PCP

Lecture 20

And Hardness of Approximation

Promise Problems

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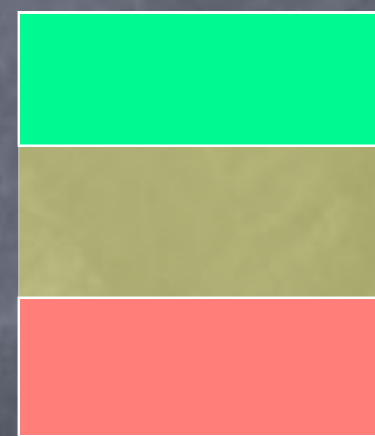

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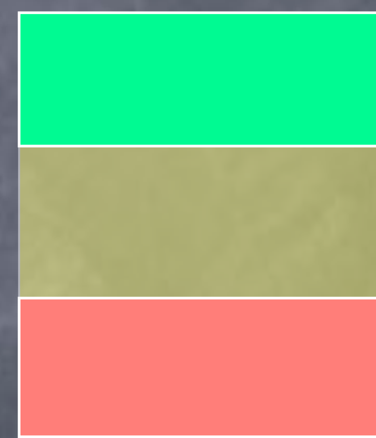

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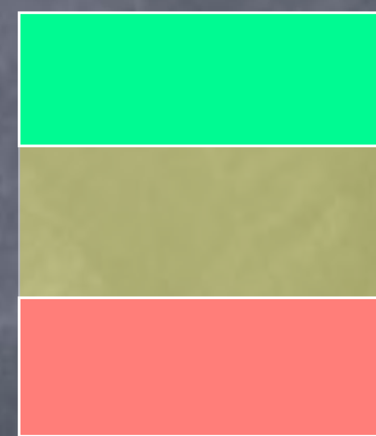

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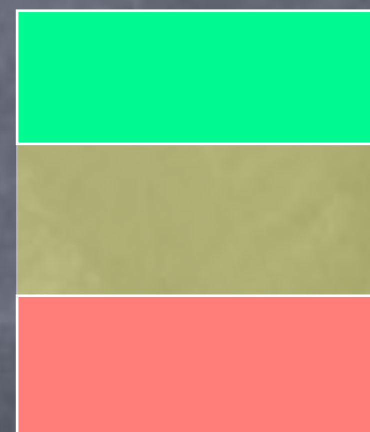
  - For inputs outside the two, don't care
    - We're "promised" that such inputs are not given



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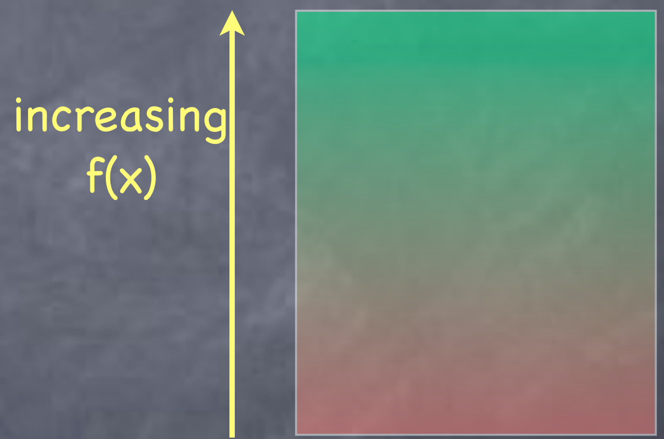

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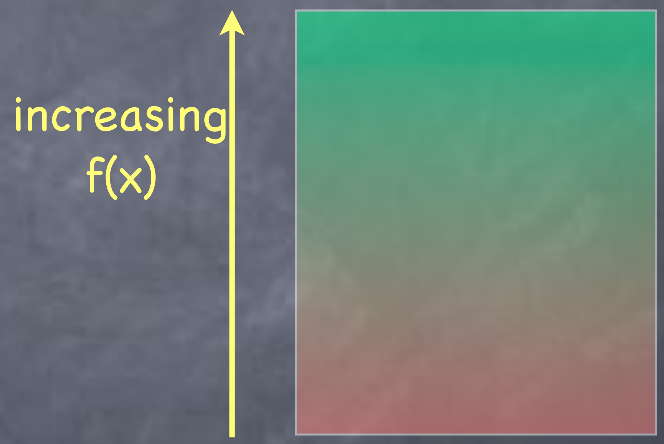

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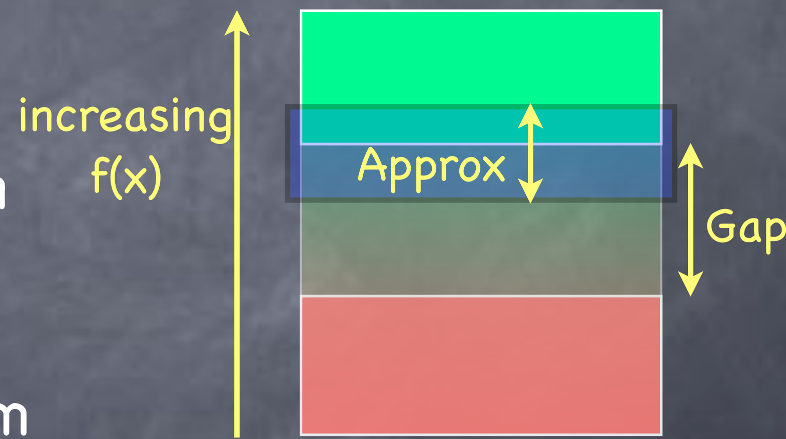
  - The more the gap the more loose the approximation can be



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  - Proof can be limited to be at most $q2^r$ bits long

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  - PCP(log m,q): q-query (non-adaptive) verifier, tosses at most log m coins

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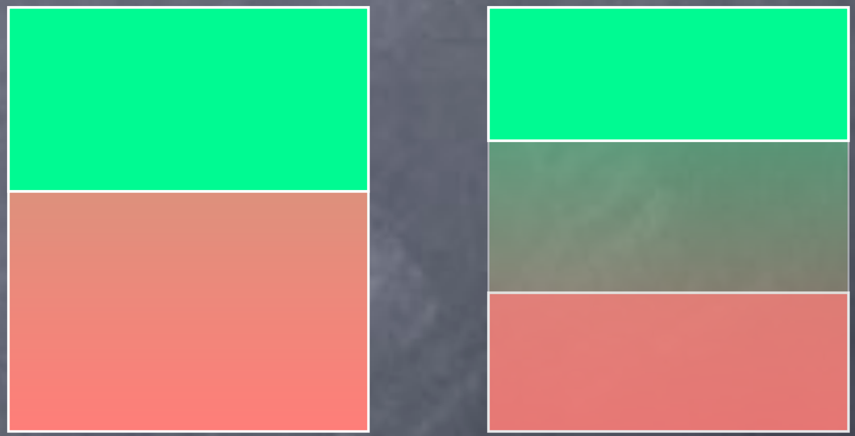

