# Circuit Lower-bounds

Lecture 25

Weak circuits are indeed weak

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- (Only sketches/partial proofs. See textbook or lecture-notes from linked courses)

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  - How shallow can a poly-sized circuit family for PARITY be?

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    - By "restricting" to n inputs
    - And showing how to rewrite with depth d-1, staying poly sized

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  - If polynomial size and constant depth ($AC^0$), stays so

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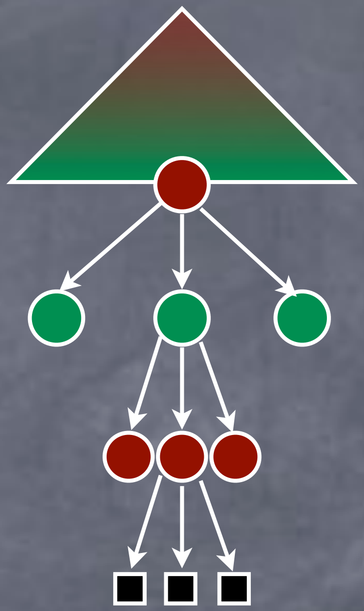

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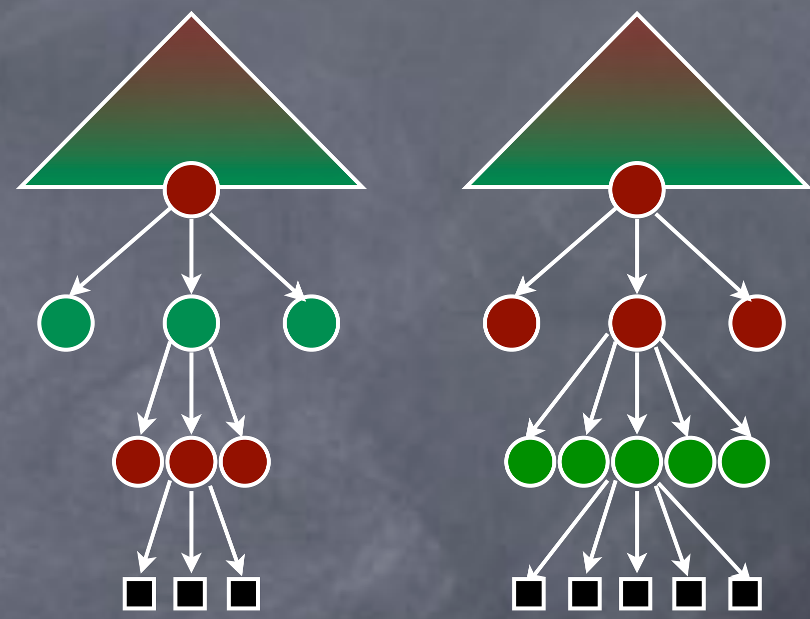

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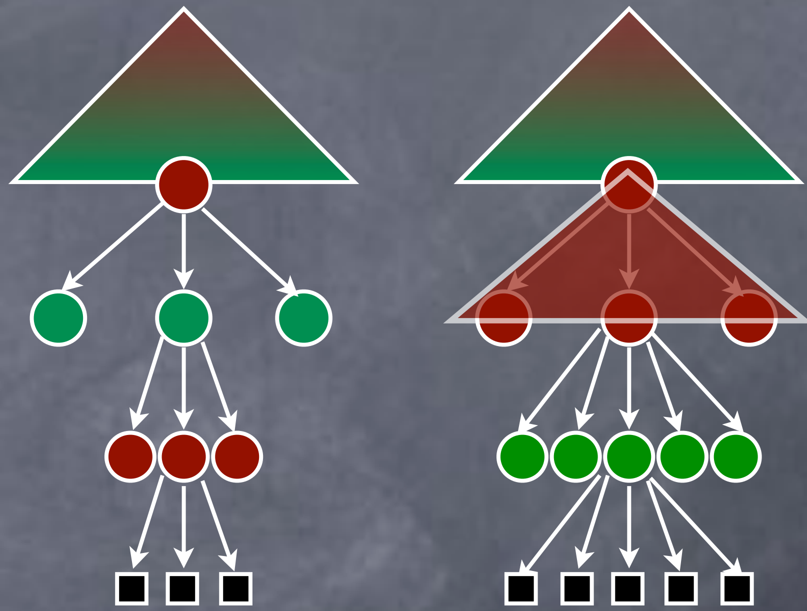

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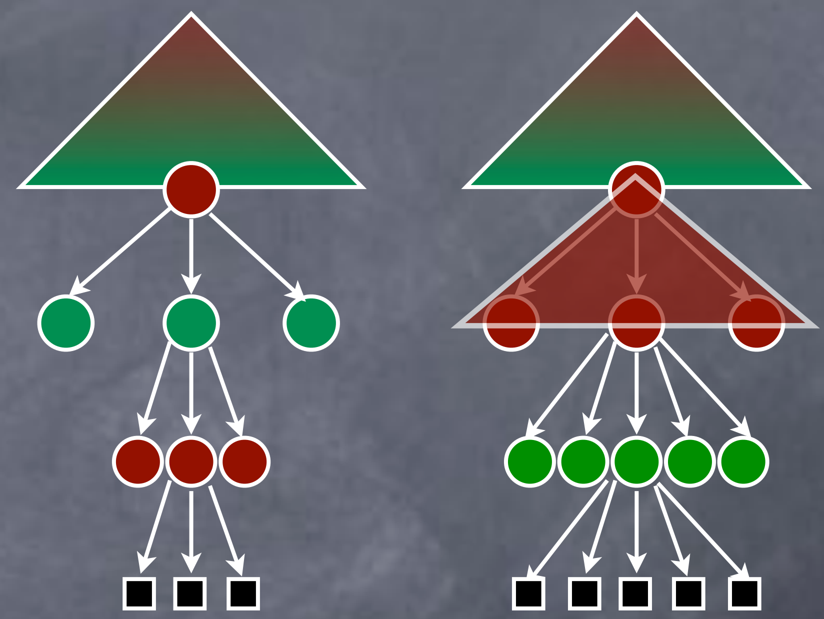

