Computational Complexity

Lecture 4
in which Diagonalization takes on itself,
and we enter Space Complexity

Meta-Questions

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"Real" Questions

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Equivalence carries over!

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    - There exist languages A, B such that $P^A = NP^A$, but $P^B \neq NP^B$!

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  - EXPTM = { $(M,x,1^n)$ | TM represented by M accepts x within time 2^n }

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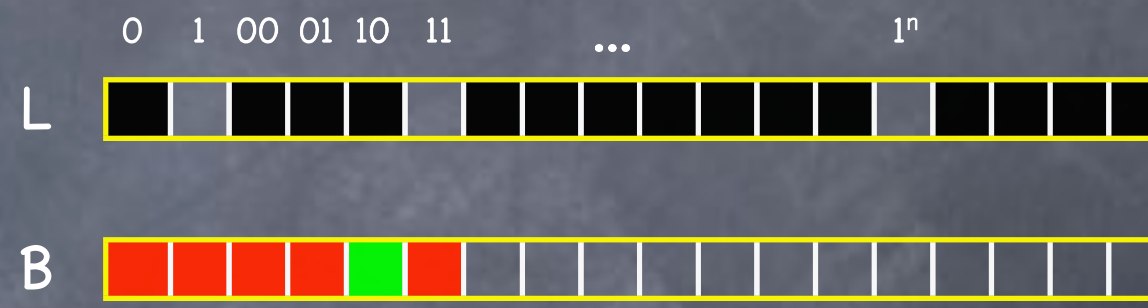

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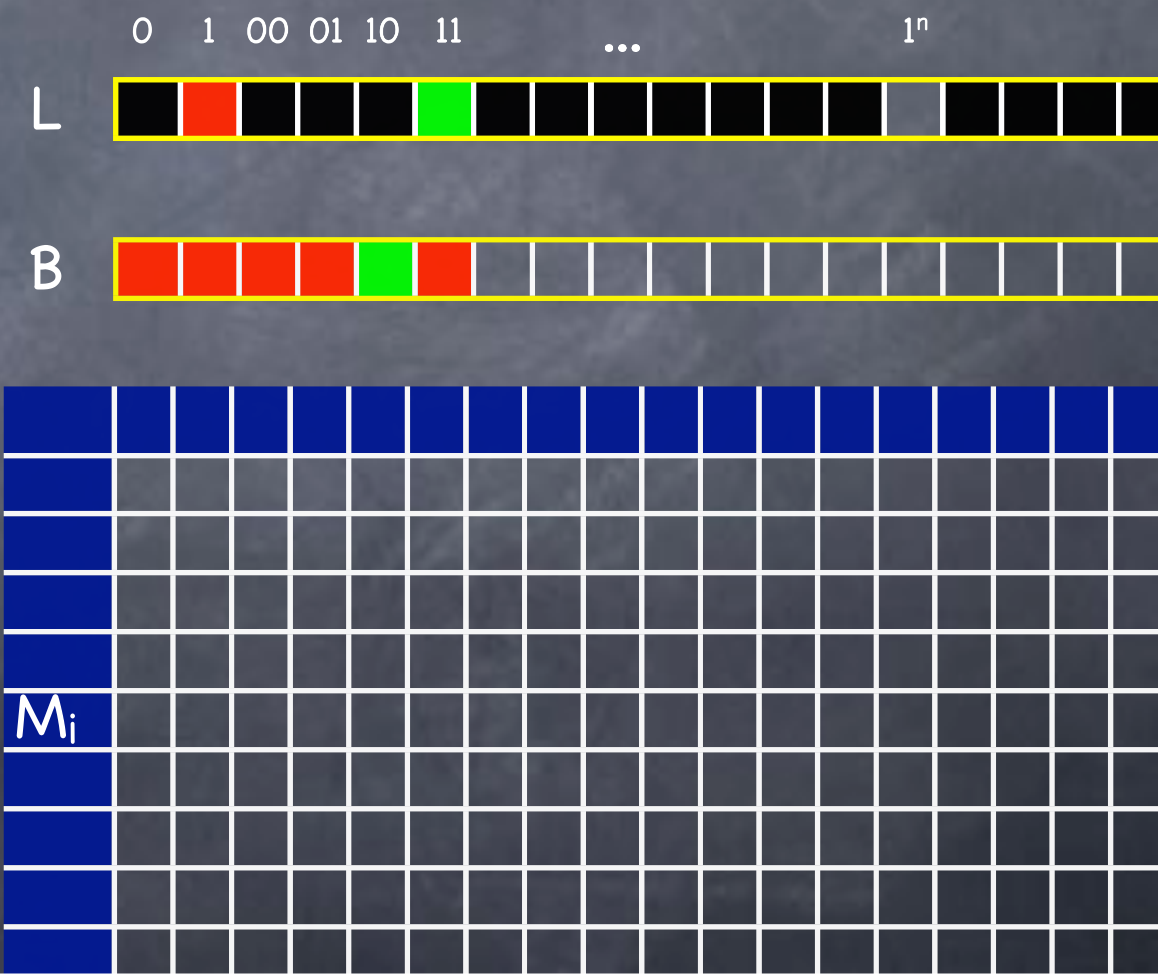