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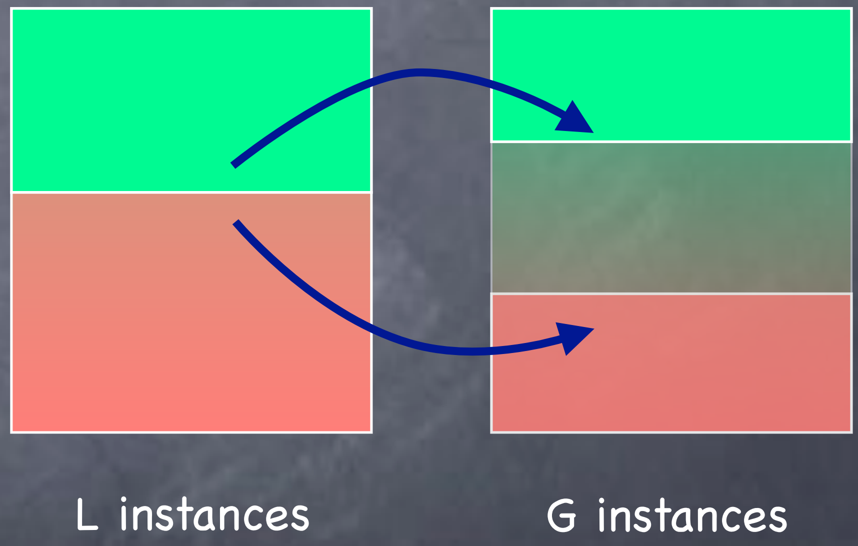

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- Note: PCP(log n, poly) $\subseteq$ NP
  - So, NP = PCP(log n, 1)

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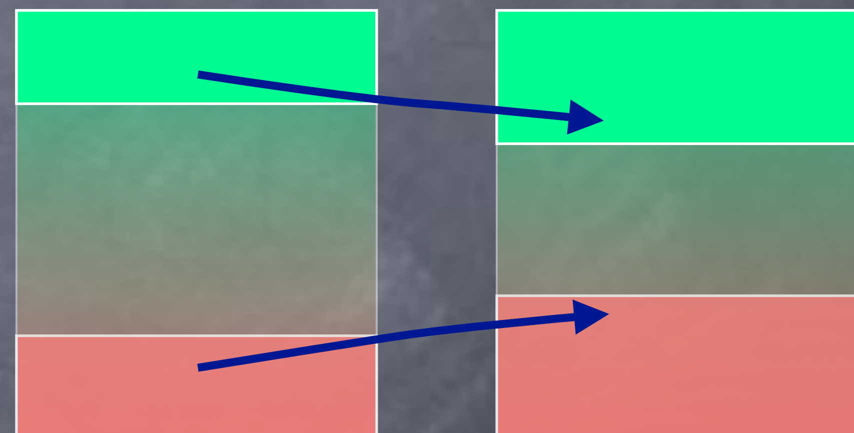

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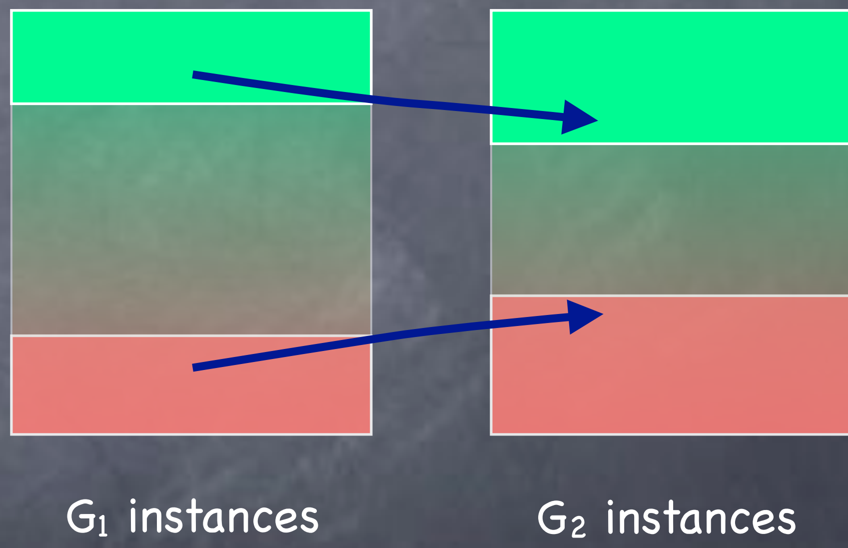
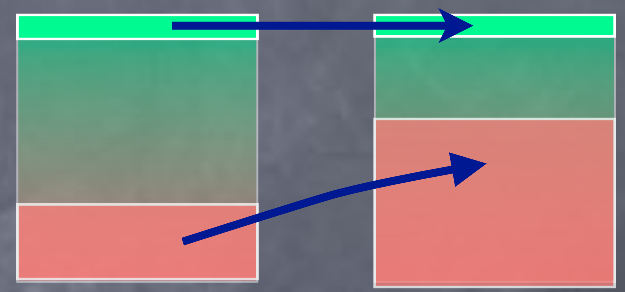


Max-qCSP to Max-3SAT



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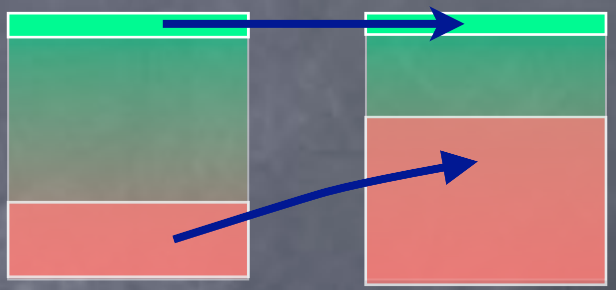

