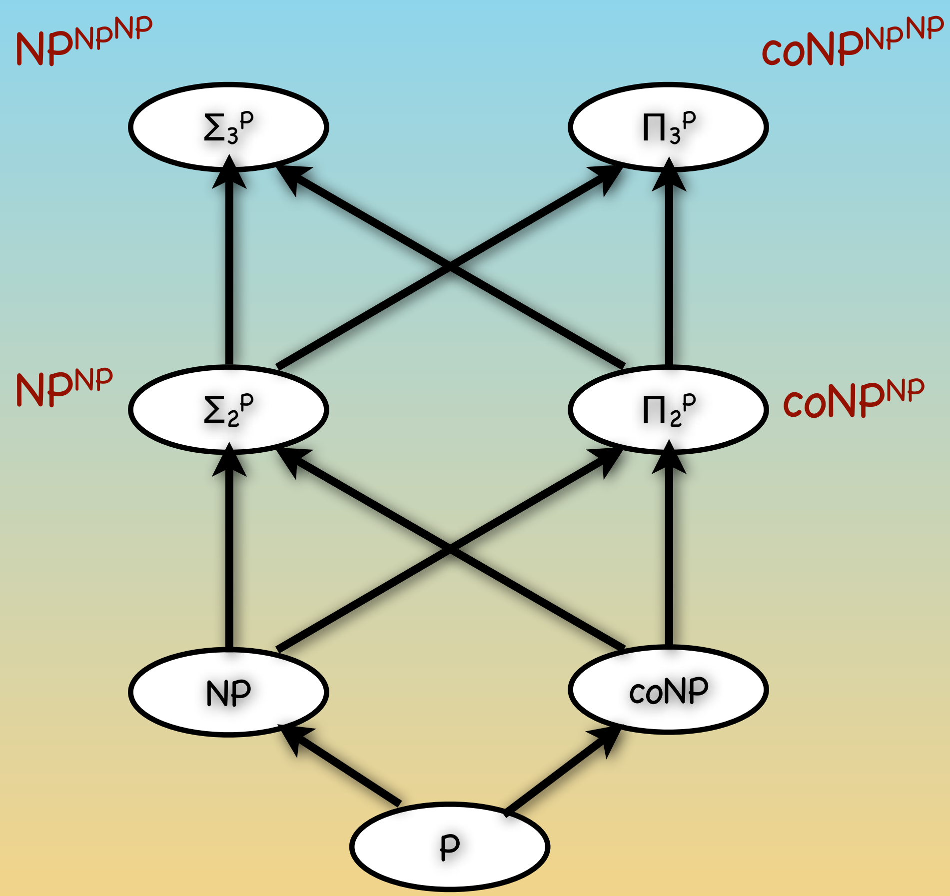
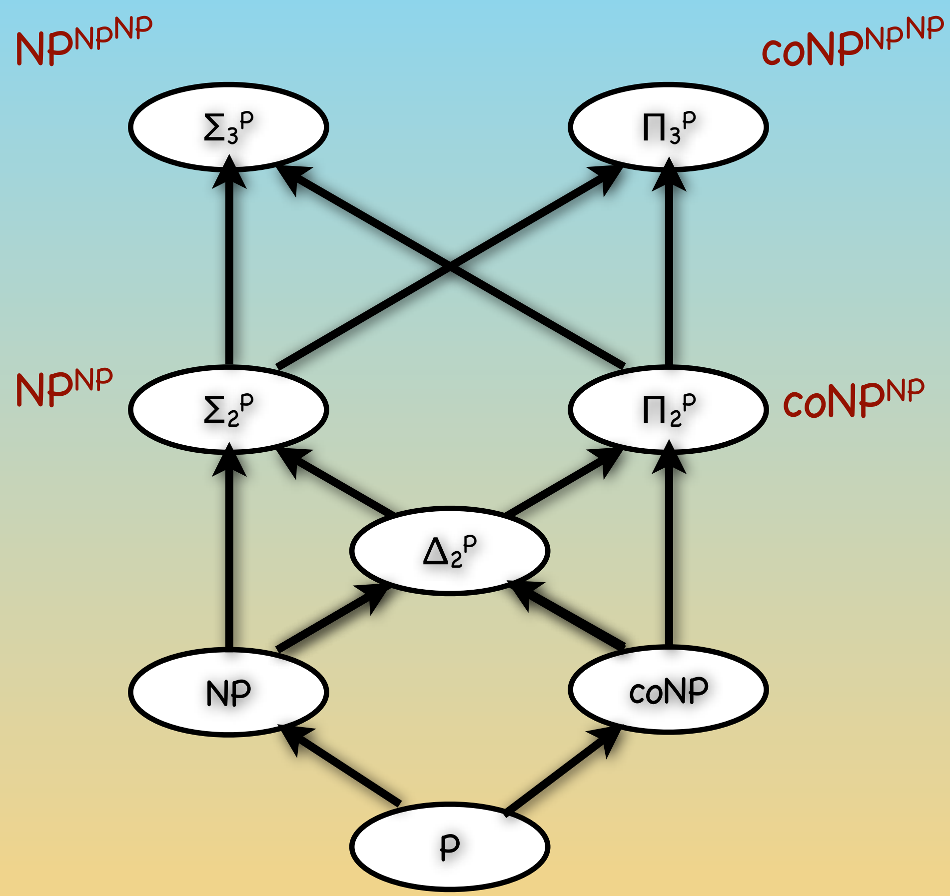


