# Complexity of Counting

Lecture 21

#P

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  - Computed by a TM running in polynomial time

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    - e.g.: Number of satisfying assignments to a boolean formula
    - e.g.: Number of inputs in a language L that are less than x (lexicographically)

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- Easy to see: FP $\subseteq$ #P (interpreting numbers as strings suitably) [Exercise]

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      - HAMILTONICITY(G) ⇔ #CYCLES(G) $\geq n^{n\hat{}2}$

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    - $\#P \subseteq FP^{PP}$ [exercise] (and $PP \subseteq P^{\#P}$ [why?])
    - So if PP = P, then #P = FP (and vice versa)

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    - Permanent (for binary matrices) is #P-complete

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      - $\mathrm{Perm}(A) = \Sigma_\sigma W(\sigma)$ over all cycle covers $\sigma$ of directed graph $G_A$ (with edge-weights from A)

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  - Almost Karp-reduction (need to rescale)

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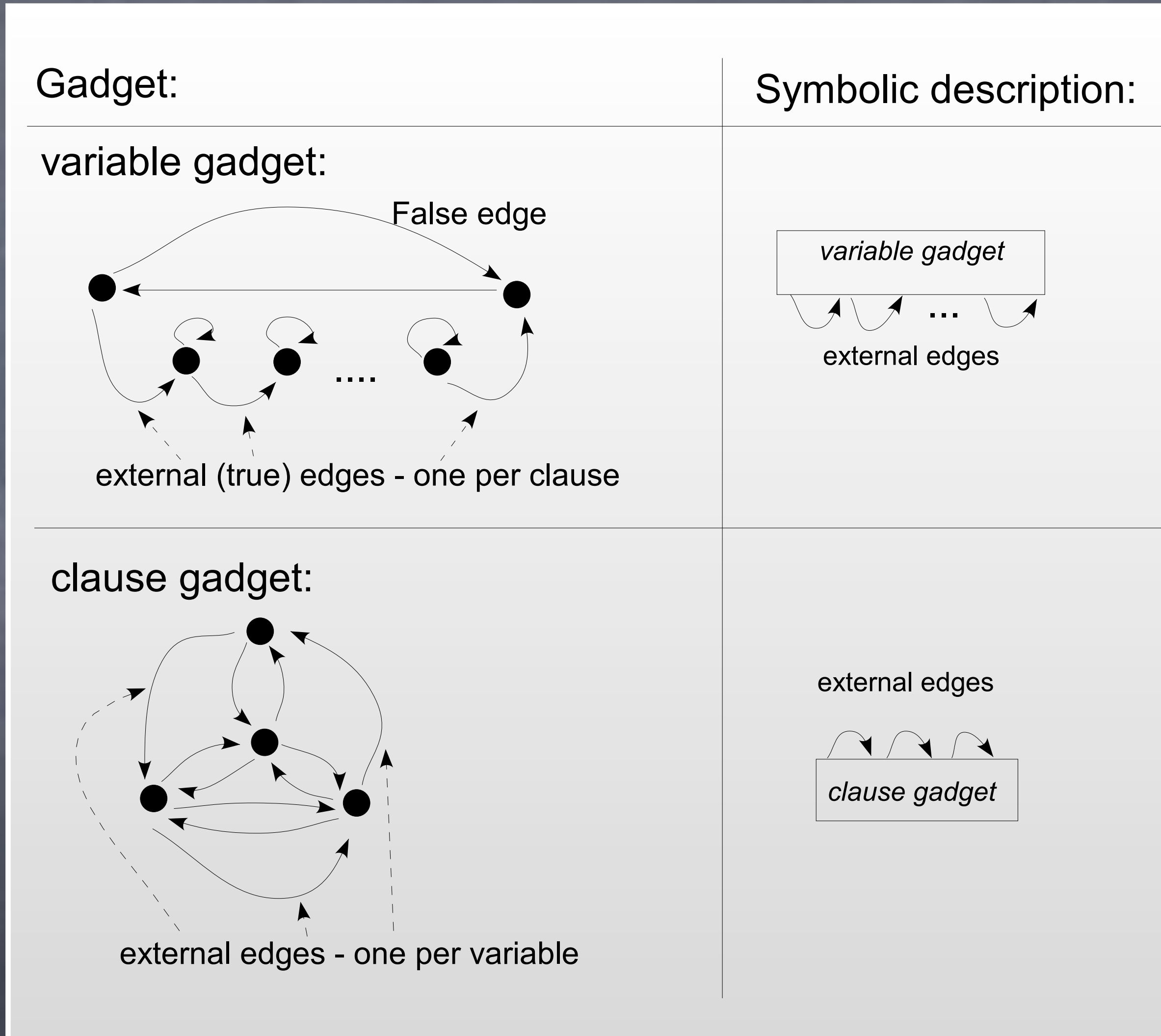