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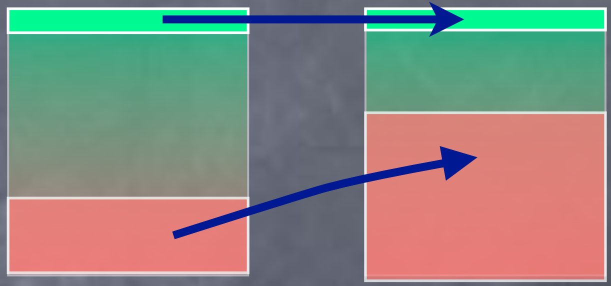

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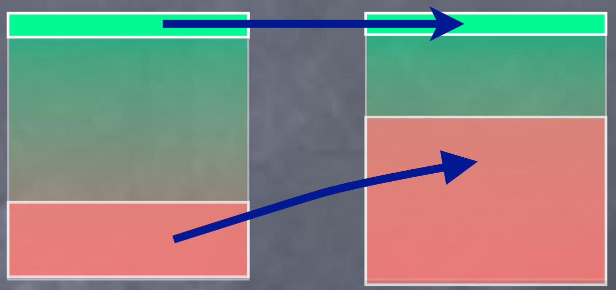

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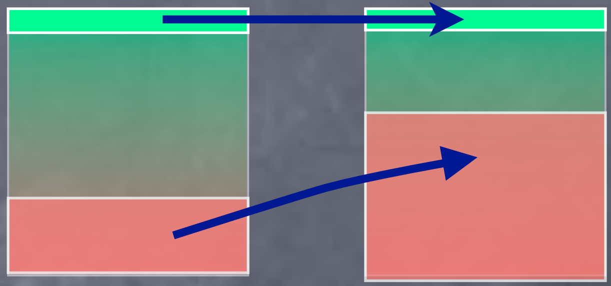

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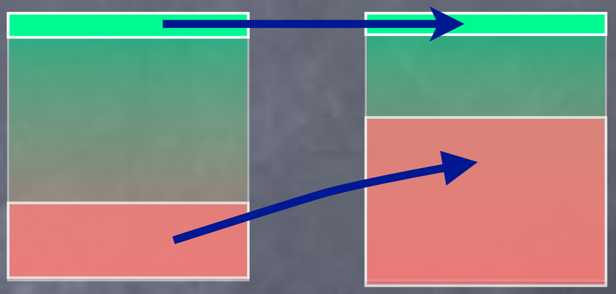

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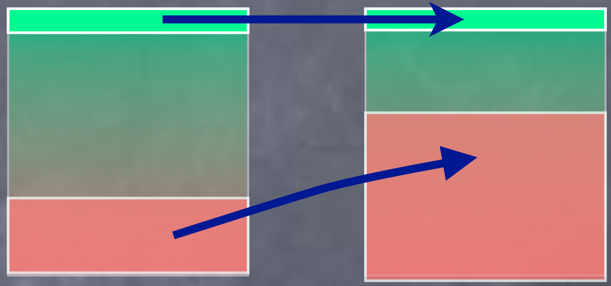

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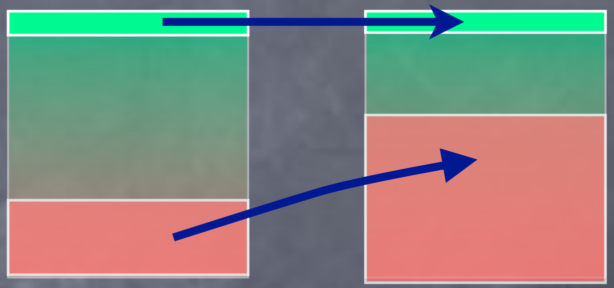

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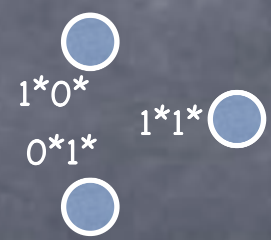

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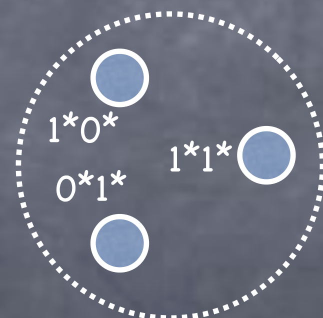

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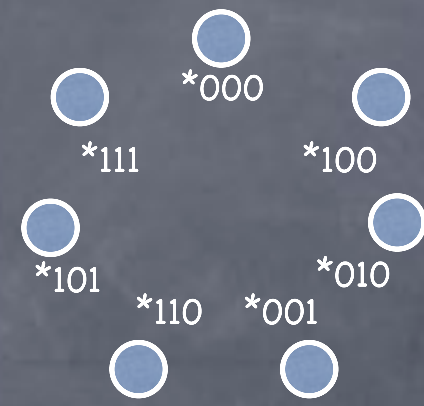

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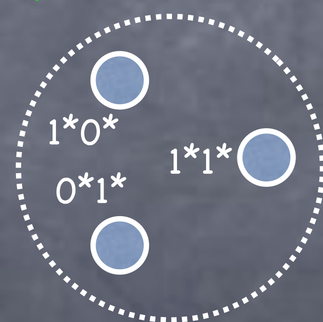

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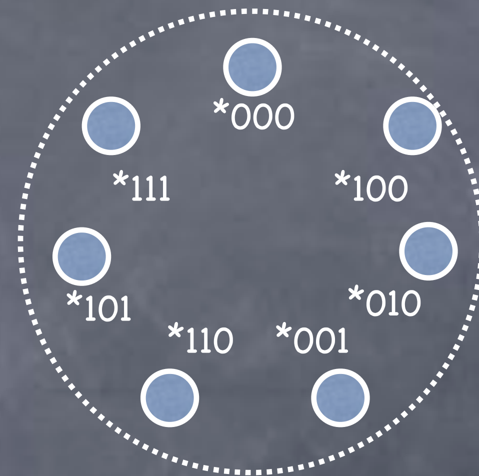

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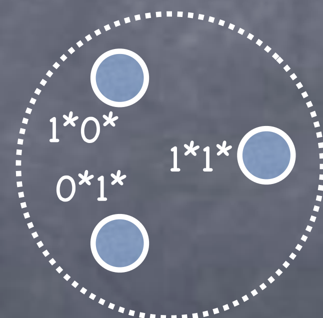