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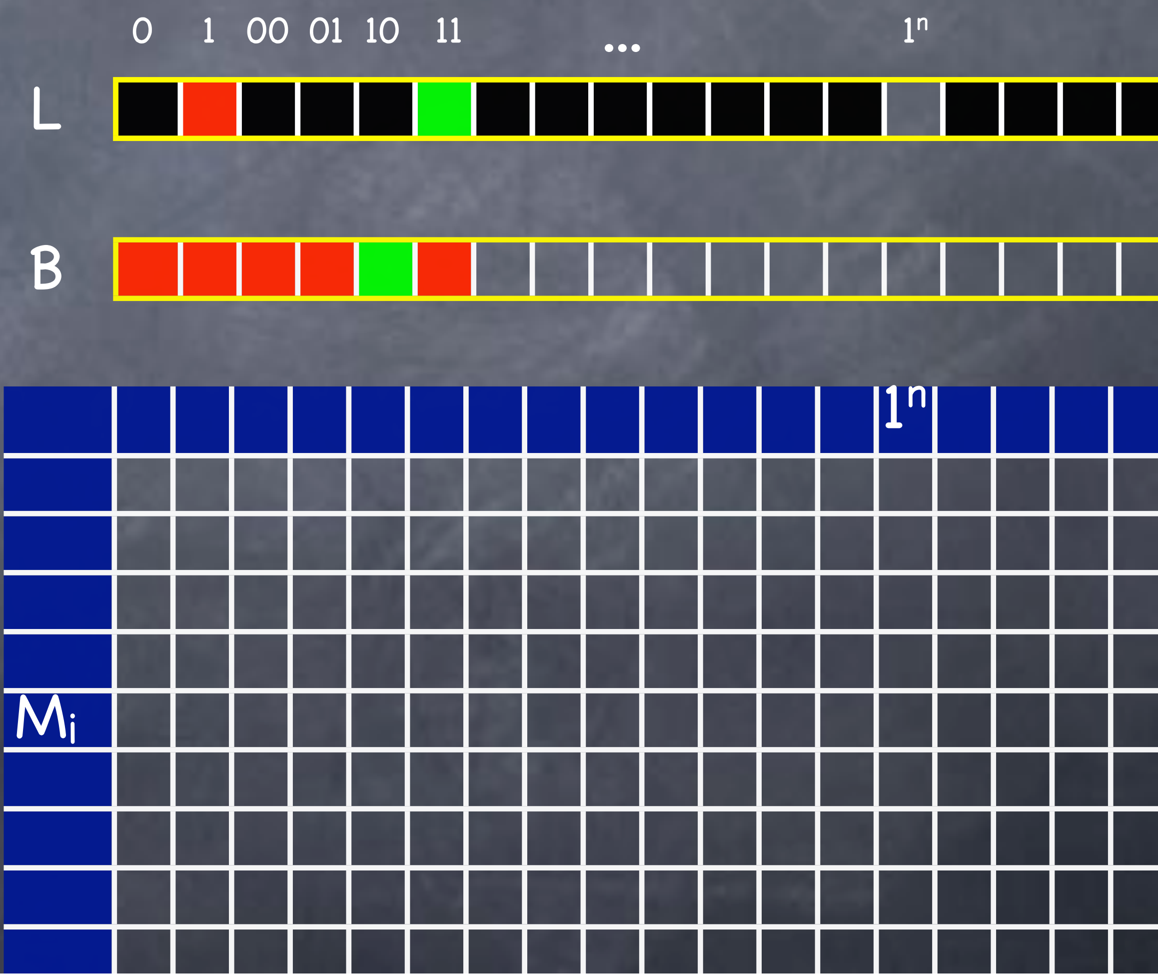