$NP^{NP^{NP}}$
$coNP^{NP^{NP}}$
Σ_3^P
Π_3^P
NP^{NP}
Σ_2^P
Π_2^P
$coNP^{NP}$
NP
coNP
P

PH



NP^NP^NP
coNP^NP^NP
Σ_3^P
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NP^NP
Σ_2^P
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coNP^NP
Δ_2^P
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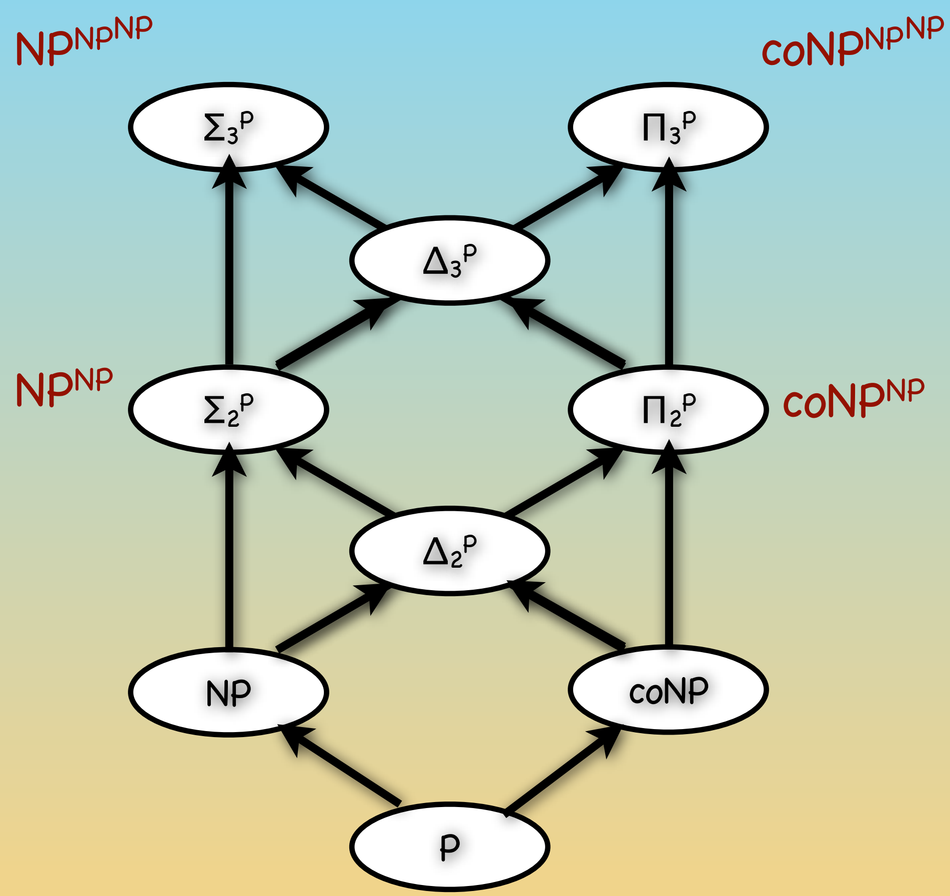