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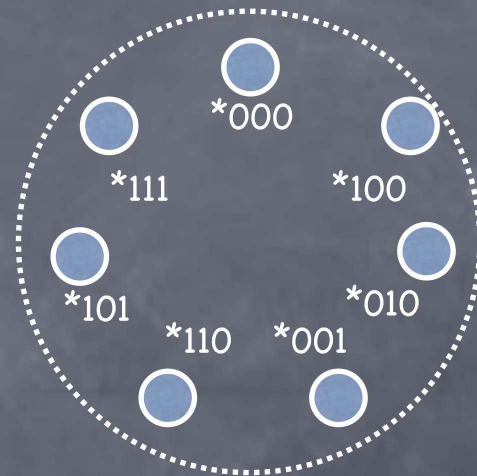

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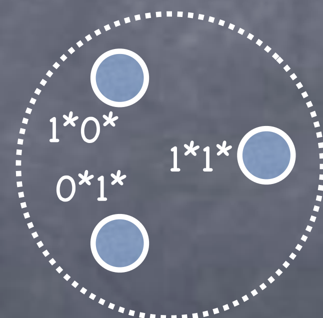

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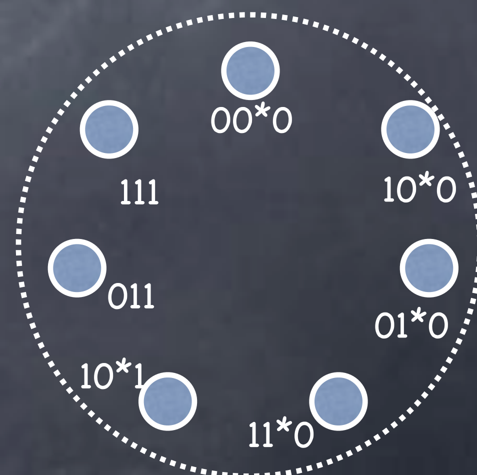

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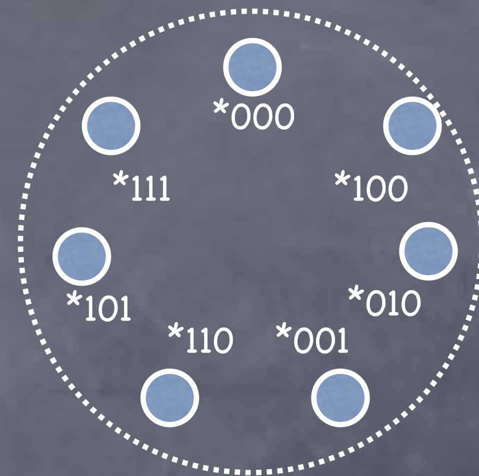
  - vertices: each clause's sat assignments (for its variables)
  - edges between consistent assignments

$(x \vee \neg y \vee \neg z)$



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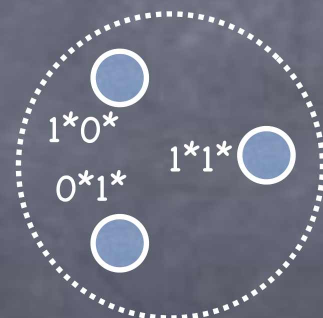

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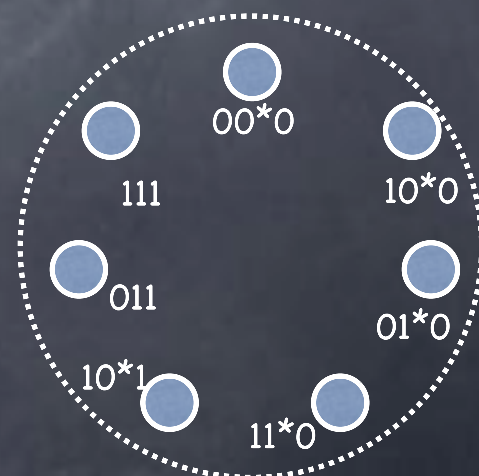

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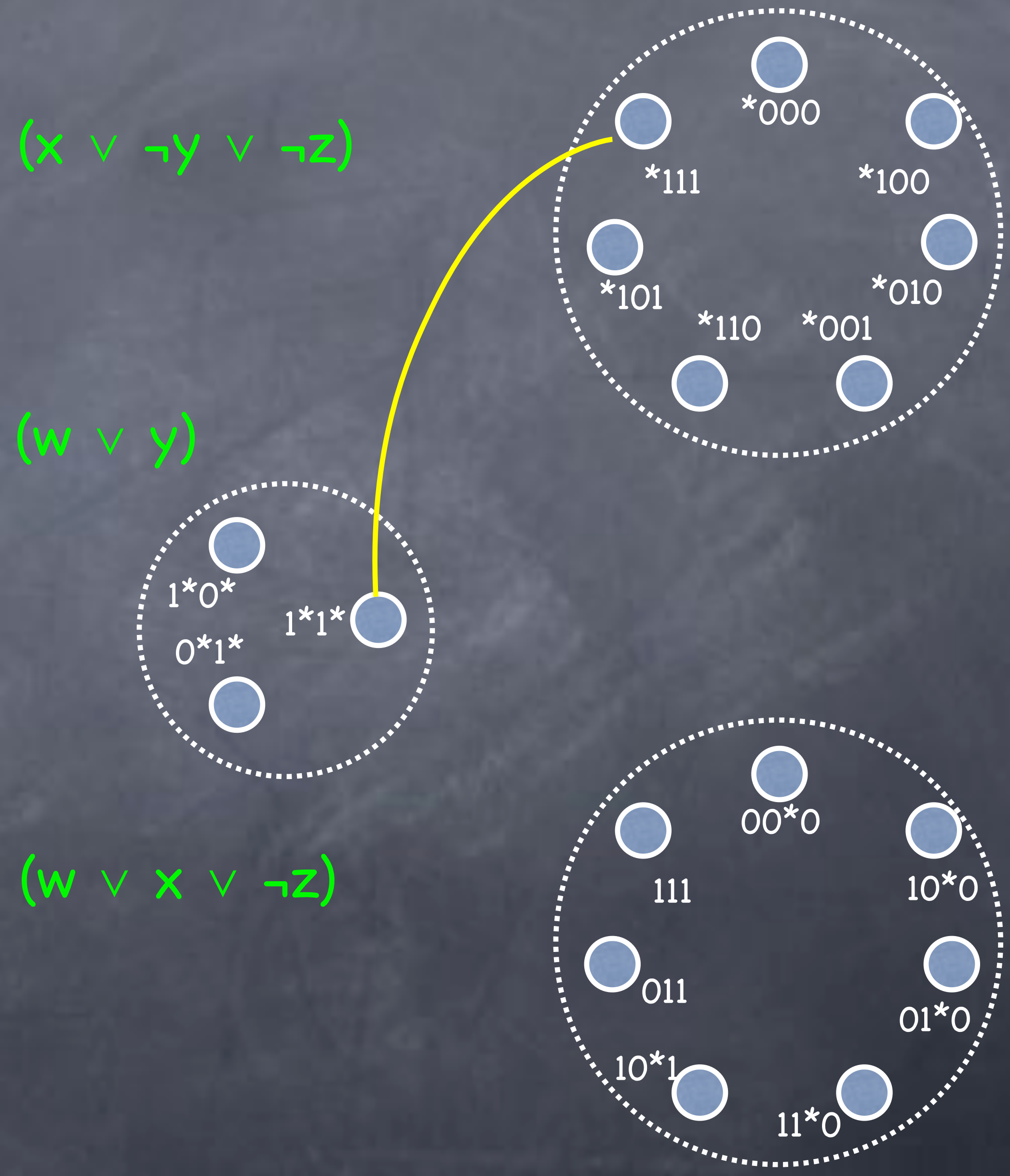