[Figures from the textbook]

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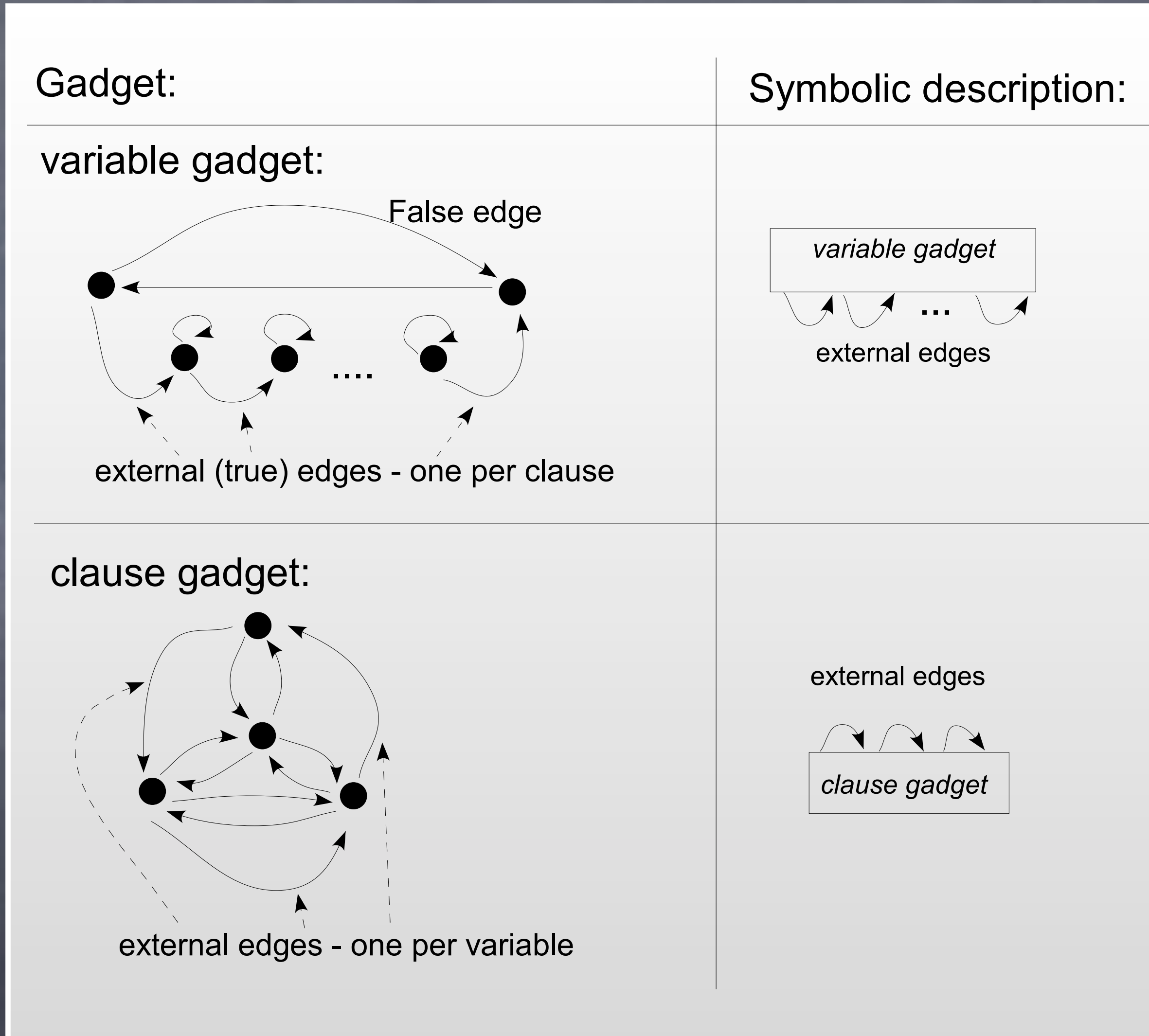


