Decision Trees

Lecture 23

To left or to right

Decision Trees

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  - Number of bits of input read

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    - Simpler combinatorial structure (need not understand P vs. NP etc.)

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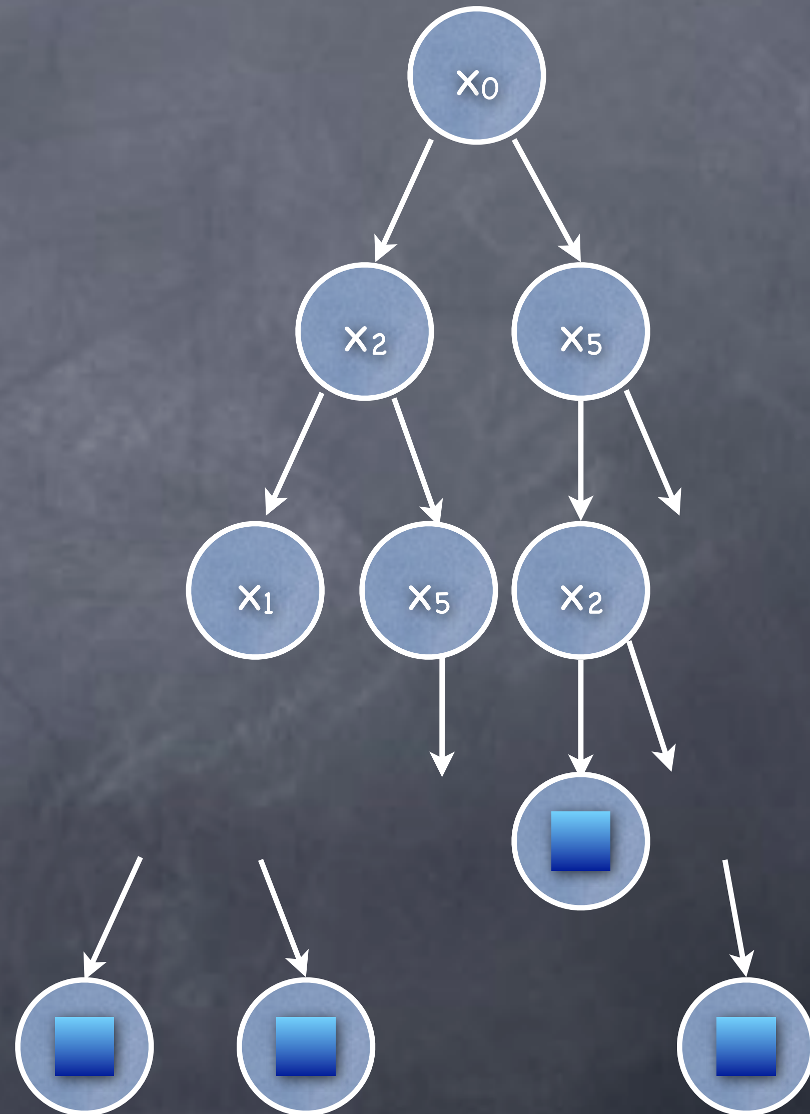