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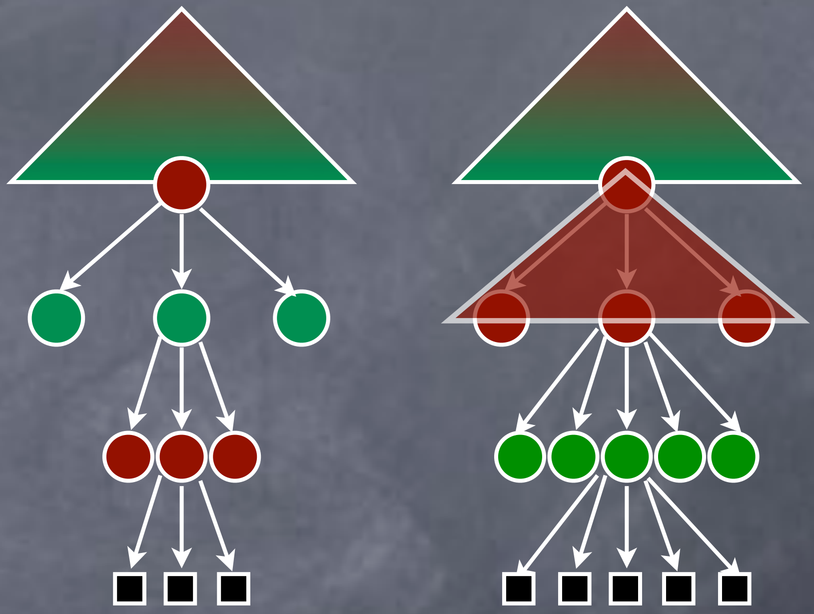

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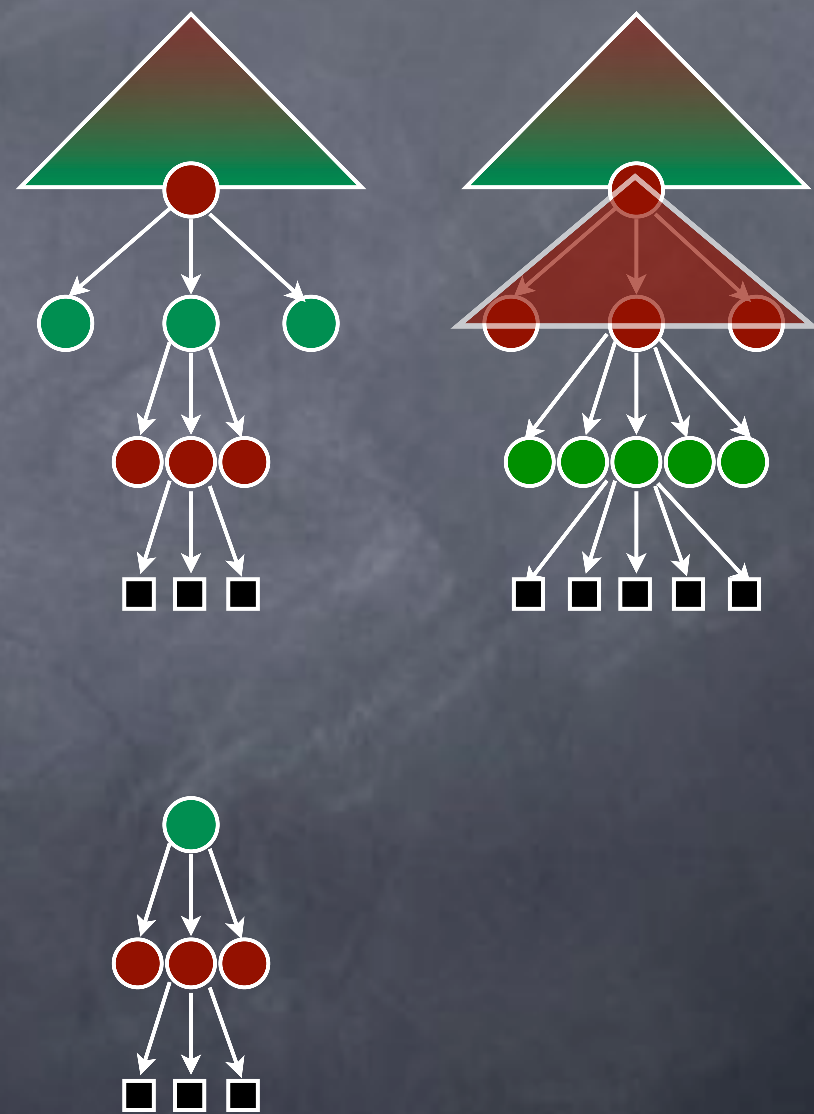