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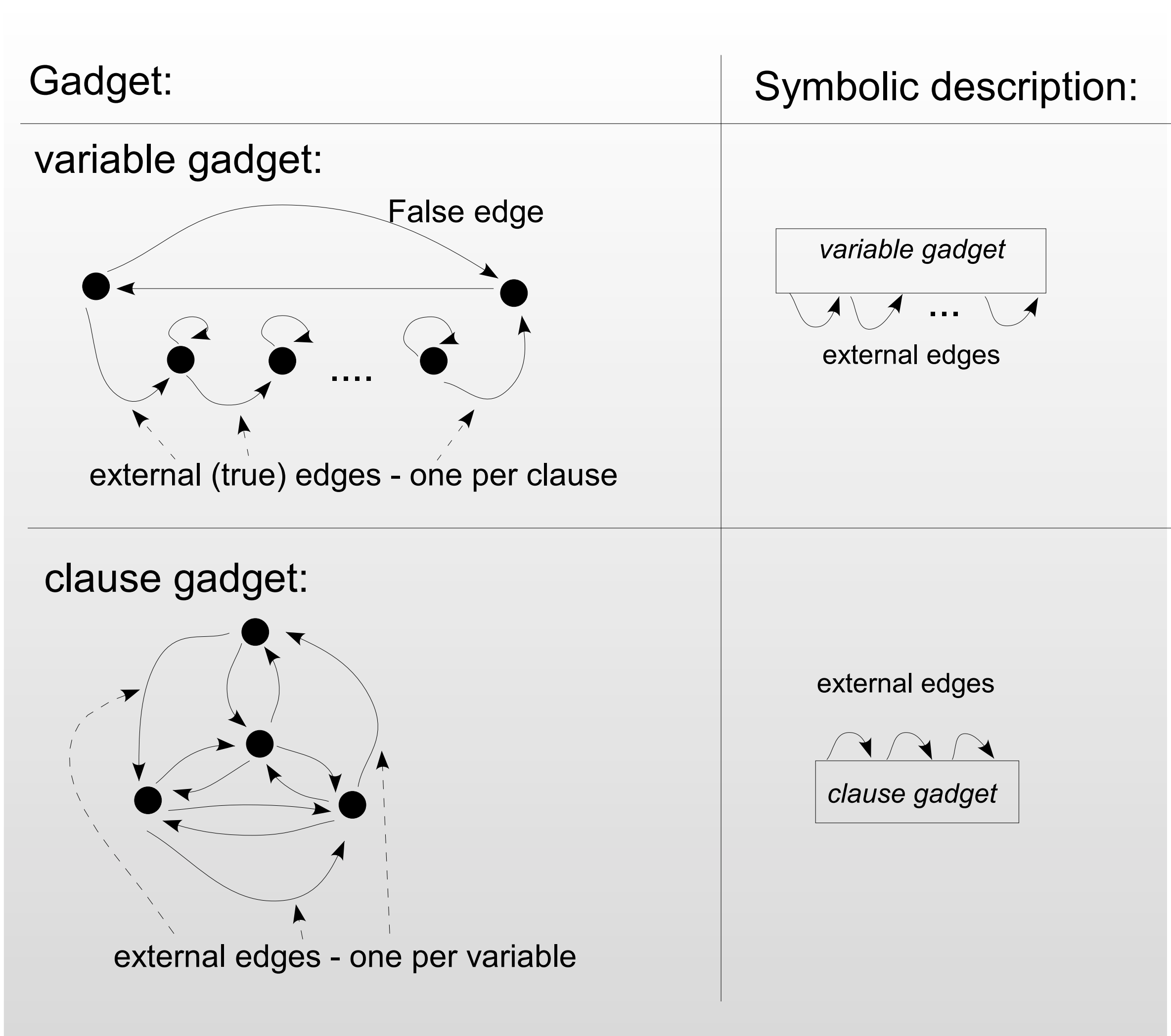