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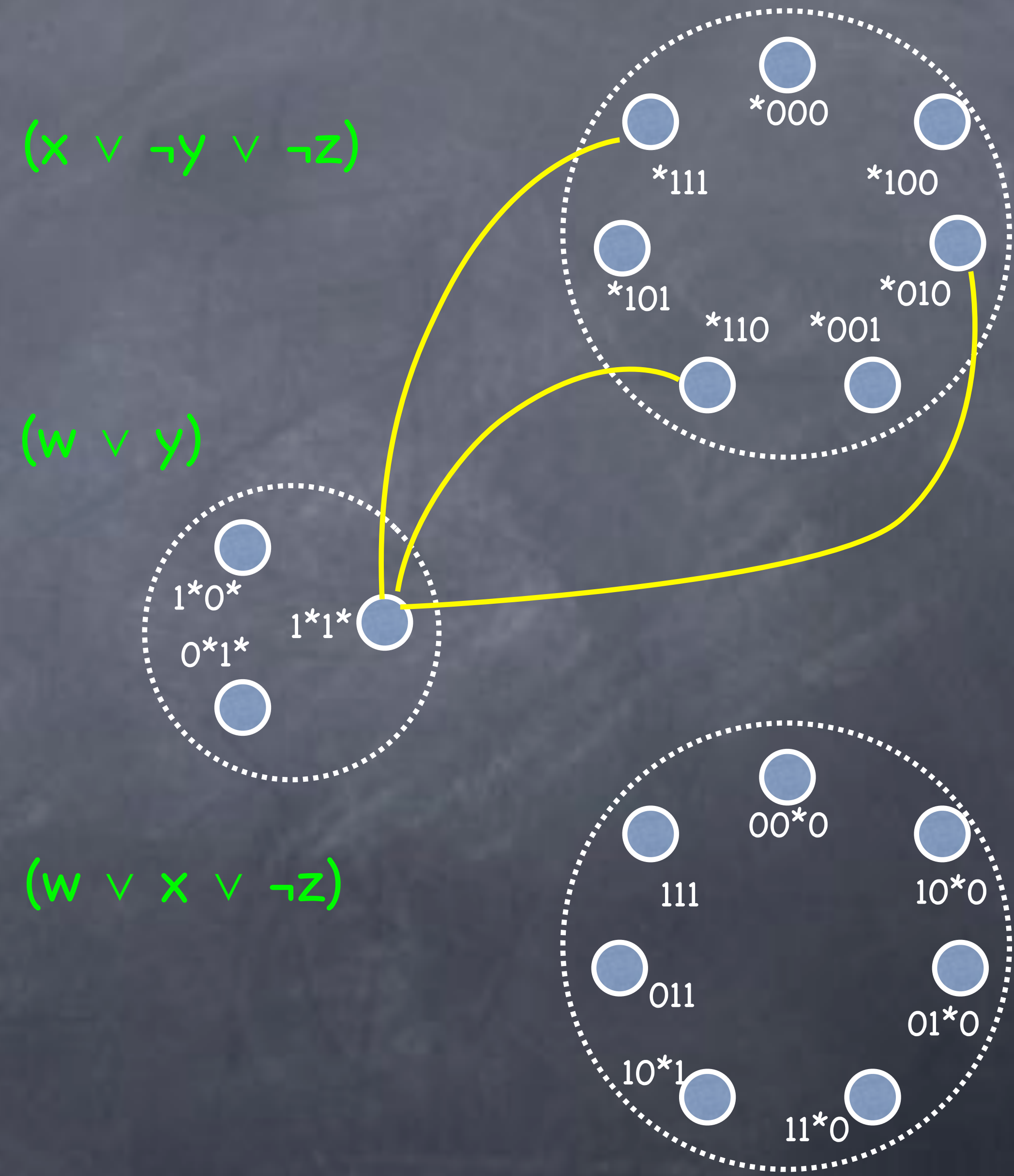