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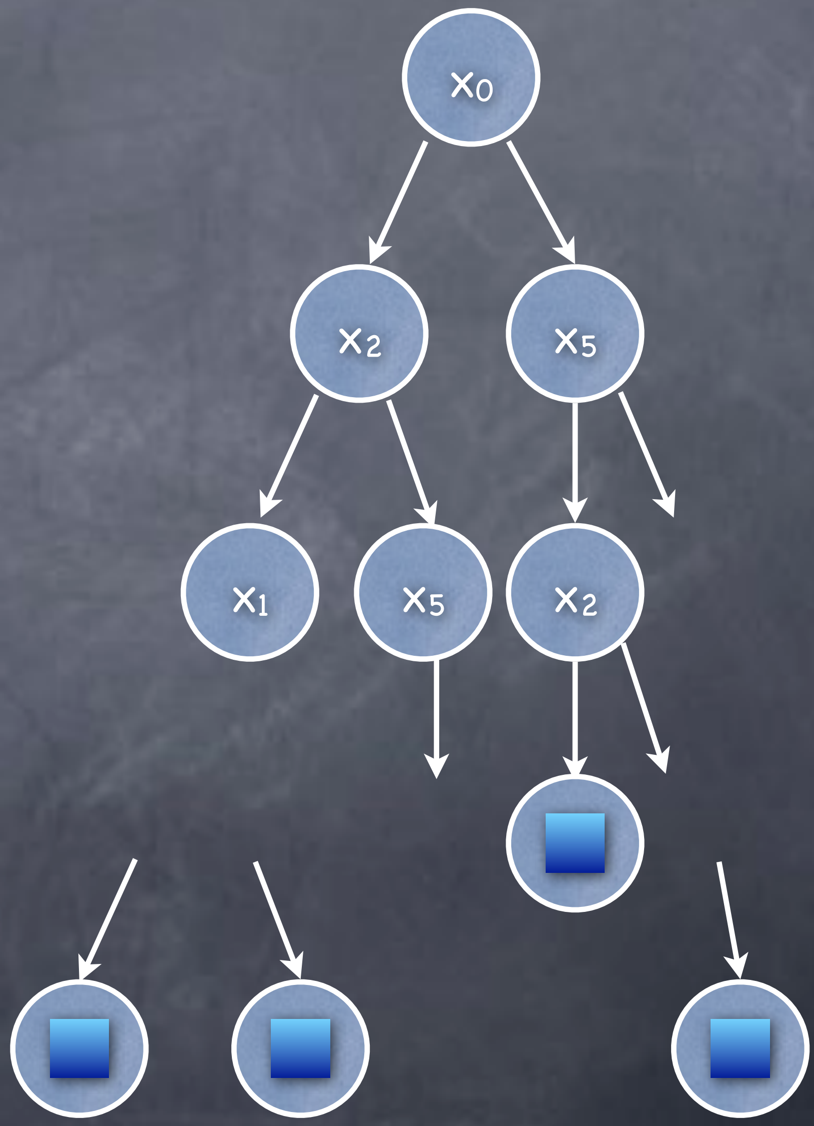

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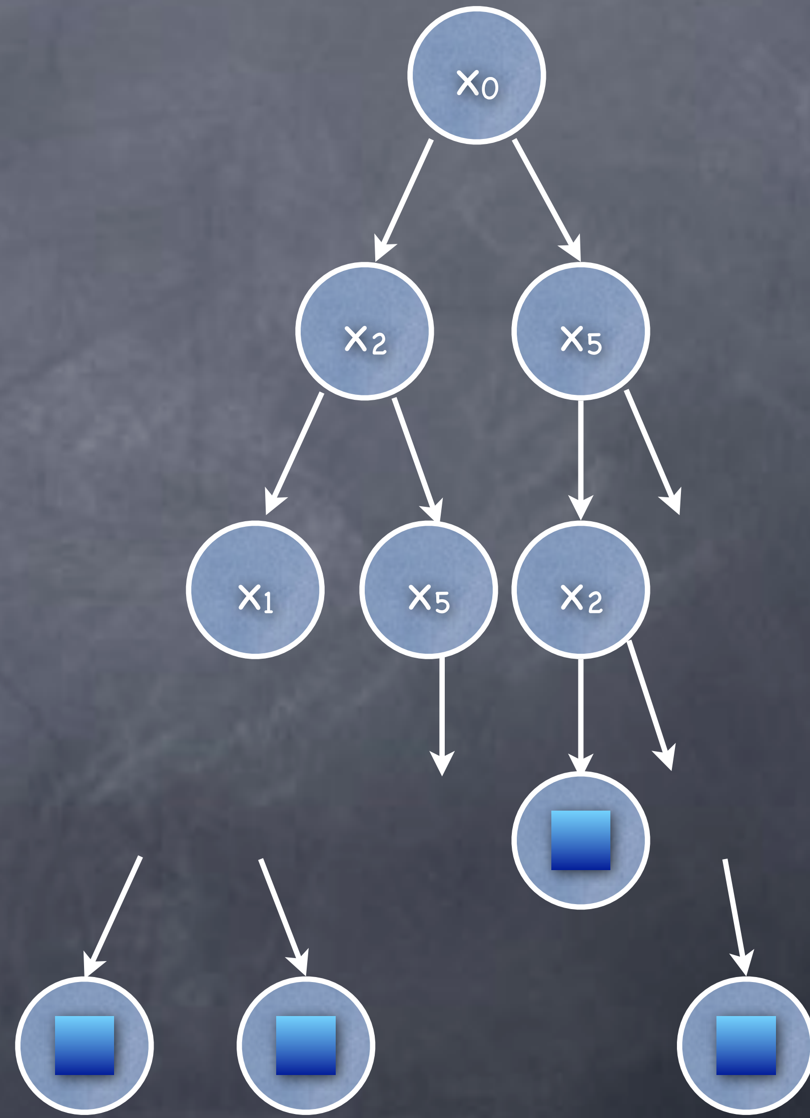