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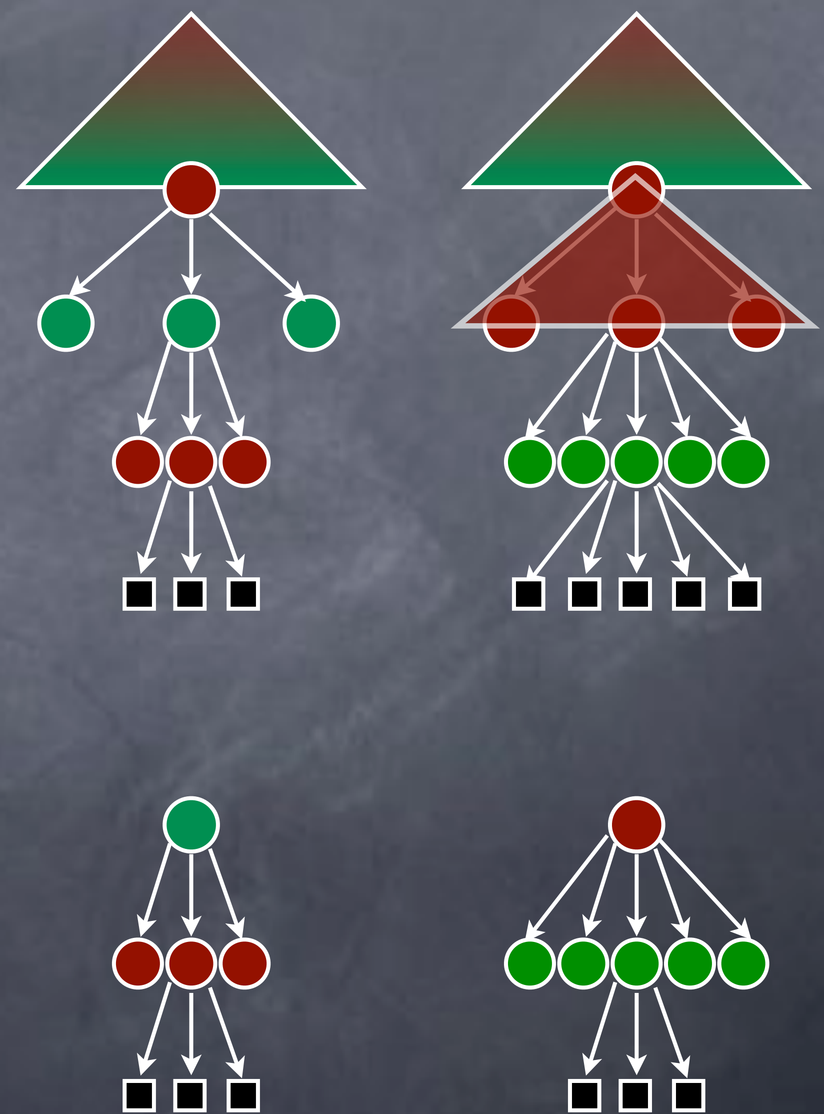

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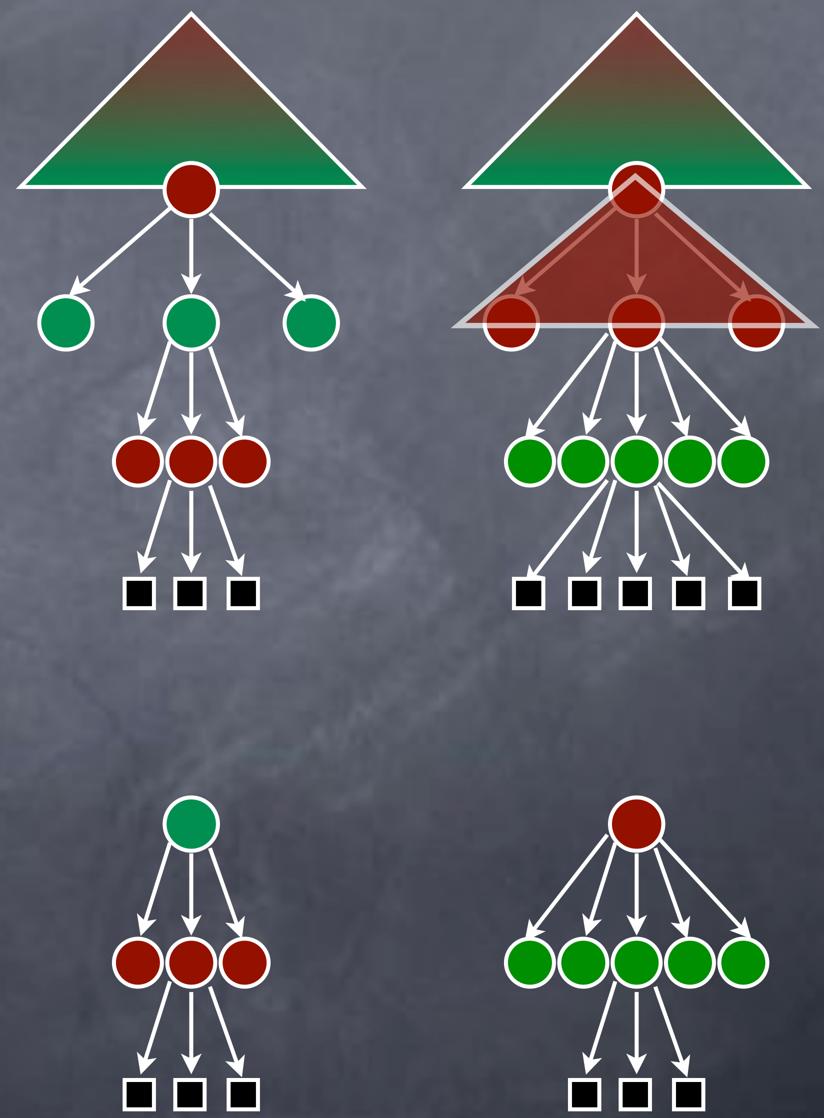