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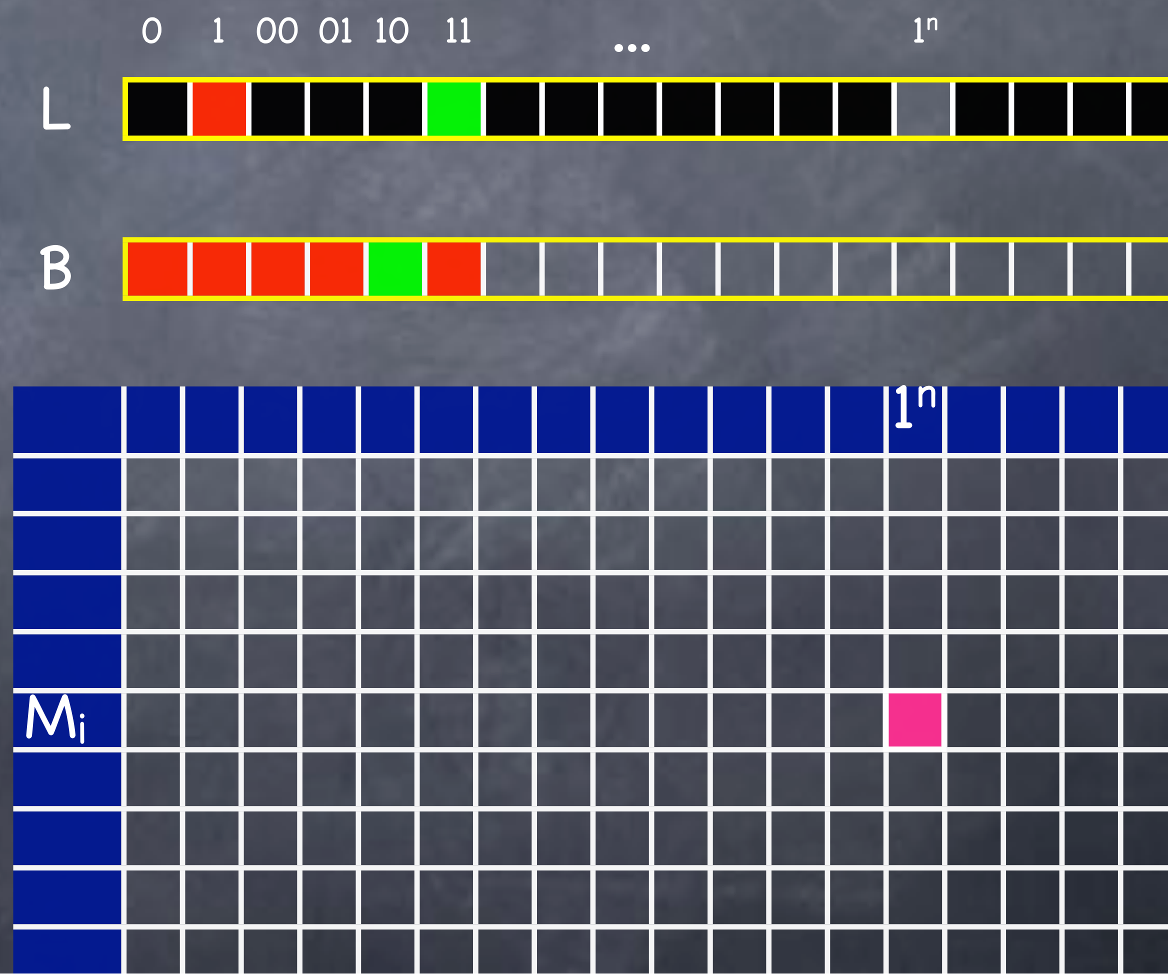

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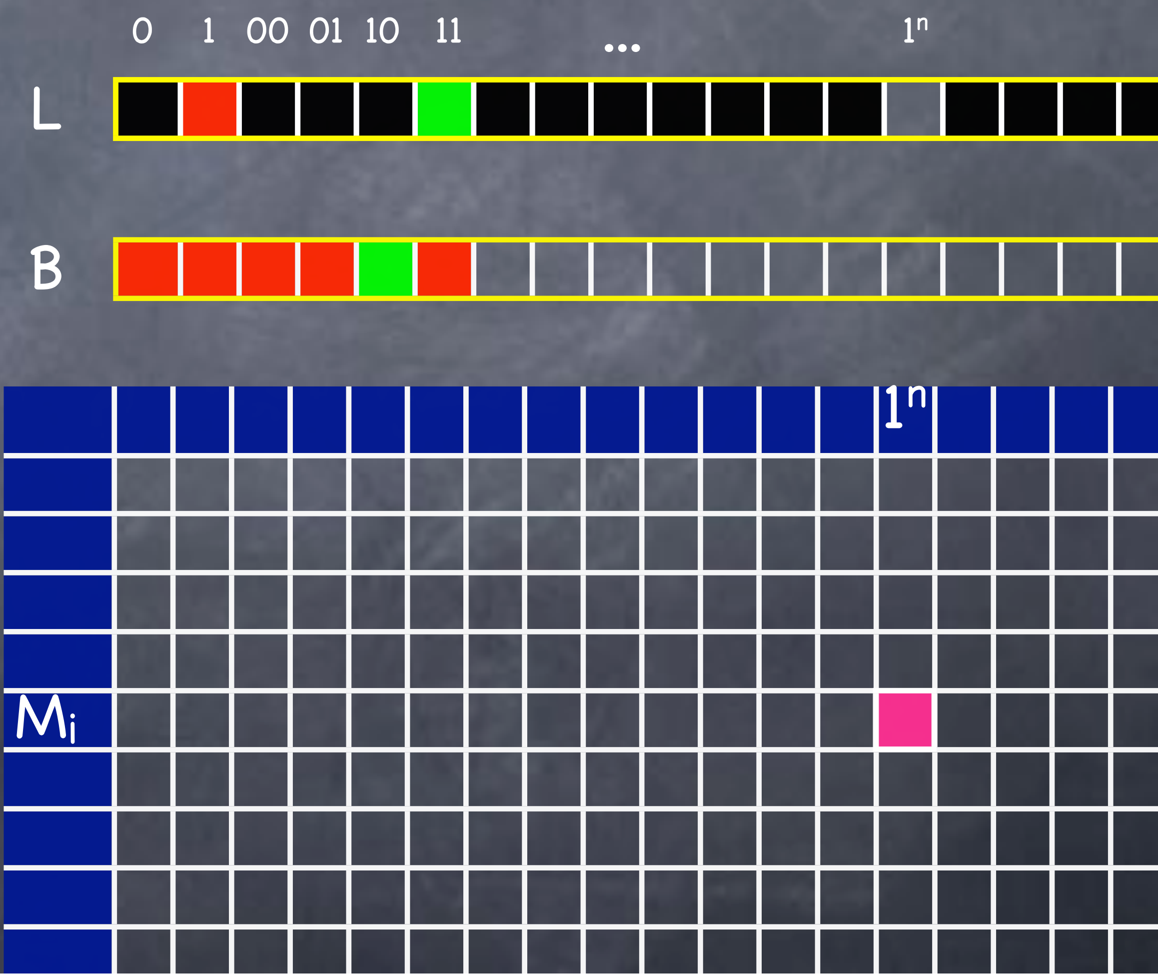