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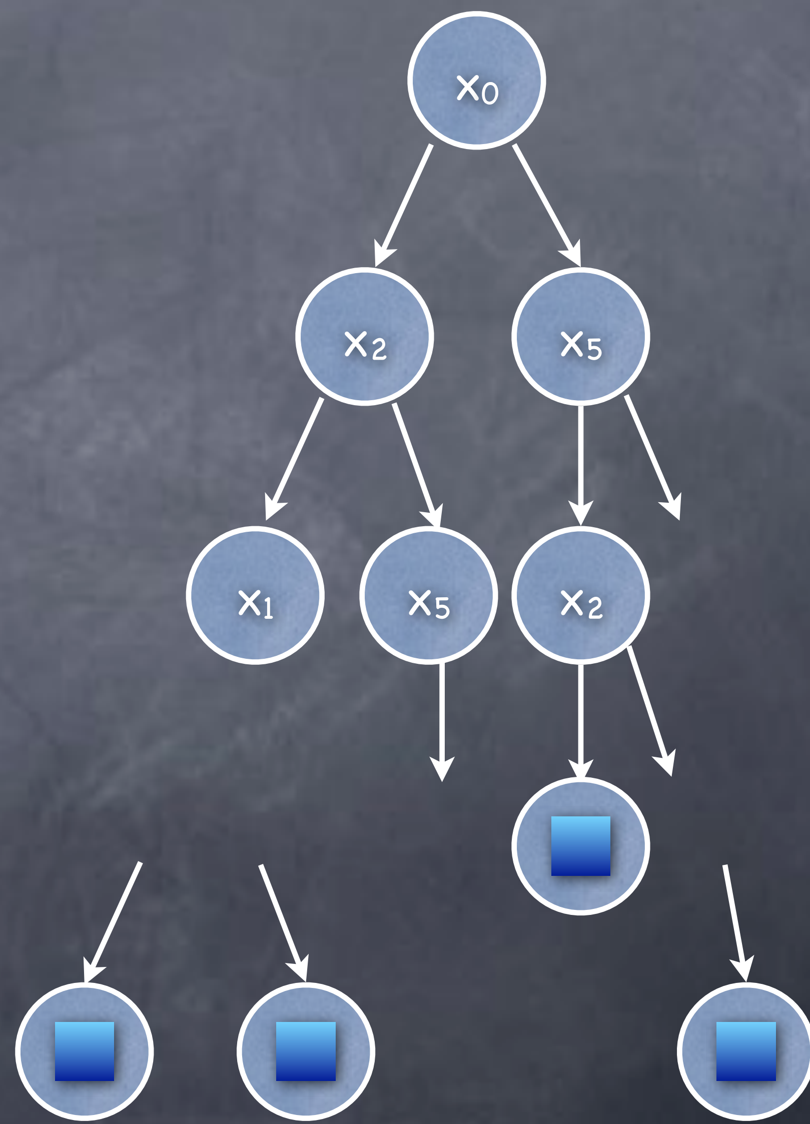
- DTree(L) = $\min_{\text{alg } A} \max_{\text{input } x} T_{A,x}$ where $T_{A,x}$ is the number of bits of x read by A



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  - CONNECTED(G) = 1 if G is the adjacency matrix of a connected graph
- We are interested in showing DTree lower-bounds for these problems

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    - Before n nodes, set of inputs contain 0^n and another input, no matter what bits where queried at the nodes

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  - Until then, graph can be connected or disconnected: by setting all unqueried edges to Yes or all to No

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  - $\mathrm{Maj}(x) = 1$ iff #1s in x > #0s (assume $|x|$ odd)
    - Adversary strategy: alternately answer 0 and 1

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    - Exercise

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- 1-Cert(L): $\max_{x \in L} \min_{c:\ x|c \Rightarrow x \in L} |c|$ (e.g. 1-Cert(OR) = 1)
- 0-Cert(L): $\max_{x \notin L} \min_{c:\ x|c \Rightarrow x \notin L} |c|$ (e.g. 0-Cert(OR) = n)

$$\mathrm{DTree}(L) \leq \mathrm{0Cert}(L) \times \mathrm{1Cert}(L)$$

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  - Clearly correct. Number of bits read?

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  - 0-certificate: enough variables so that can evaluate just one input wire for AND gates, and all input wires for OR gates
  - 1-certificate: enough variables so that can evaluate just one input wire for OR gates, and all input wires for AND gates
  - If regular AND-OR tree, 0Cert(L) x 1Cert(L) = number of leaves = DTree(L)

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  - "Sensitivity" is a lower-bound on DTree(L)
- Will explore some in exercises

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  - Flip all coins up front and then run a deterministic computation
    - i.e., randomly choose a (deterministic) decision tree

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- Question: How to prove lower-bounds against randomization?