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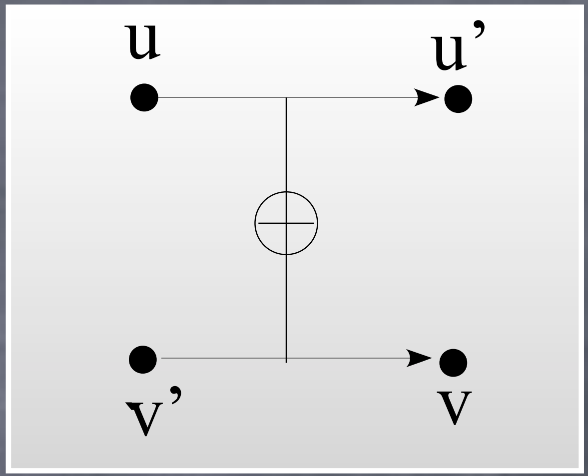

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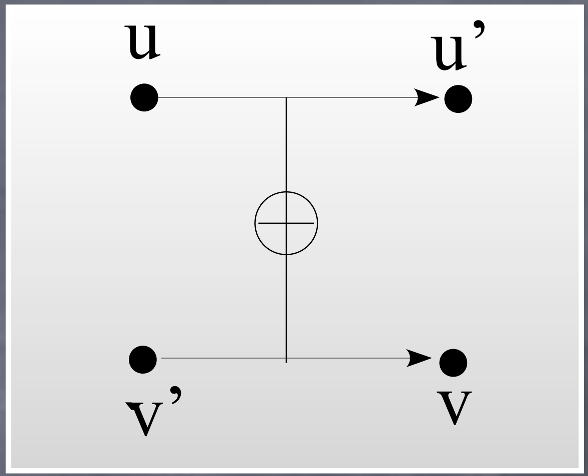

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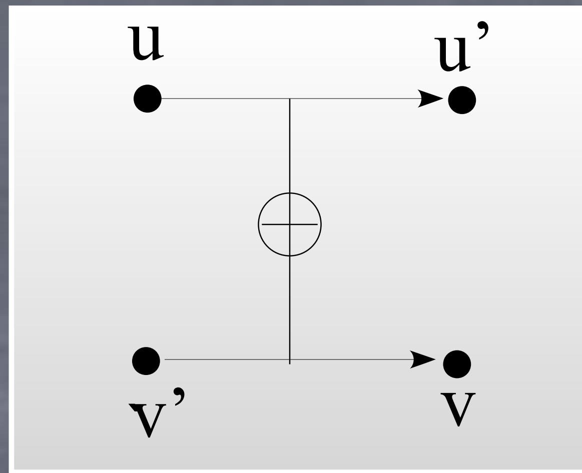