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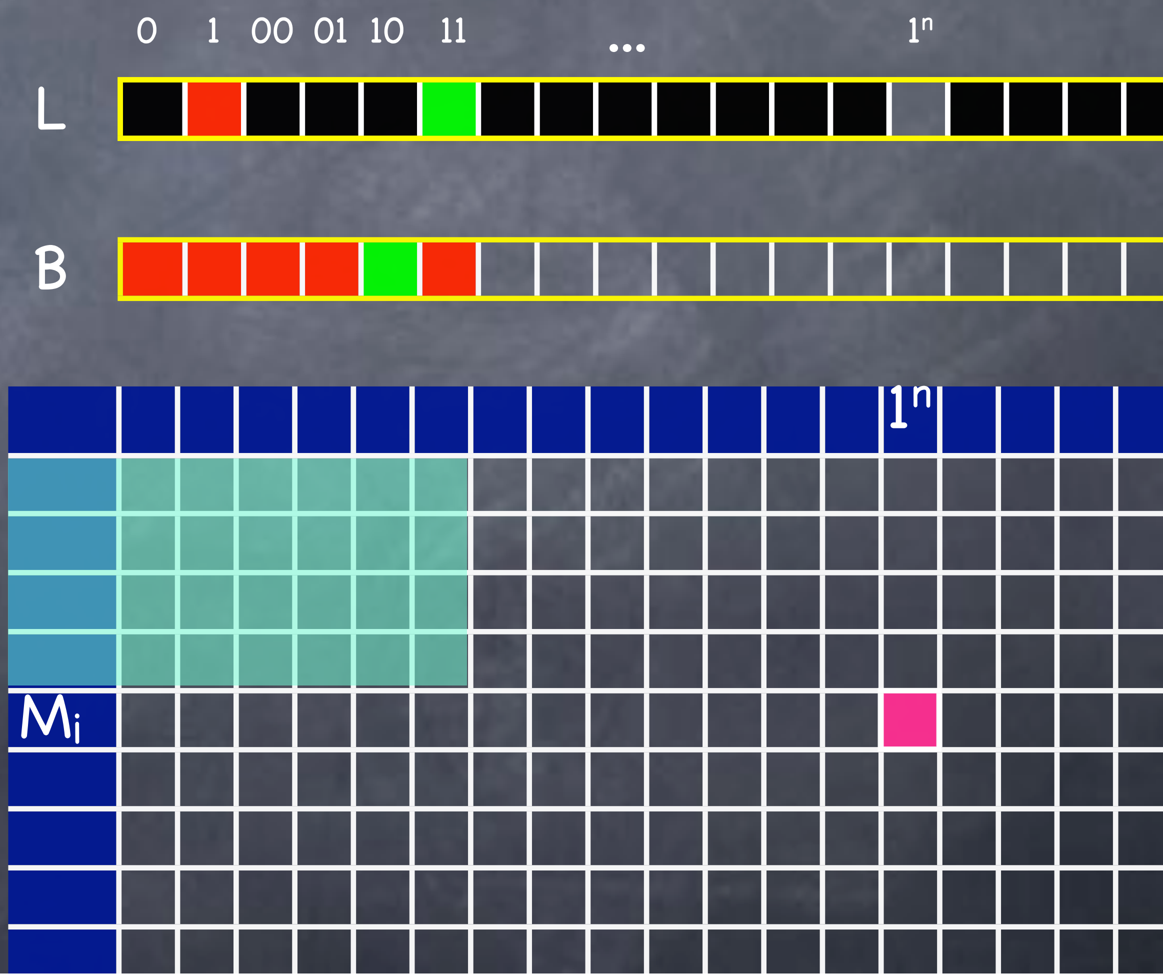