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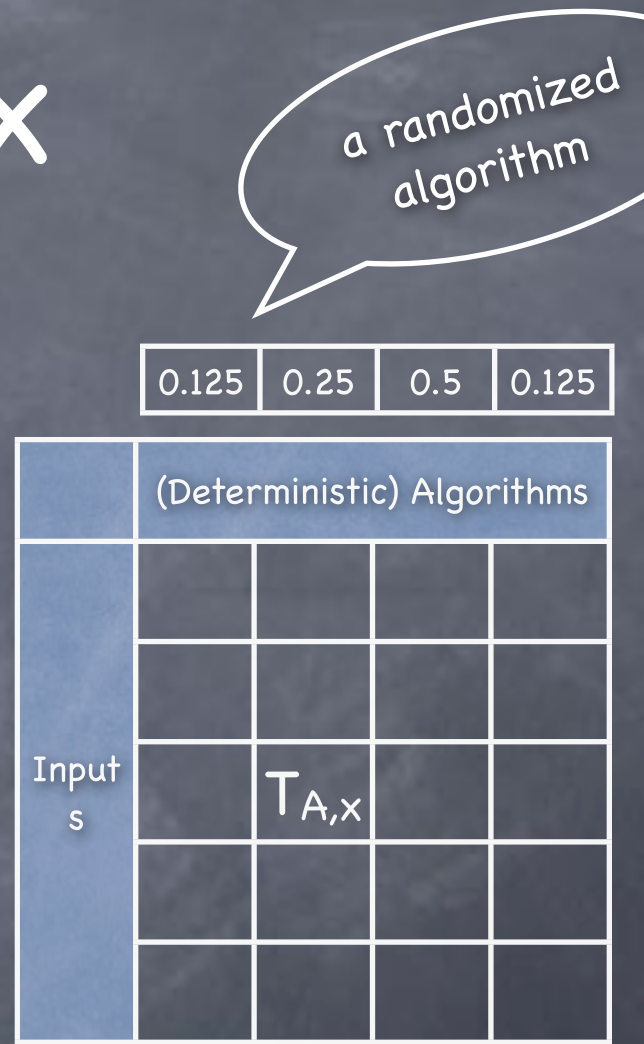

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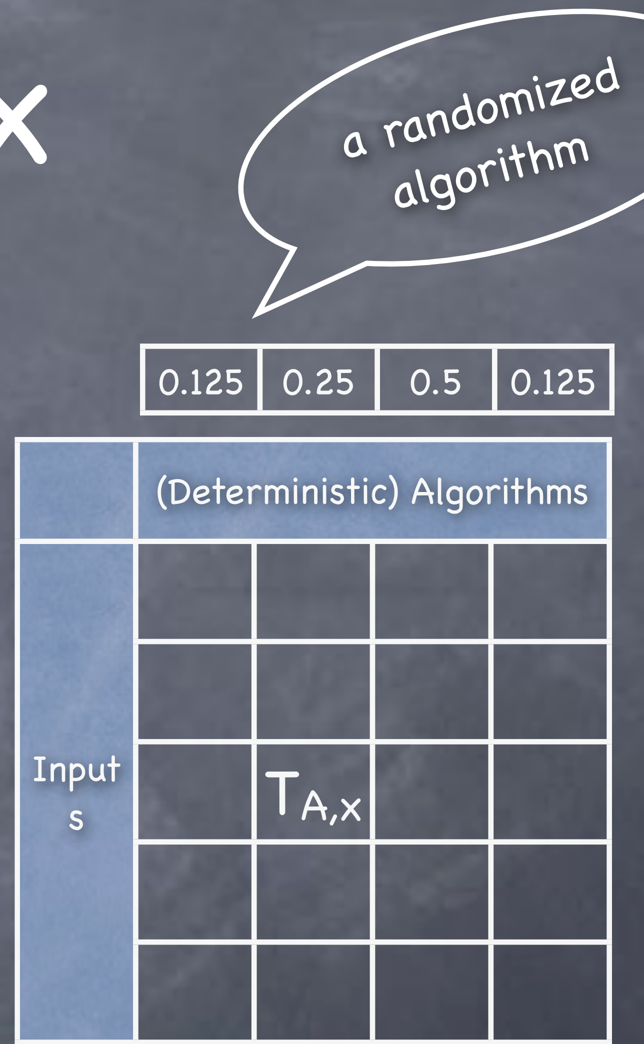