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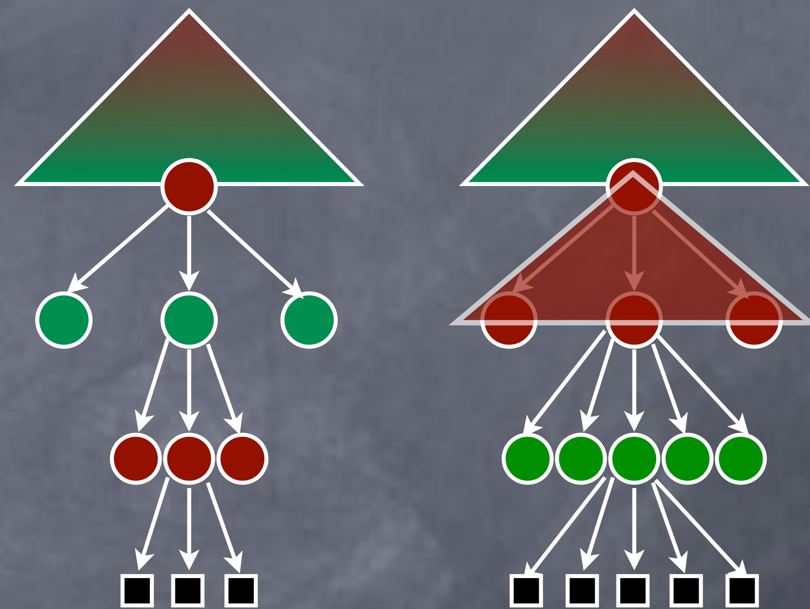

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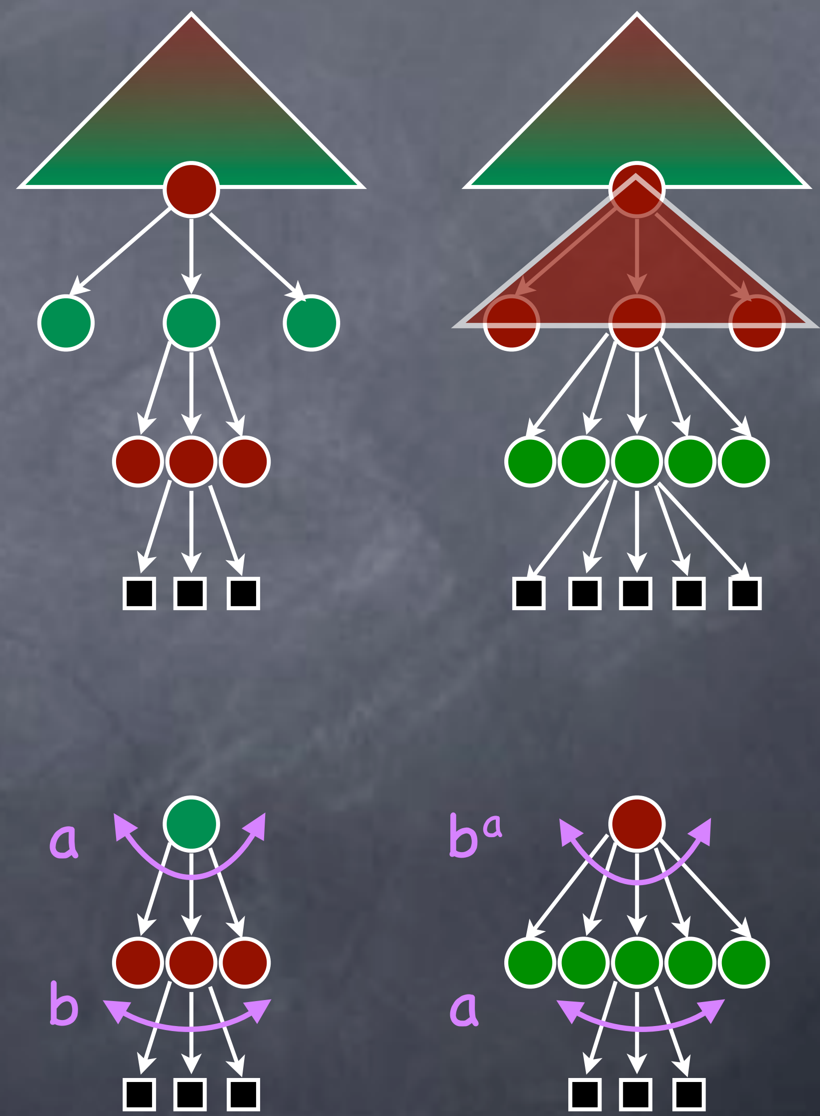