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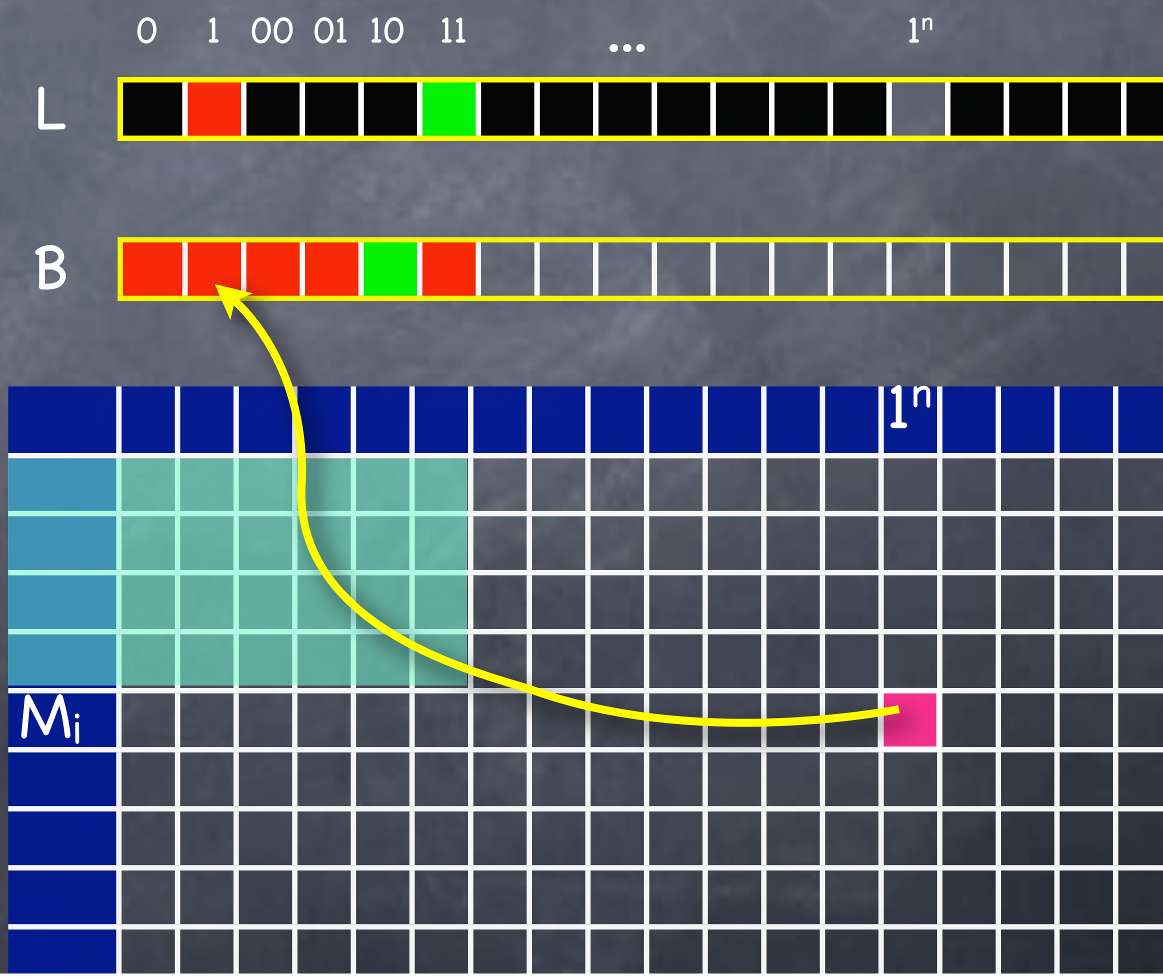

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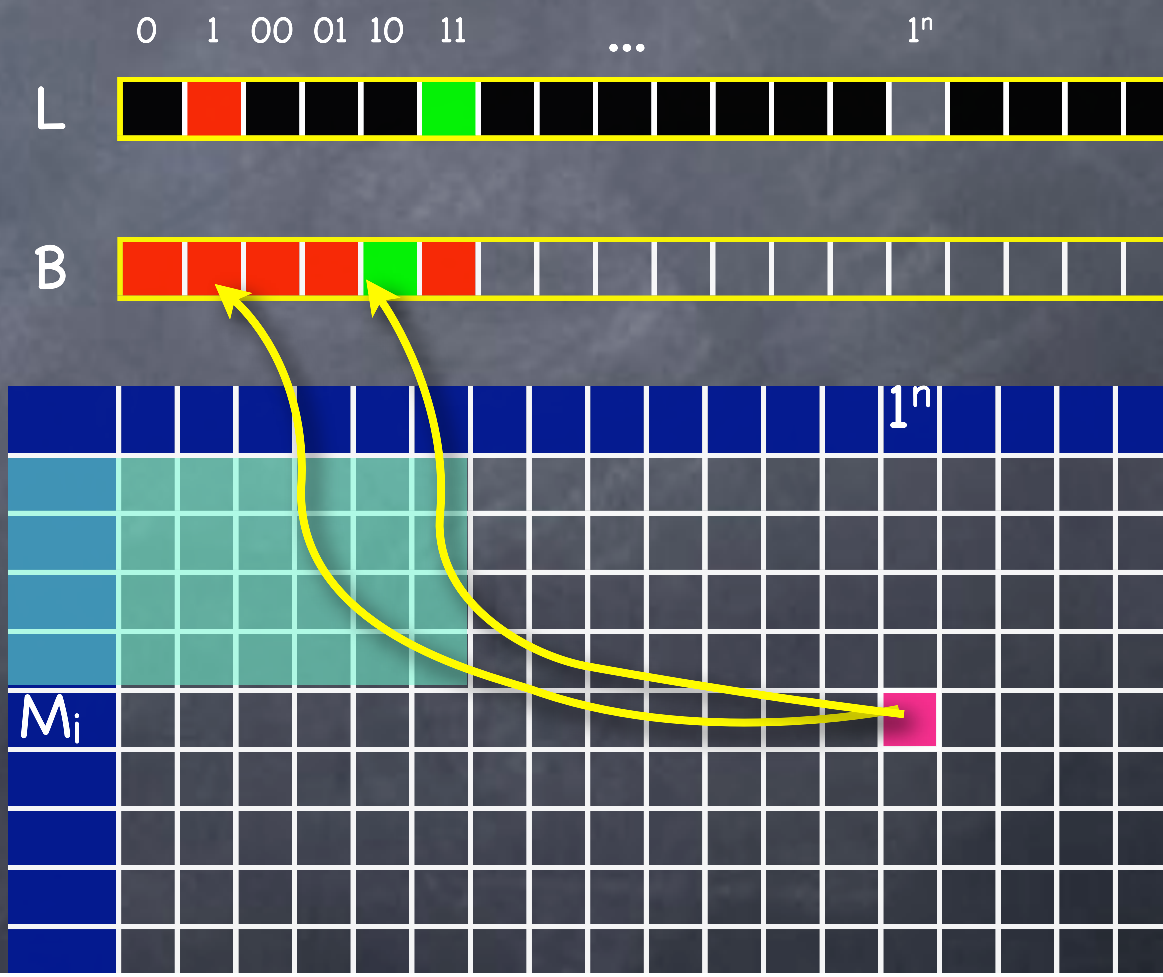