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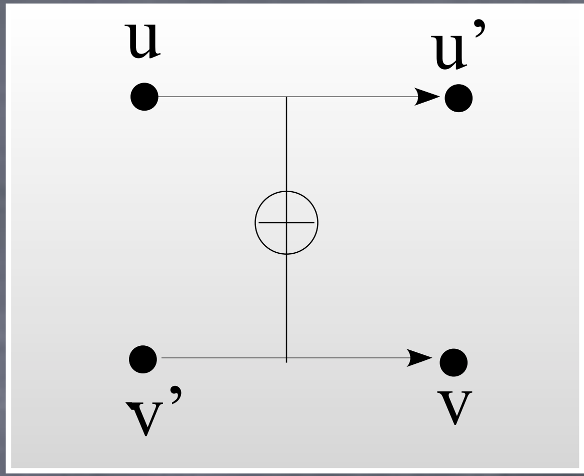


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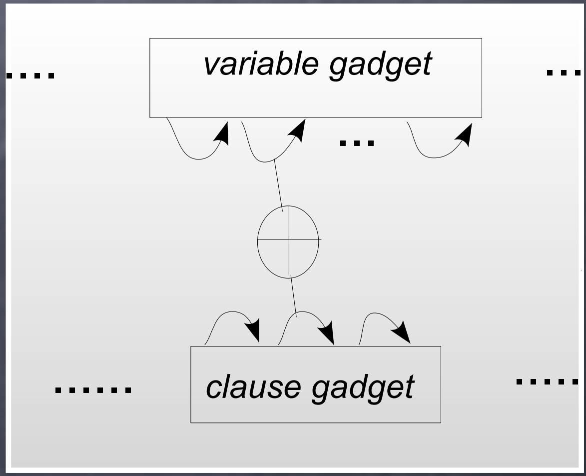

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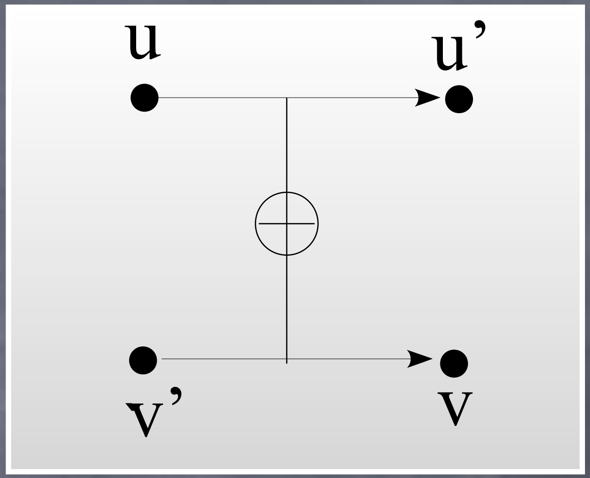


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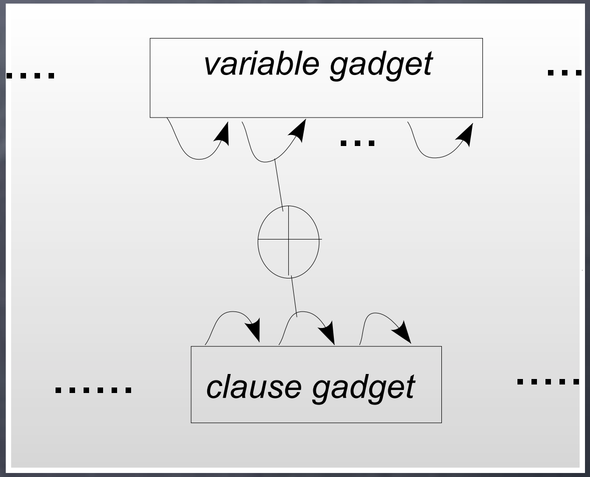