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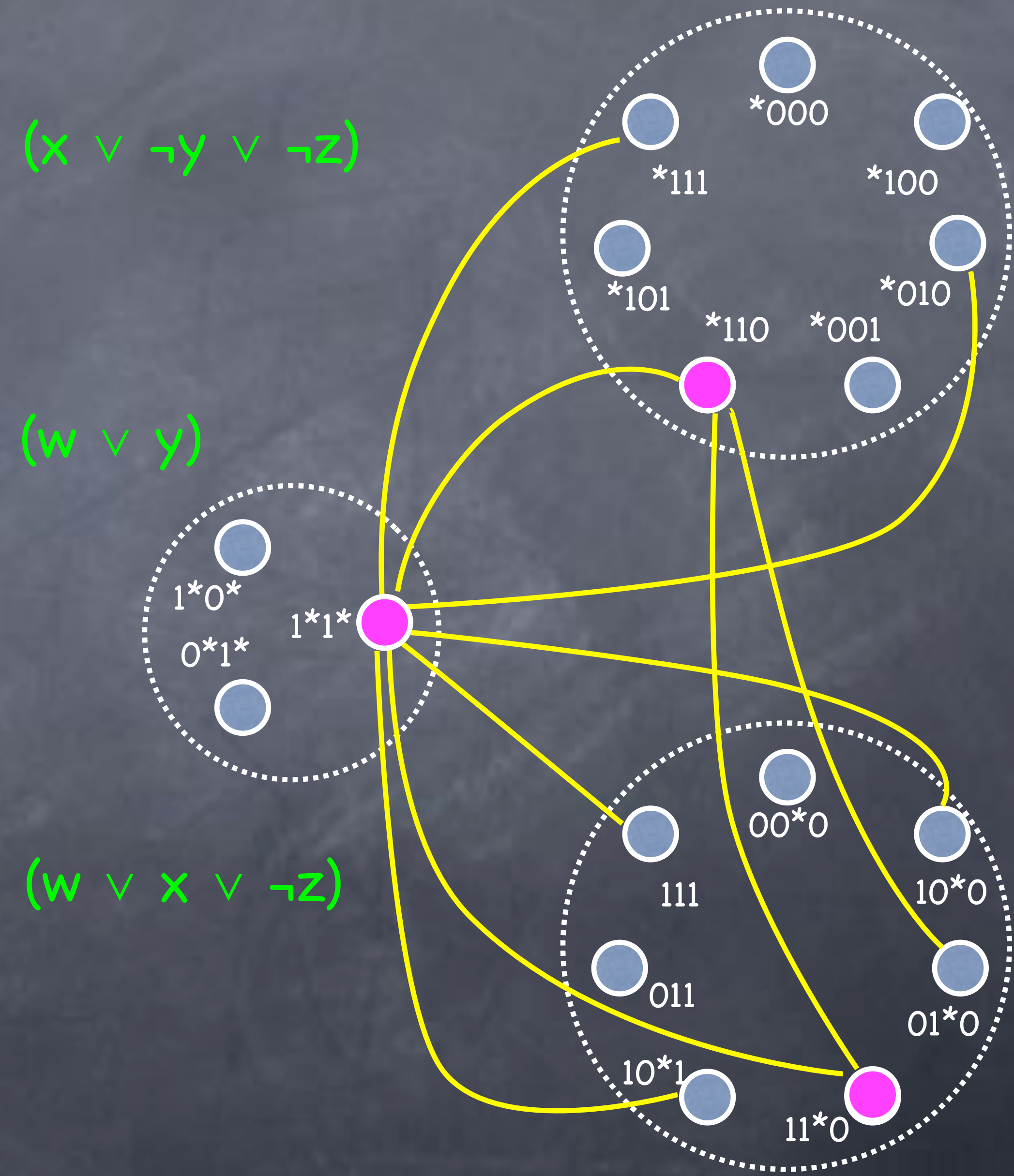

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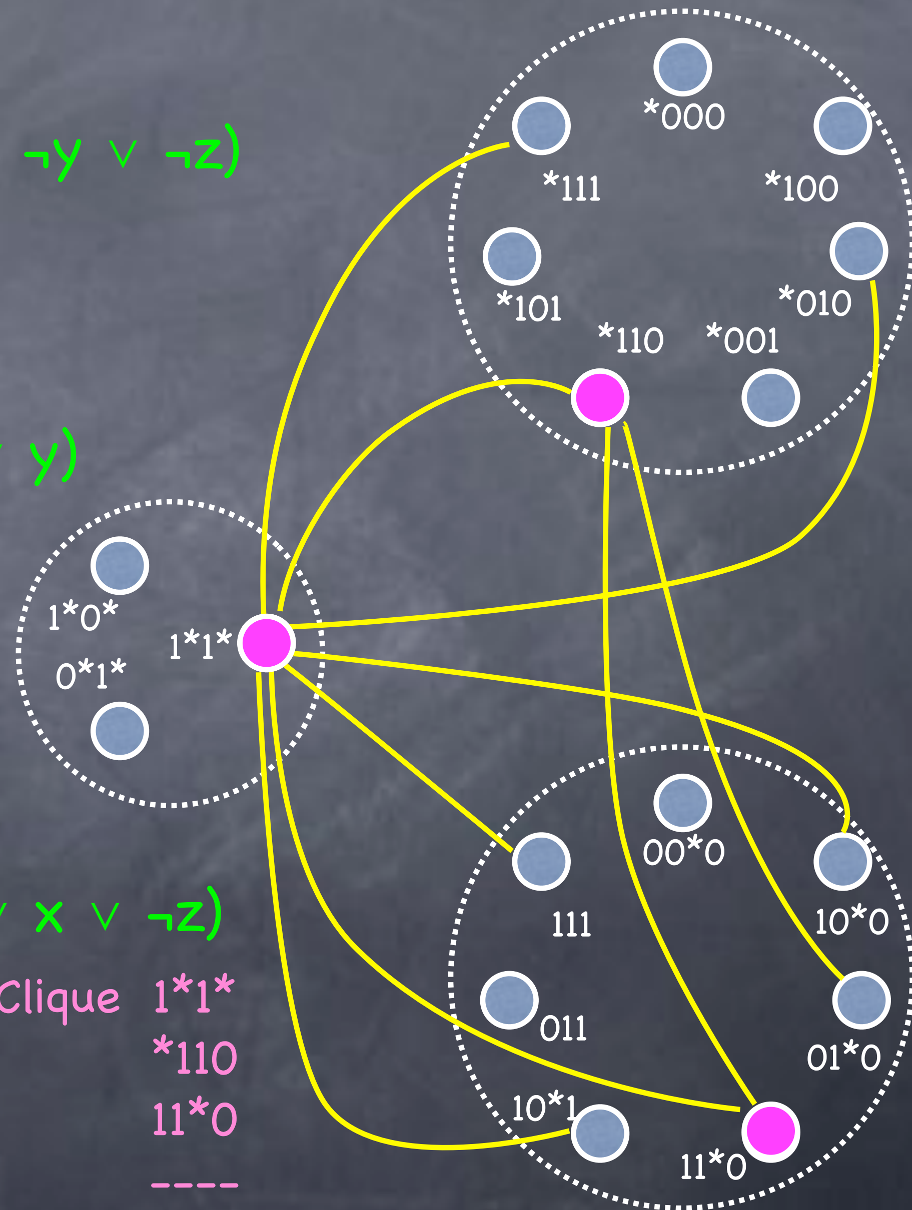