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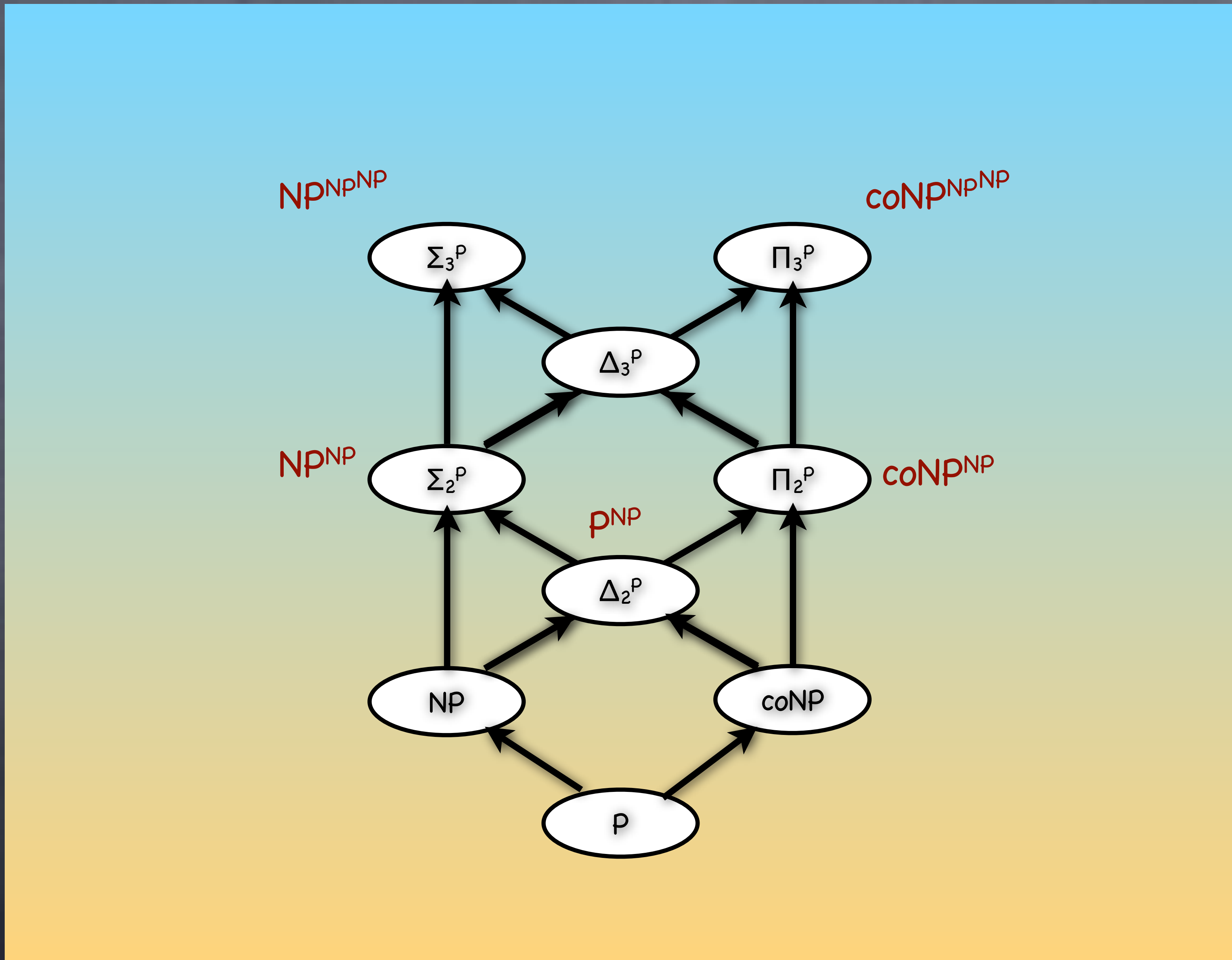