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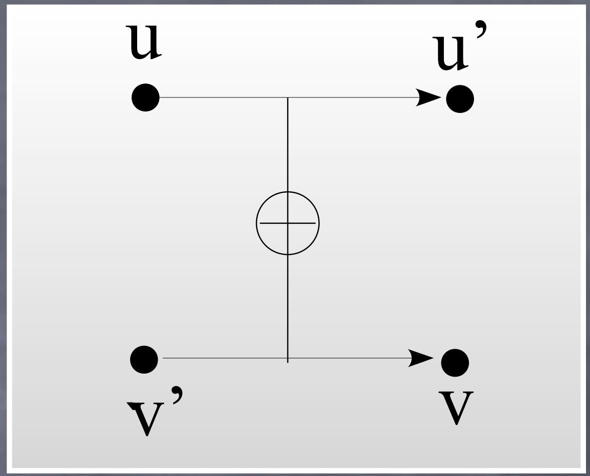


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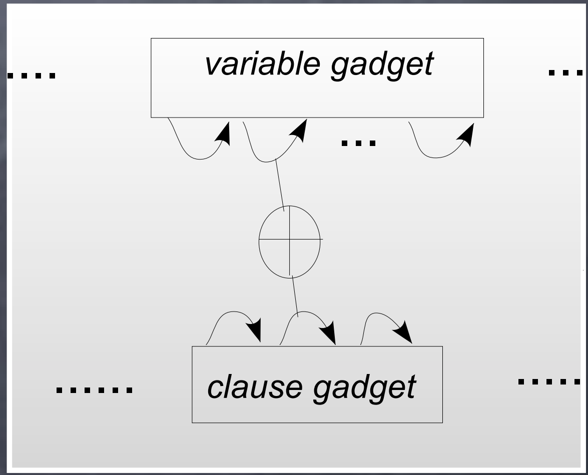

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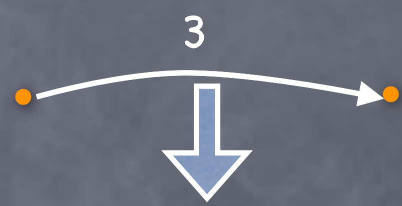

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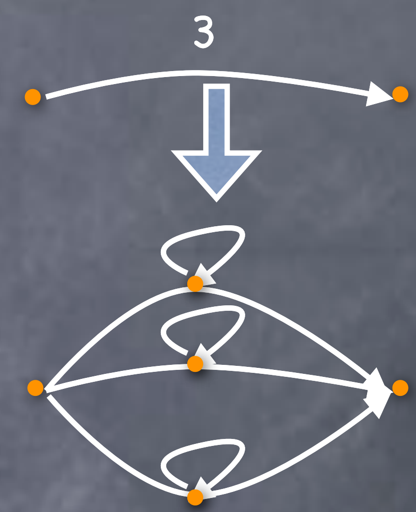

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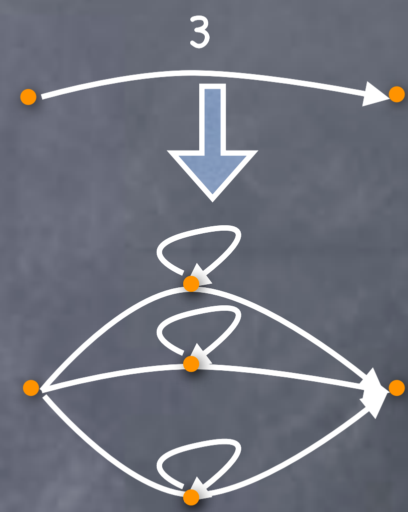

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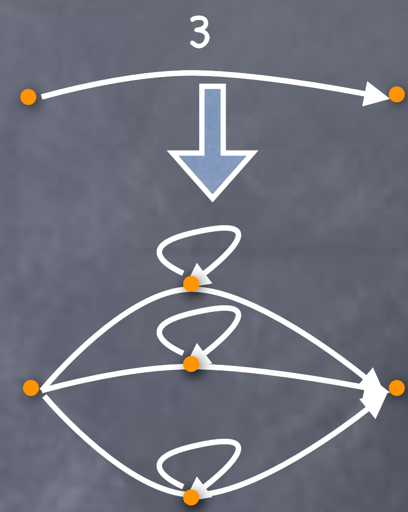