3-Clique 1*1*
*110
11*0
----
sat assignment 1110

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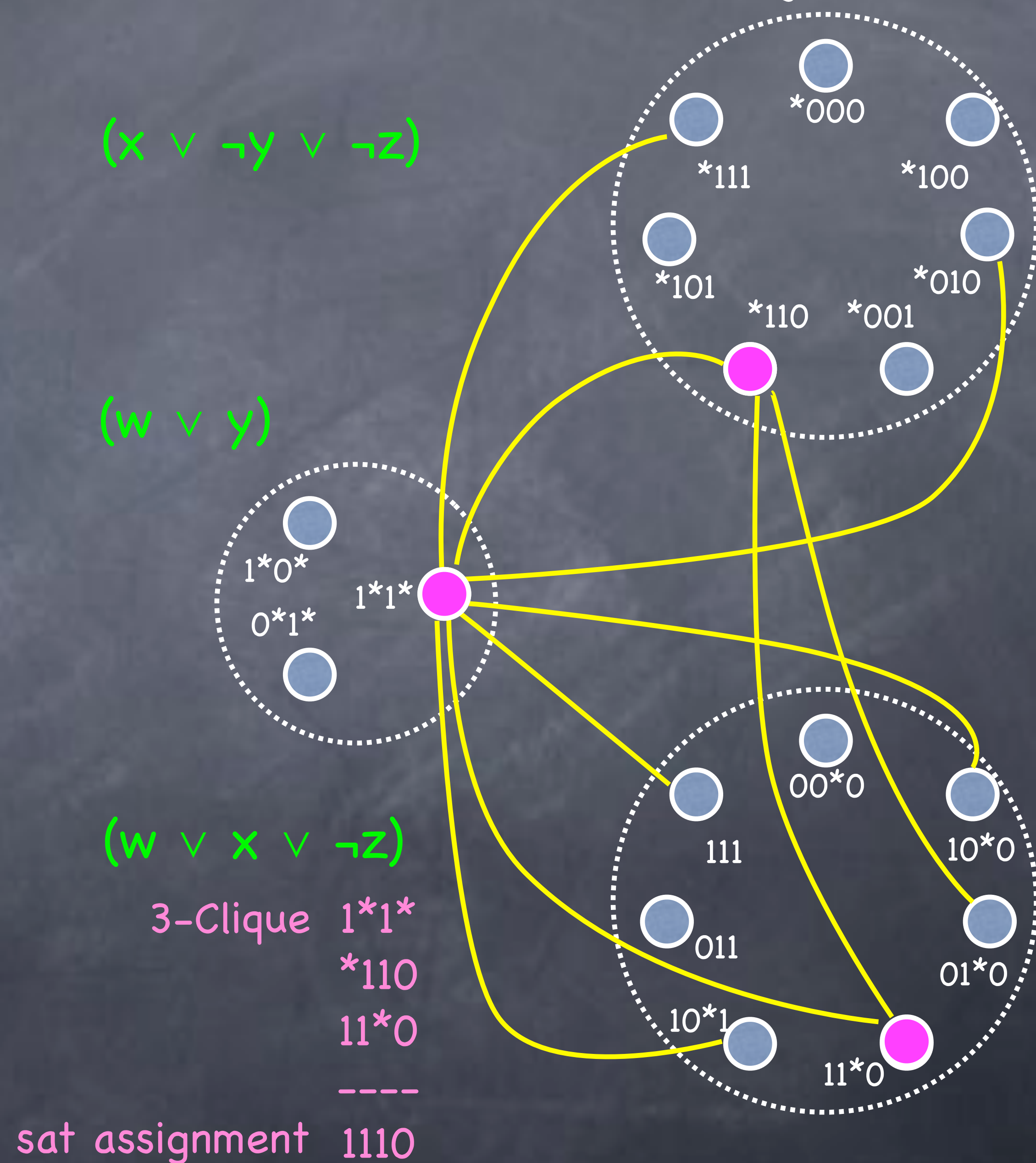

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  - Need to check that the encoded vector is the evaluation of the clauses on an assignment, and that encoding is valid

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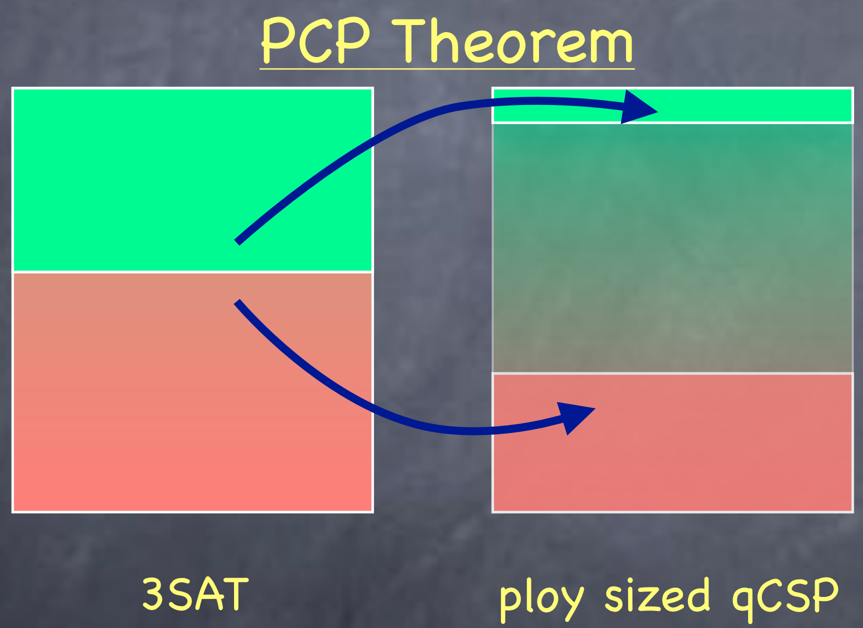