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coNP^NP^NP
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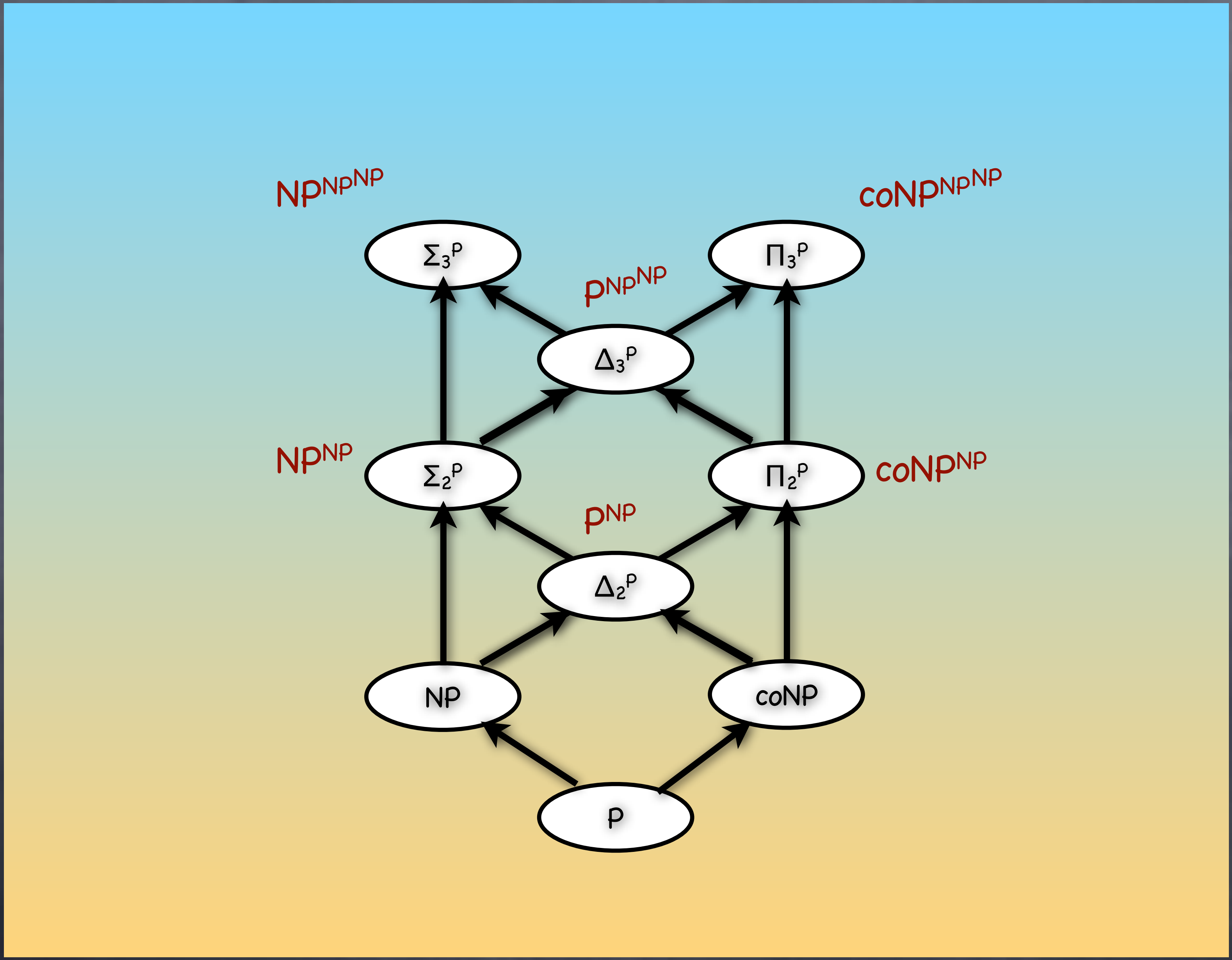

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Today

Today

- Today, more PH

Today

- Today, more PH
  - Oracle-based definitions (in particular $NP^{NP} = \Sigma_2^P$)

Today

- Today, more PH
  - Oracle-based definitions (in particular $NP^{NP} = \Sigma_2^P$)
- Next lecture, more PH

Today

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  - Time-Space tradeoffs