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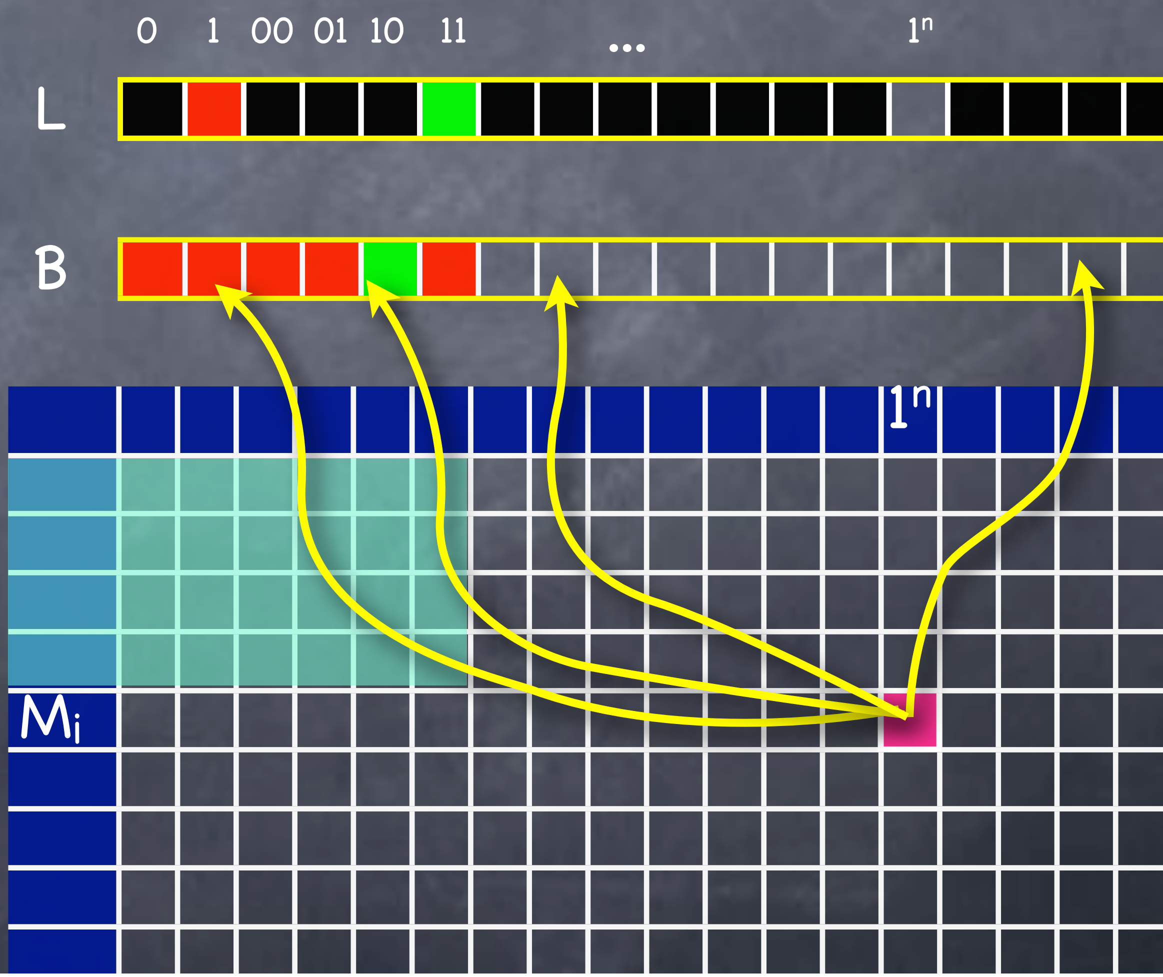