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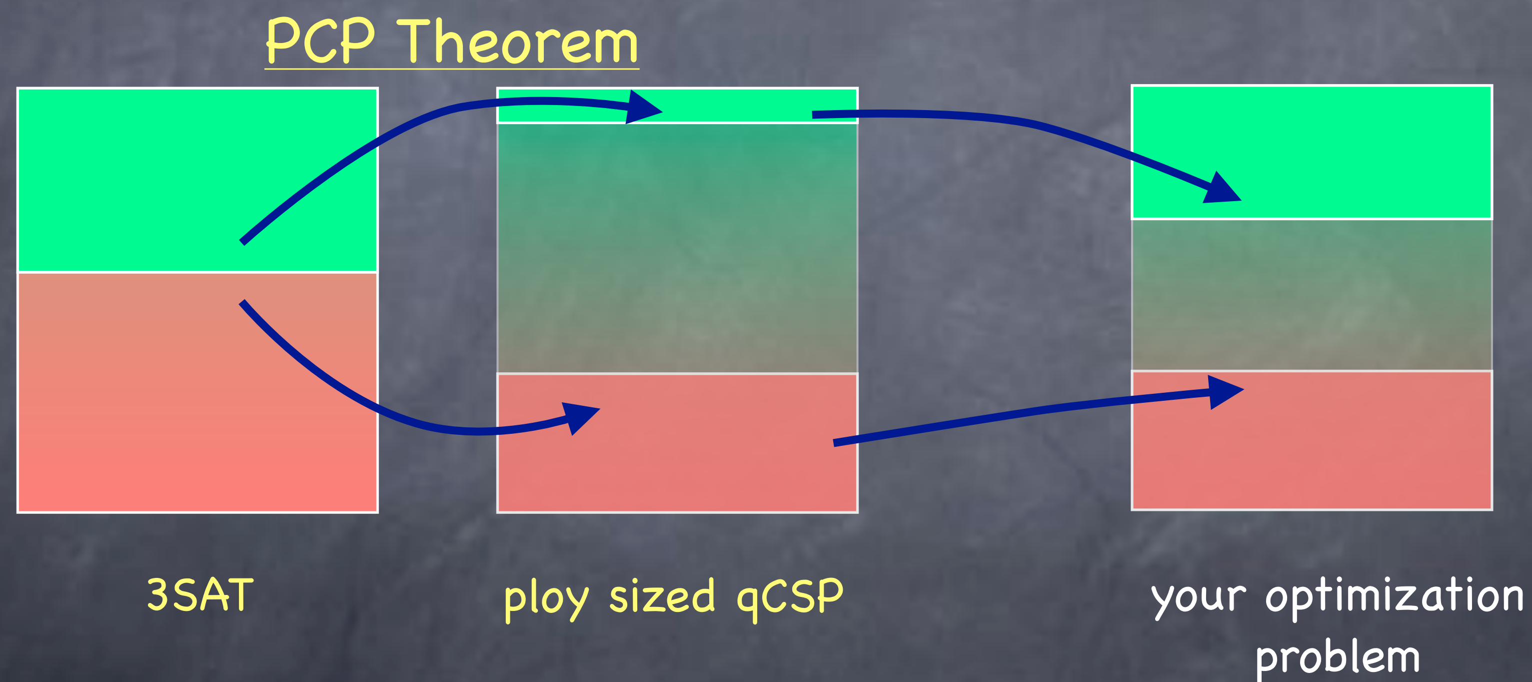

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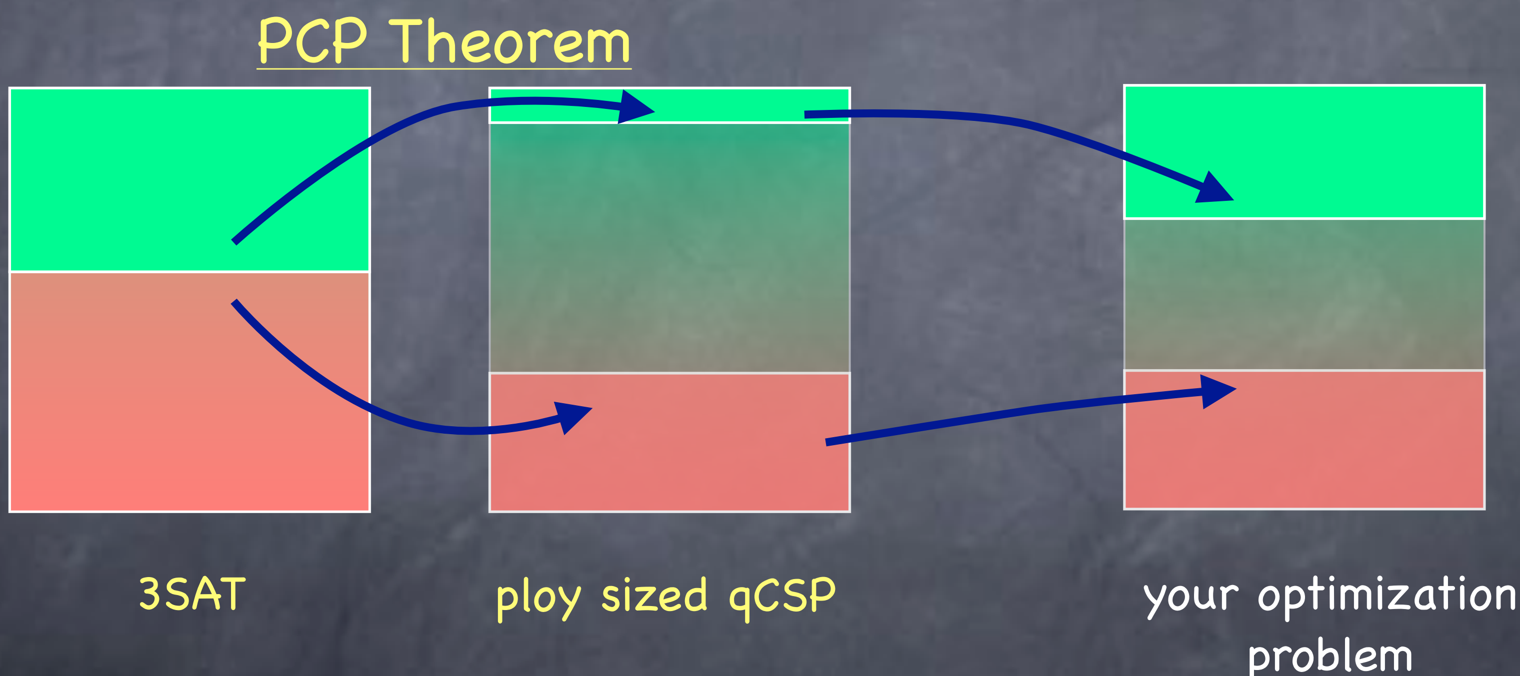

- Variants of these reductions to get different hardness results for different approximations