Lecture 26 Weak techniques are indeed weak!

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- What does a proof look like (often)?
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 - Being able to show that for Φ might require it to be a nice (natural) property

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 - Require at least 1/N fraction to have Φ

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Switching Lemma: Depth d AC circuit becomes depth 2, restricted to n⁸ vars. Can fix to 0 or 1 by restricting n⁸/2 more vars.

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 - But doesn't give an "explicit" function (say NP function)

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 - But a natural property that avoids P/poly distinguishes any distribution of P/poly functions from random functions

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- X, Y are ε-indistinguishable for size-S distinguishers
 if this holds for all circuits D of size at most S

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- (Strong PRF, because "usual" PRF is against poly(n)-size distinguishers who can in particular read only poly(n) positions of the truth-table)

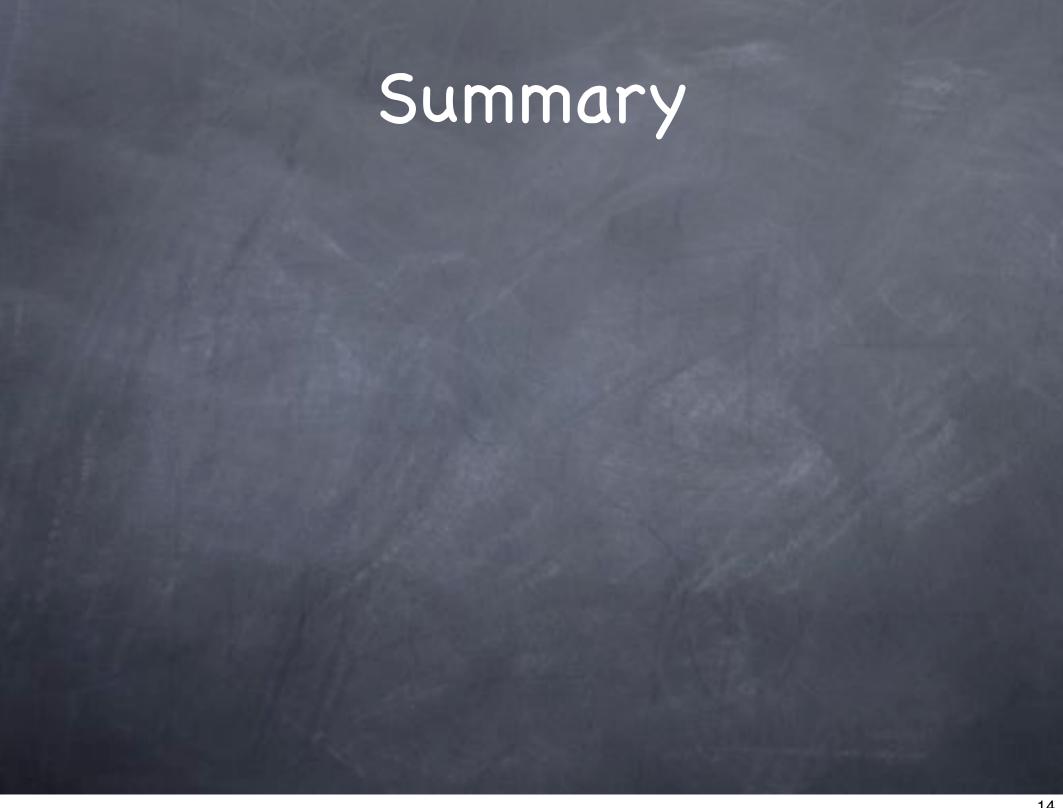
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- A natural property that avoids P/poly can be used to efficiently distinguish a distribution of P/poly functions from random functions
 - Let D(f) be $\Phi(f)$: D is of size S = poly (because Φ natural), $\Pr_{f \leftarrow all}[\Phi(f)=1] > 1/N$ (because Φ natural), and if Φ avoids P/poly then $\Pr_{f \leftarrow PRF}[\Phi(f)=1] = 0$

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- If PRFs exist, then no natural property that avoids P/poly exists



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 - Low-complexity functions which look-like a random function, to "efficient" distinguishers
- Natural proofs can't separate out P/poly as low-complexity, if pseudorandom functions exist in P/poly (as we believe)