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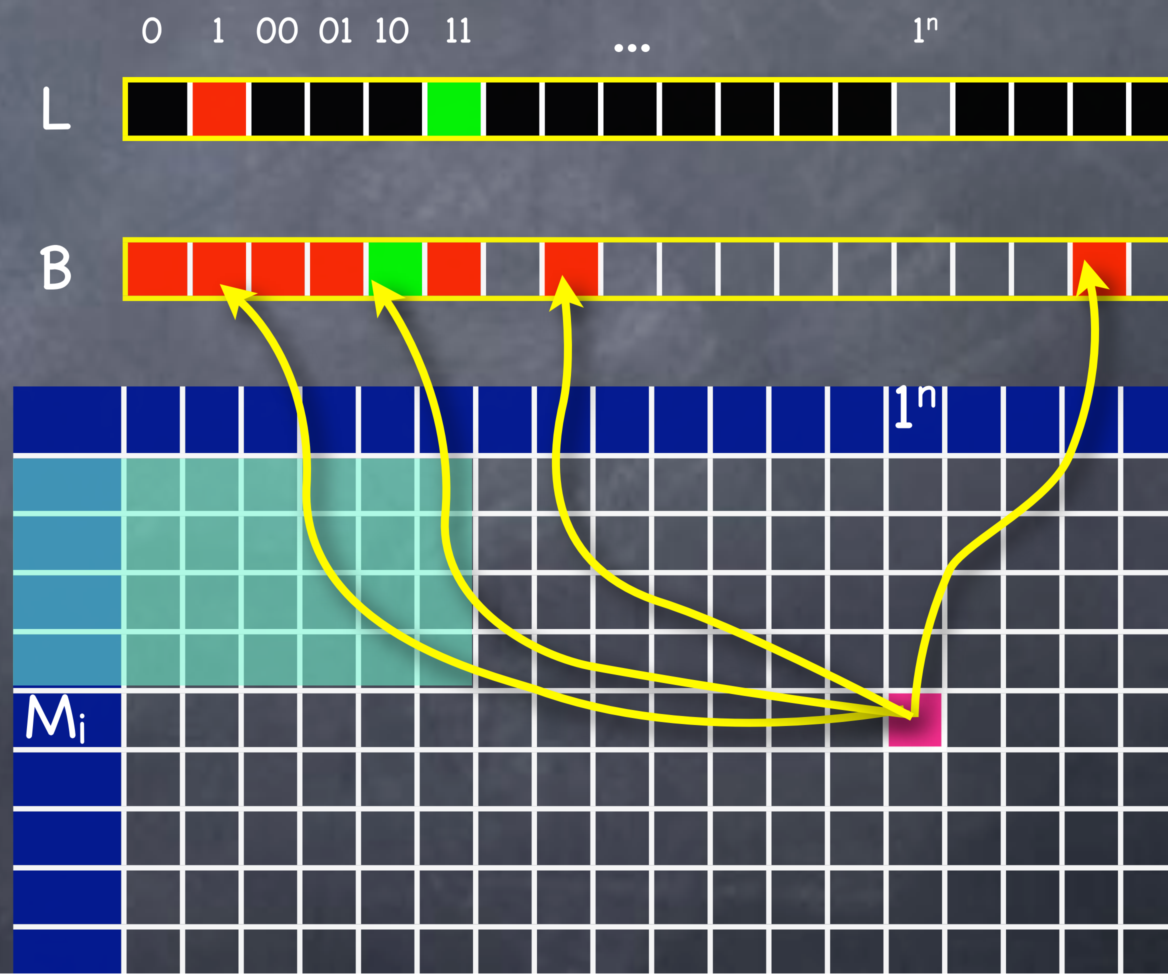

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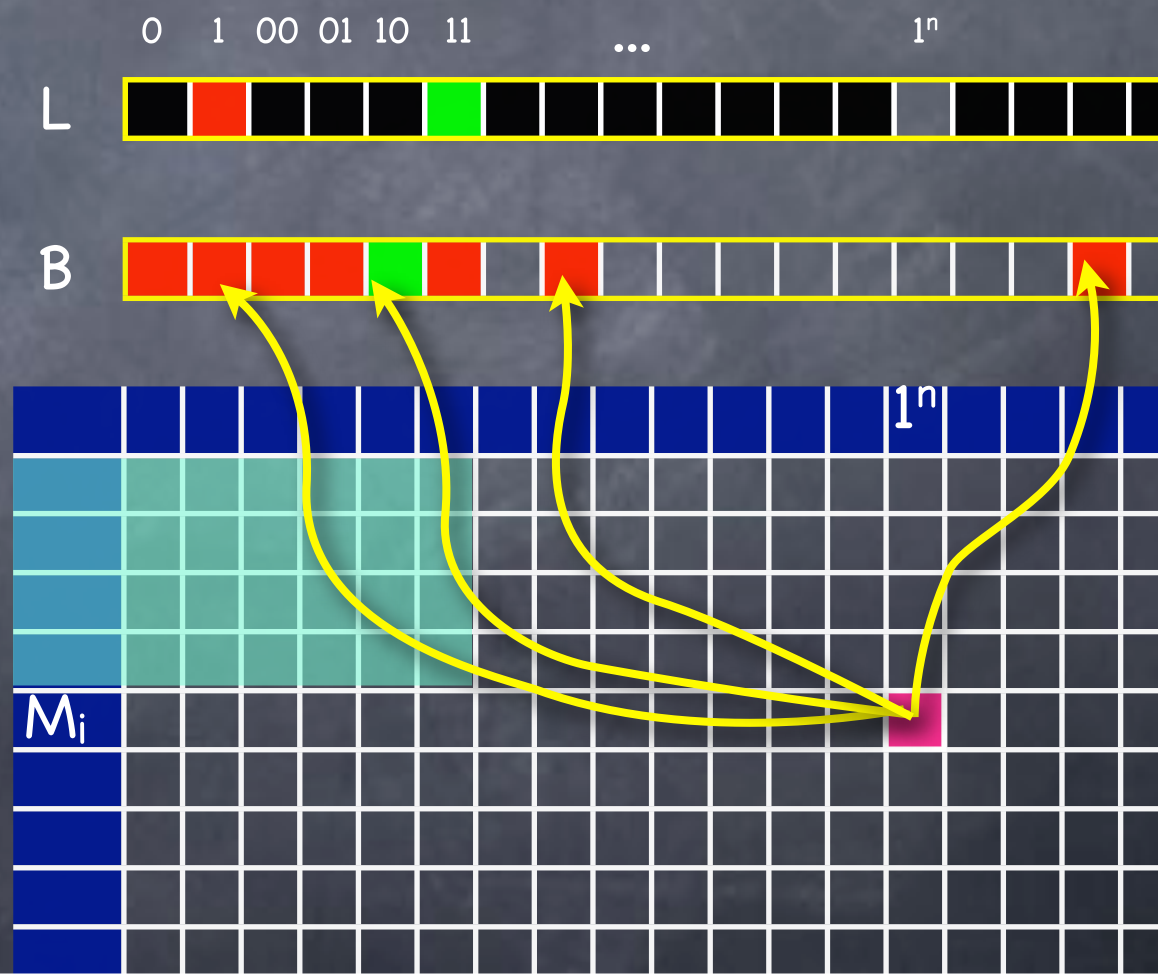