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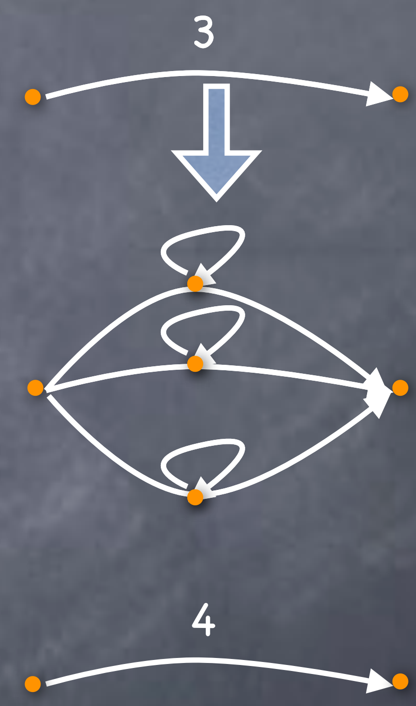

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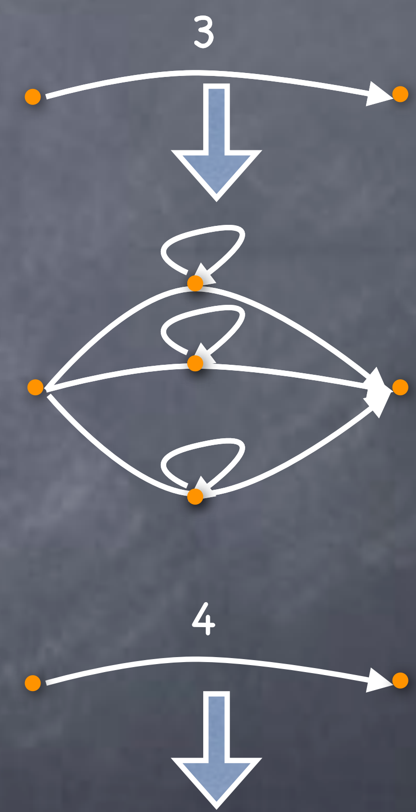
    - Also, let M be a power of 2 (M= $2^k$). Replace M by log M edges of weight 2 in series, each further replaced by +1 weight edges



# Permanent is #P-complete

- Can use binary matrix instead of integer matrix
    - First change to +1/-1 weights (adding vertices)
    - To replace -1: working modulo M+1 for large M (say M > $2^{n^2}$) does not change positive values. -1 is then M.
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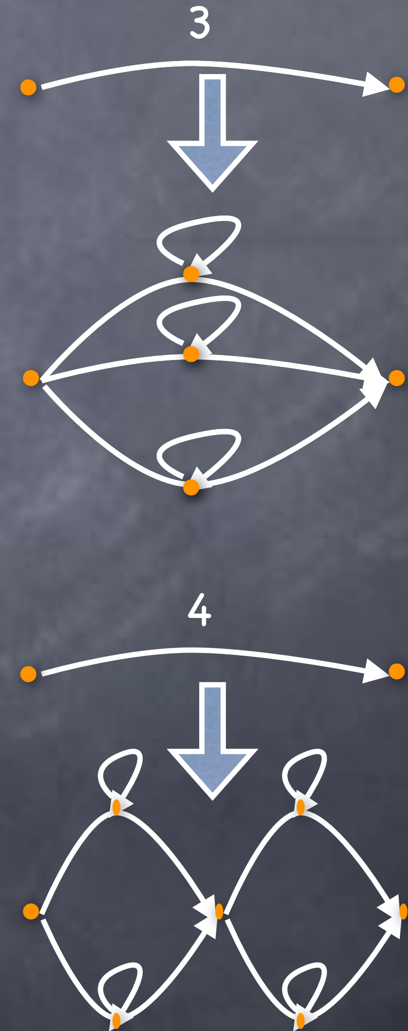

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- Next: Toda's Theorem: $PH \subseteq P^{\#P} = P^{PP}$