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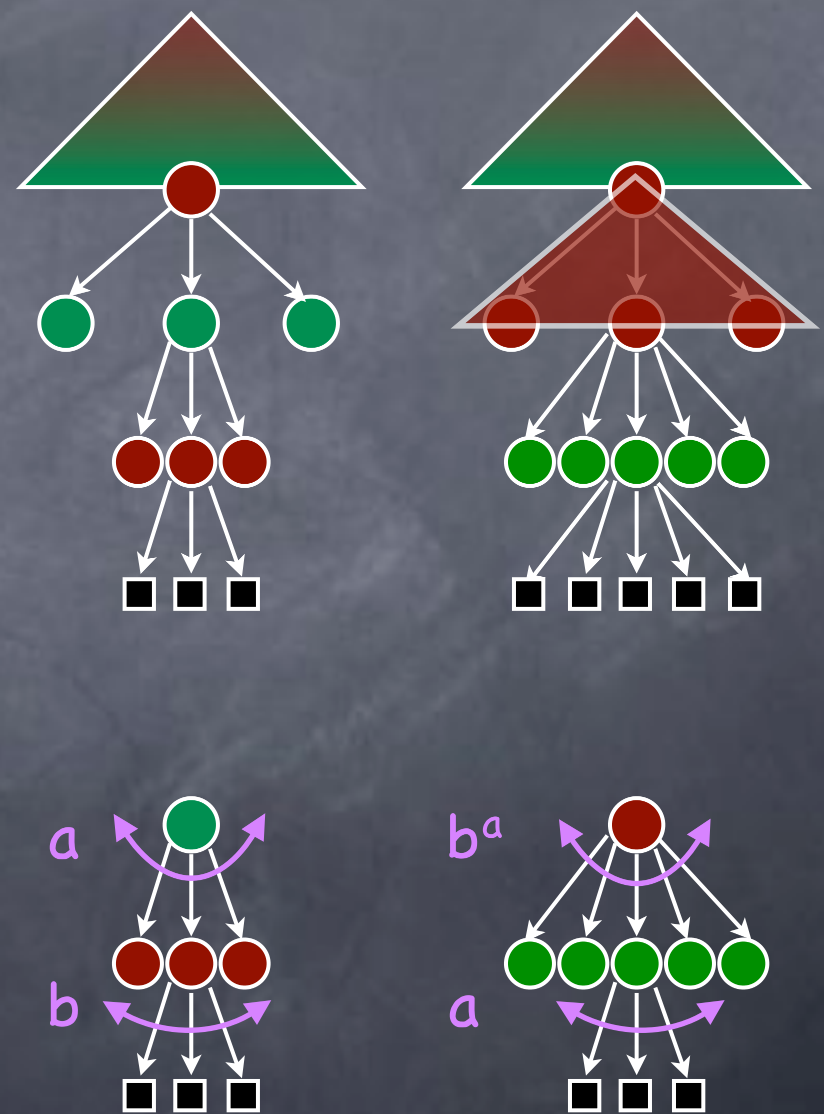

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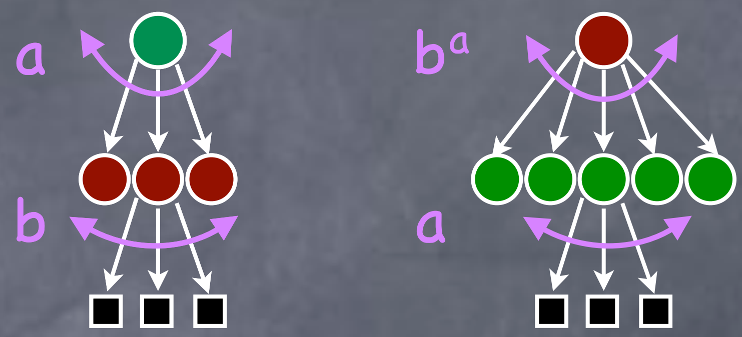

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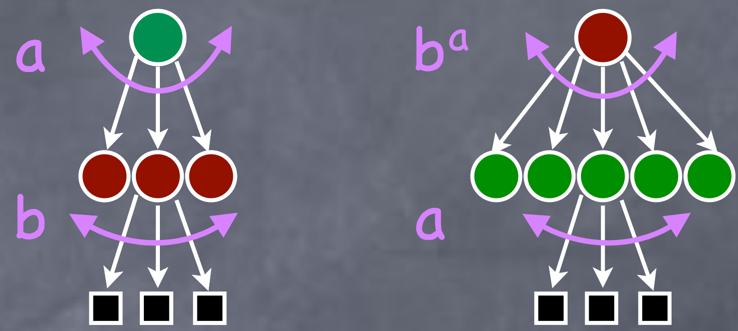

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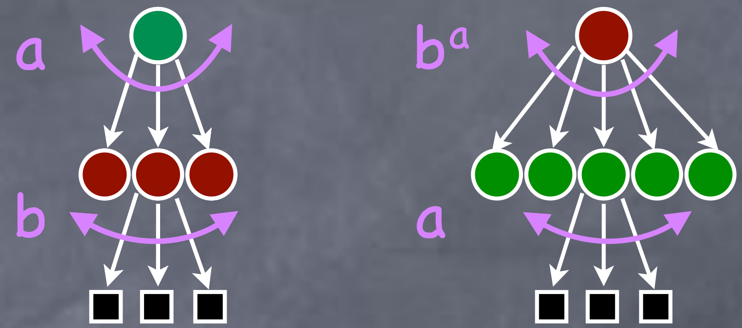

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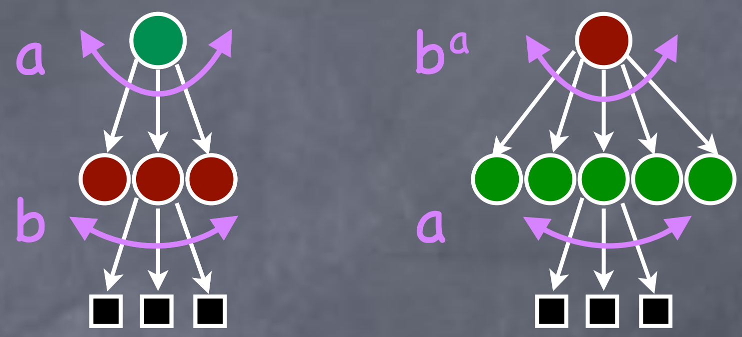