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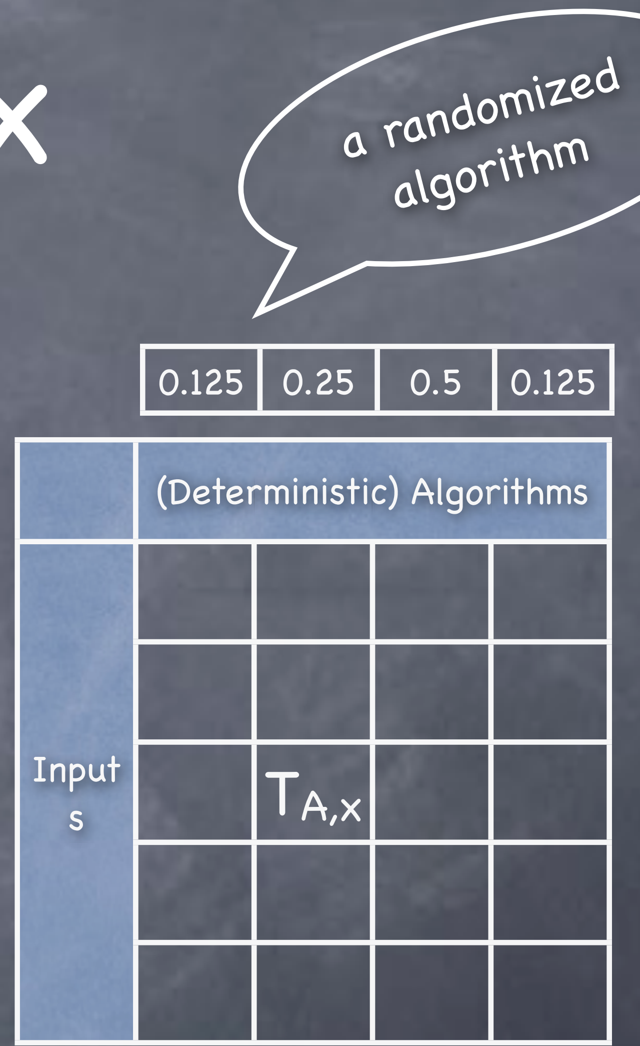

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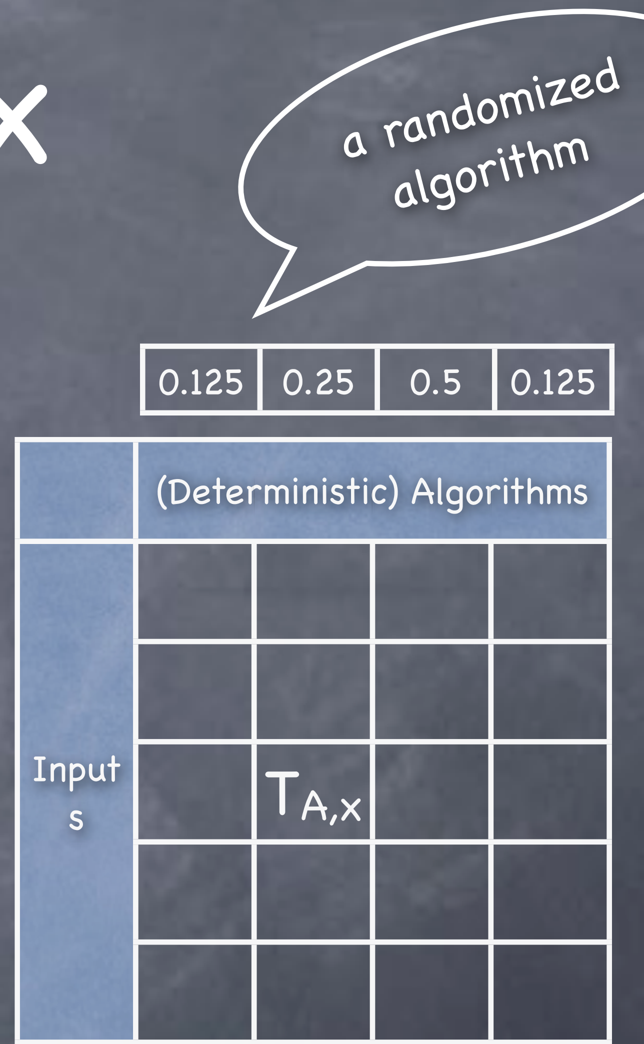
- Both have the same expected cost!! (not obvious: follows from LP duality)



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  - If every algorithm A performs badly on an **<u>input-distribution</u>** X, then a randomized combination of those algorithms also perform badly on X. If R does badly on X, on some x in its support it does at least as badly (x depends on R)
  - Useful: Can show lower-bound for randomized algorithms via lower-bound on distributional complexity for deterministic algorithms