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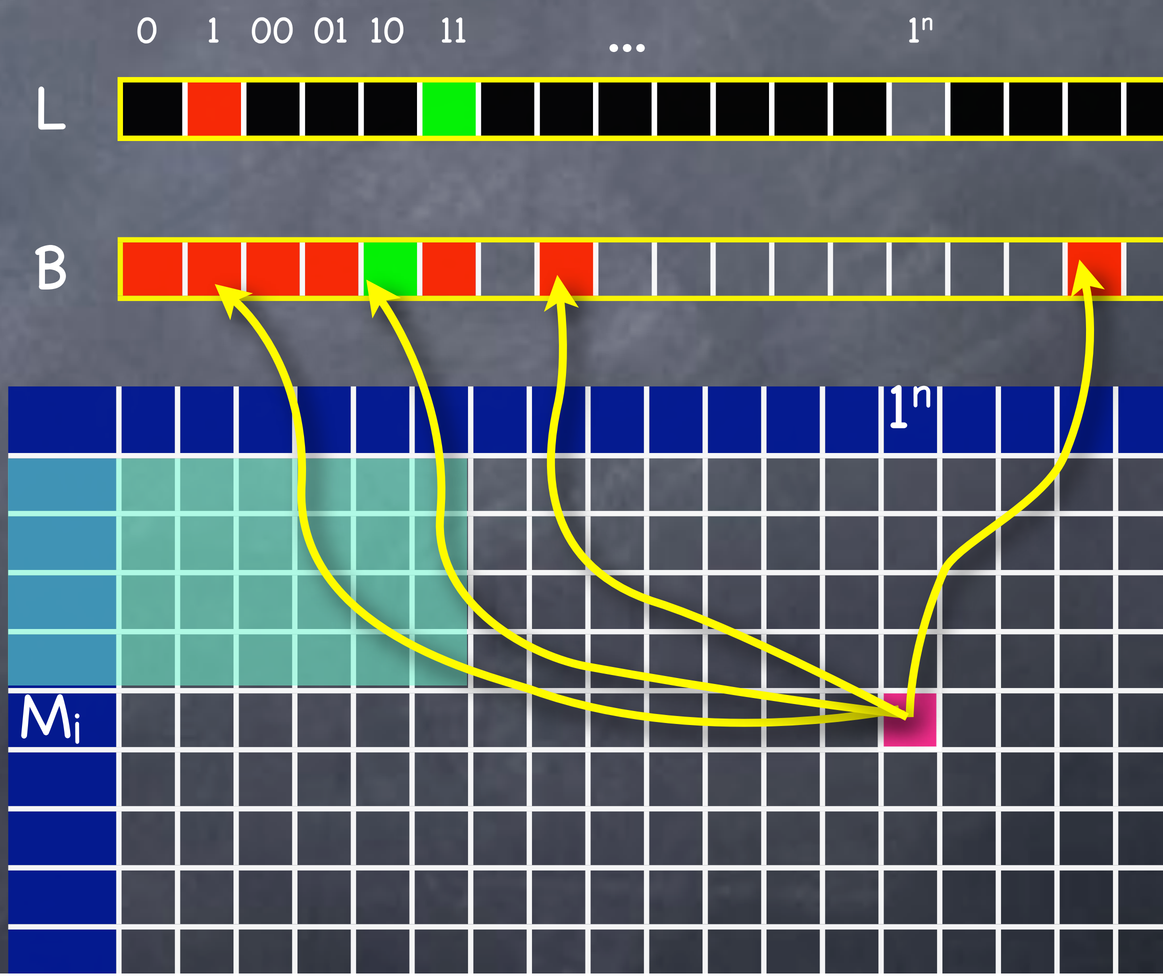
  - After M_i finished set B up to $x=1^n$ s.t. $L(1^n) \neq M_i^B(1^n)$



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  - e.g. of non-relativizing proof: that of Cook-Levin theorem

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    - Turns out, often a little memory can go a long way (if we can spare the time)

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- "L and NL are to space what P and NP are to time"

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  - Tight hierarchy: if $T(n) = o(T'(n))$ (no log slack) then $\mathrm{DSPACE}(T(n)) \subsetneq \mathrm{DSPACE}(T'(n))$
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      - No "delayed flip," because, as we will see later, $\mathrm{NSPACE}(O(S)) = \text{co-}\mathrm{NSPACE}(O(S))$!

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