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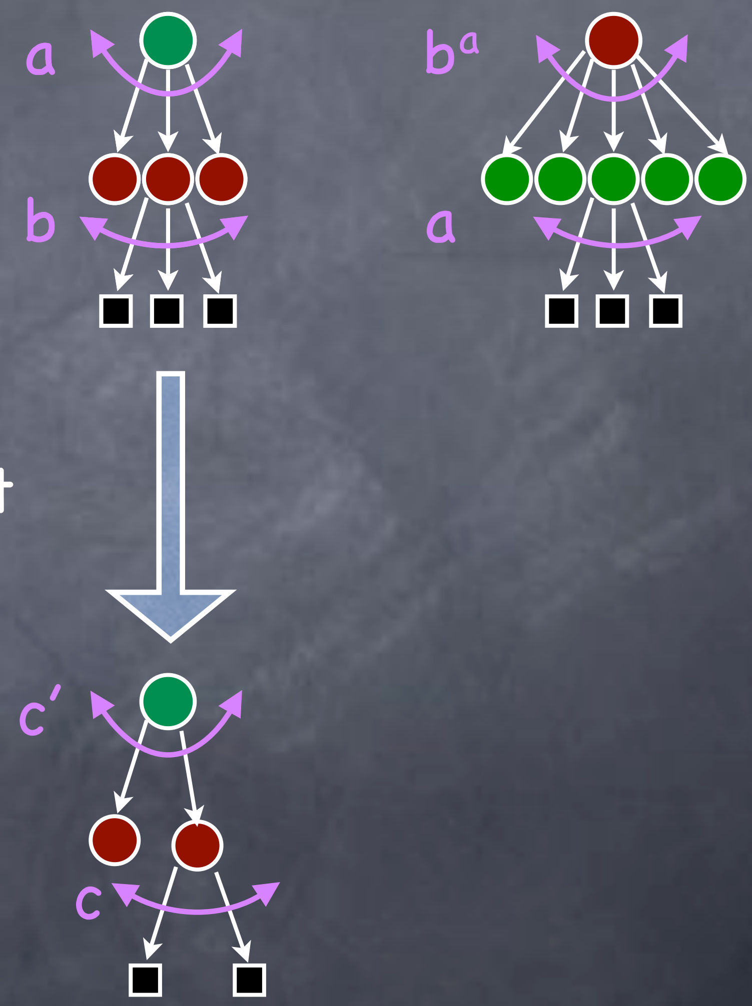

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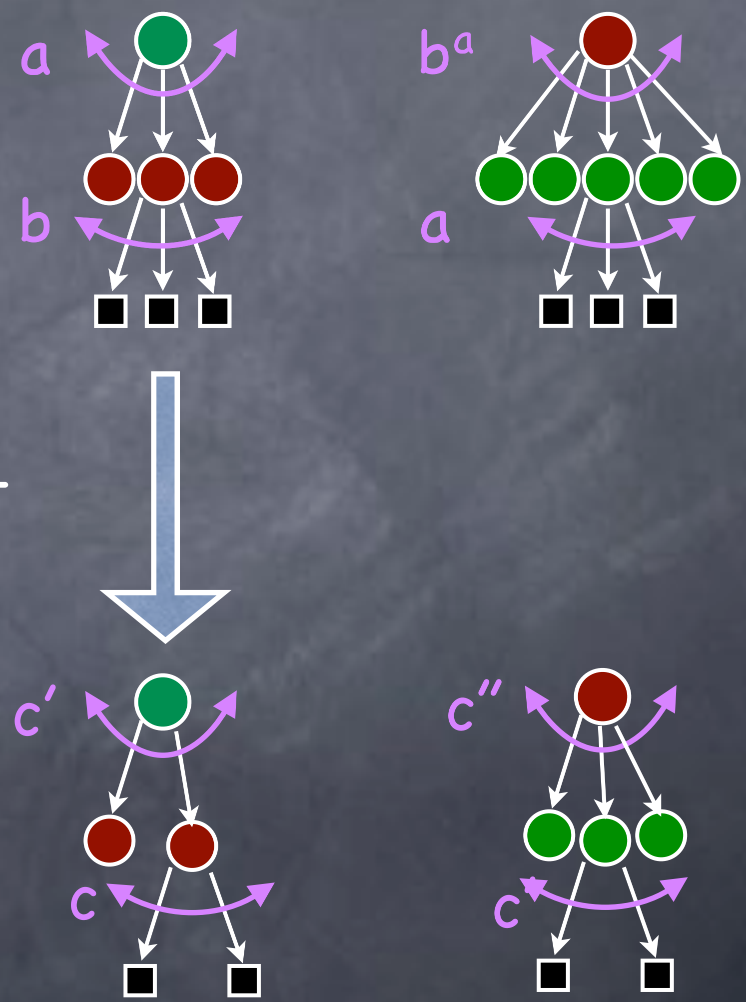

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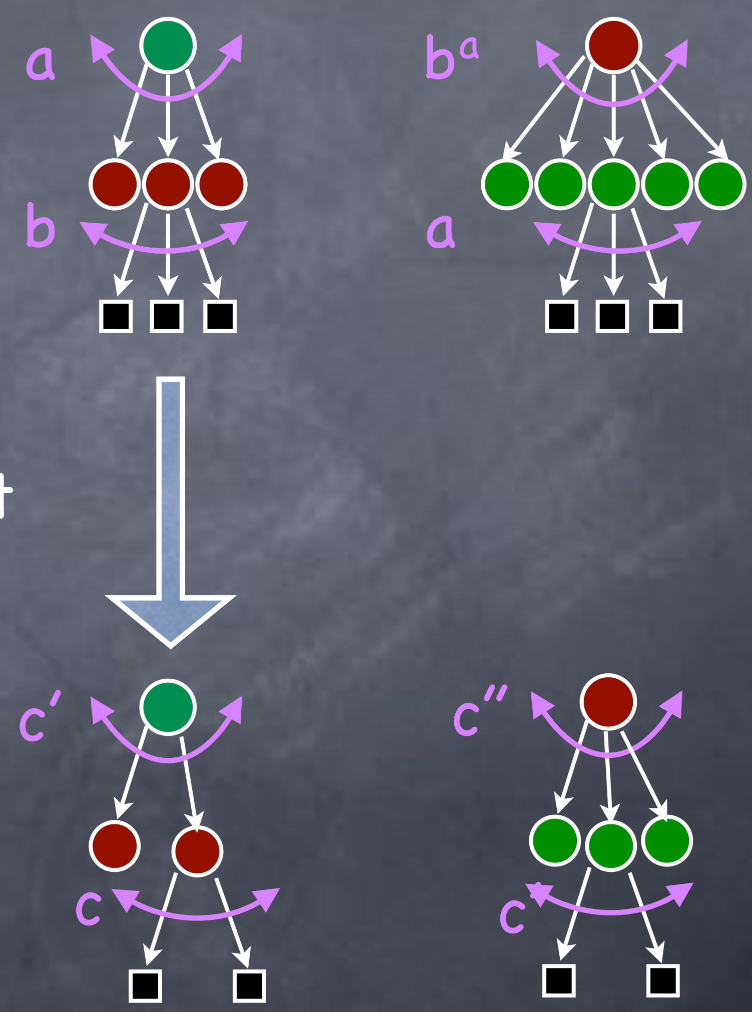

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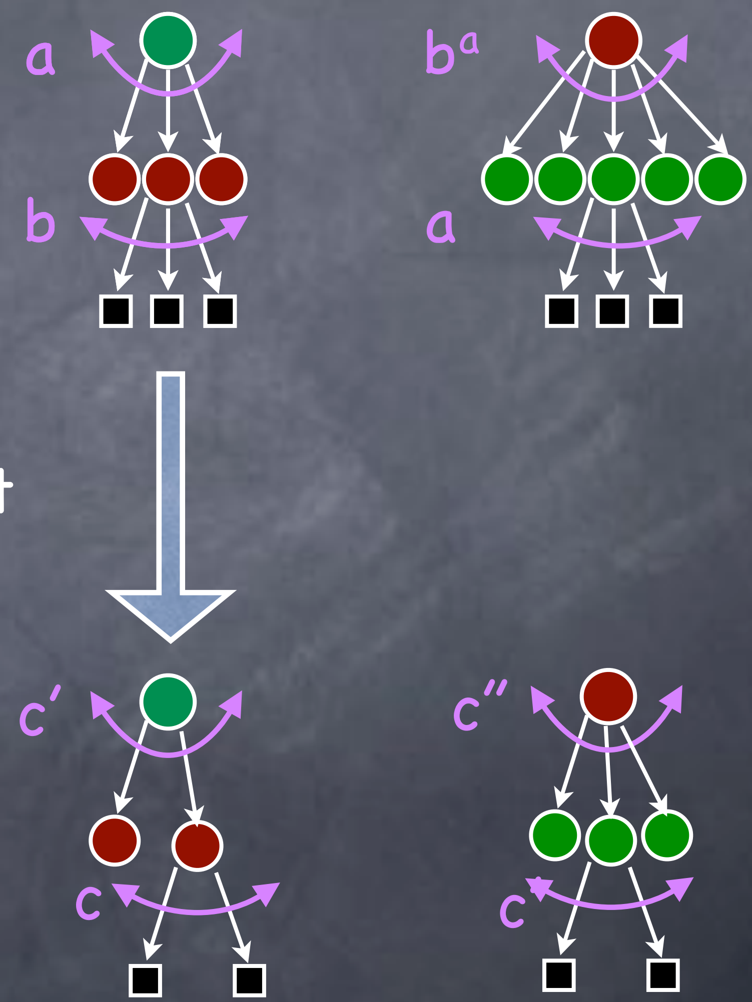

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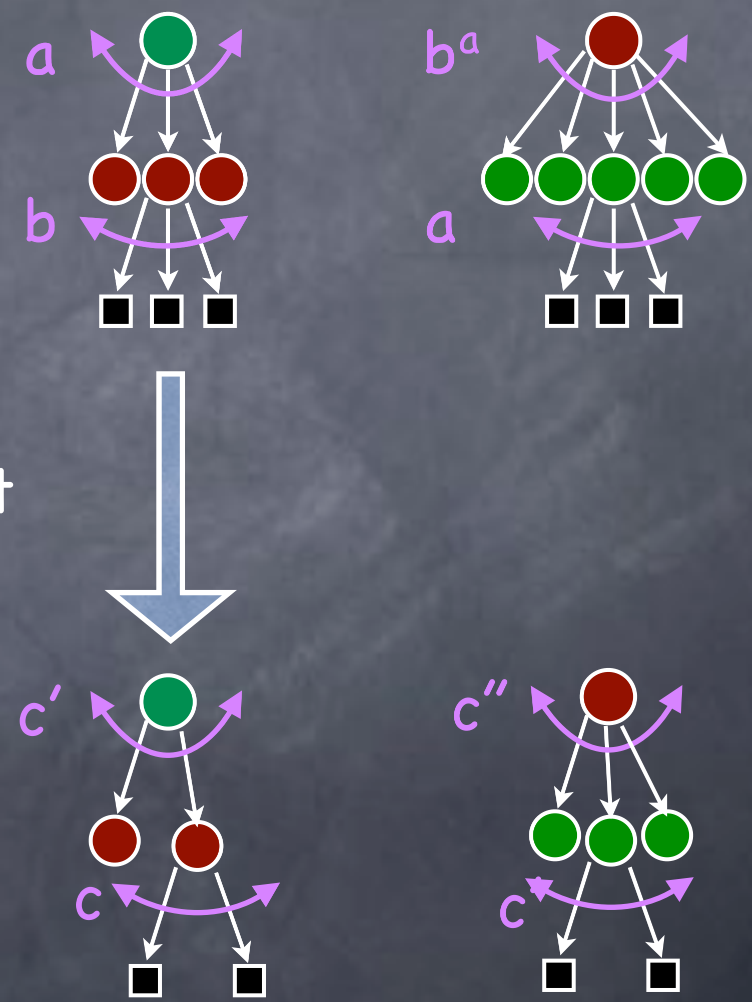

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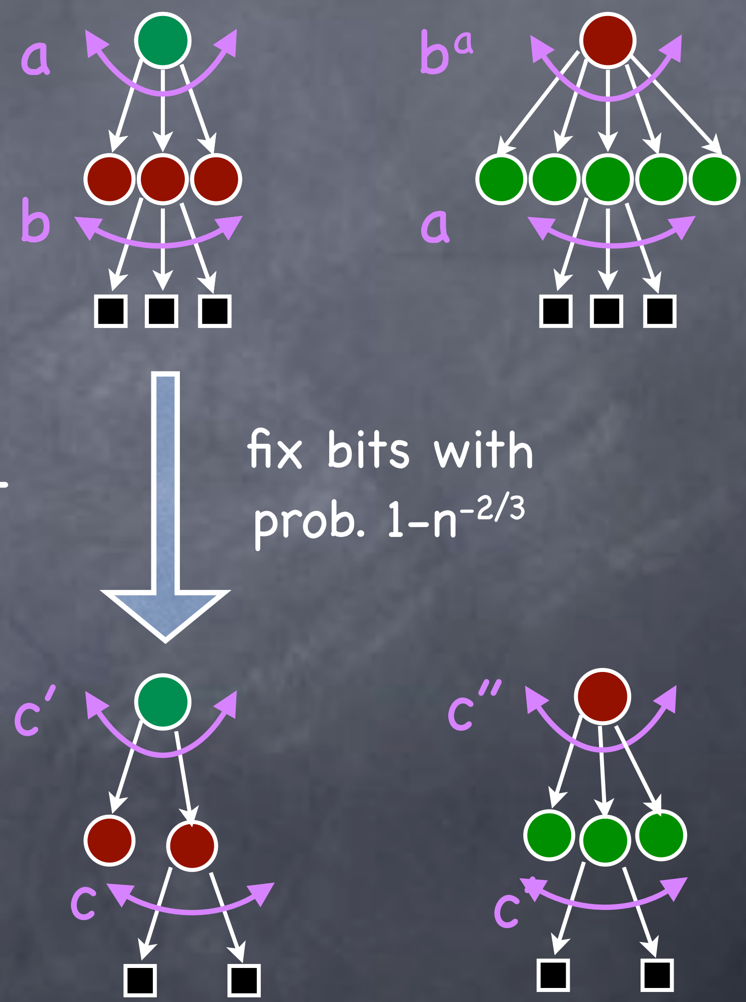

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  - Contradiction: started with minimal depth!

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  - Show that no low degree polynomial can agree with PARITY on that many inputs

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    - But high degree! Need OR to be simple!

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- Can boost probability by doing (exact) OR t times: deg < qt

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- Using $\varepsilon=1/(4s)$, degree $O(\log s)^d$ polynomial, correct w.p. $> 3/4$

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- Composing gate-polynomials into circuit-polynomial
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  - Degree multiplies: depth d circuit gives deg $O(\log 1/\varepsilon)^d$
  - Error adds (by union bound): size s circuit gives error $< s\varepsilon$
- Using $\varepsilon=1/(4s)$, degree $O(\log s)^d$ polynomial, correct w.p. $> 3/4$
  - One polynomial, correct on $> 3/4$ fraction of inputs (why?)

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    - Need $D = \Omega(\sqrt{n})$ to have enough degree D+n/2 polys.

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- $\log(s) = \Omega(\sqrt{n})^{1/d}$ or $s = 2^{\Omega(n)^\wedge(1/2d)}$ : i.e., if depth is constant then size not poly (in fact exponential)

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  - To show that f has no poly-sized monotone circuit family
  - Still possible that f may have a more efficient non-monotone circuit family (or even be in P)

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  - Pruning uses "sunflower lemma": find a sunflower and replace petals by core

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- A circuit with only approximation gates errs on a large number of the examples
  - If output identically "No" then errs on entire Yes set. Else, output wire's label has some subset X, $|X| \leq t = O(\sqrt{k})$, and then a constant fraction of No-examples get accepted