Interactive Proofs

Lecture 18
AM

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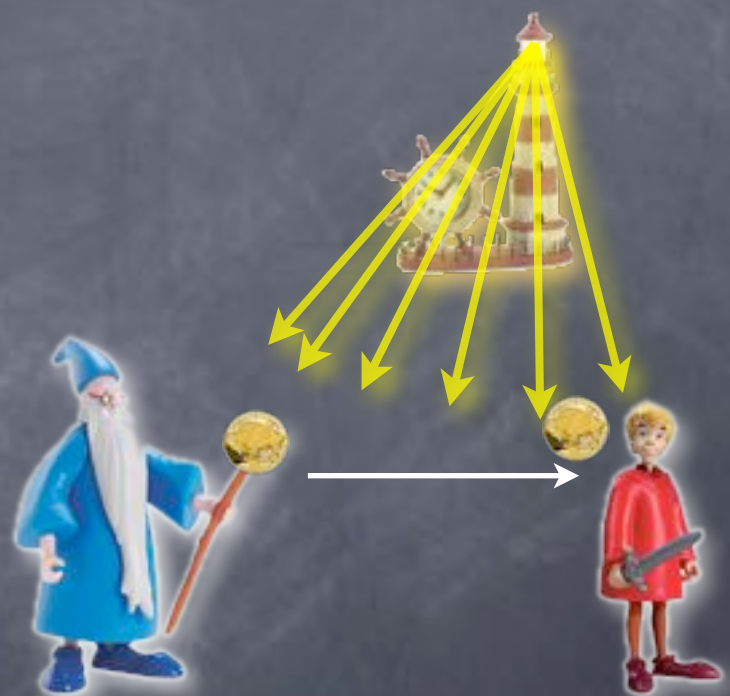

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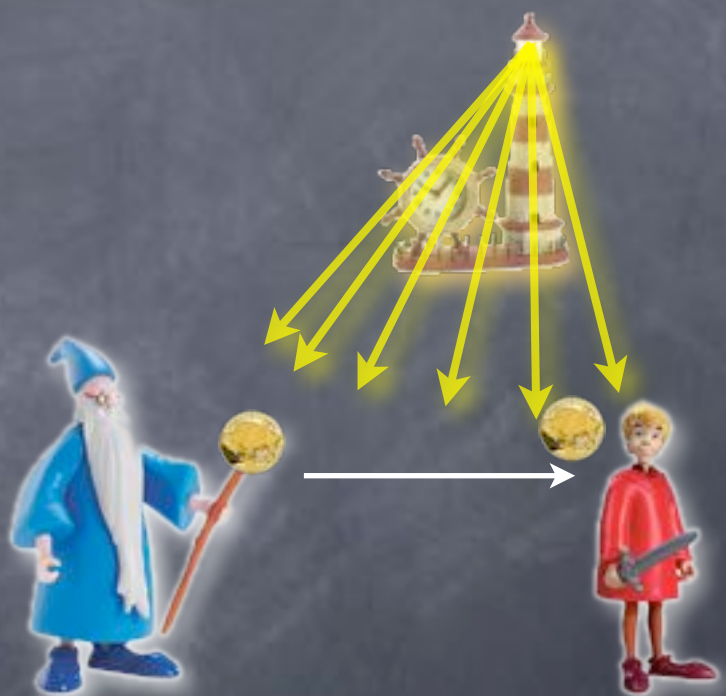

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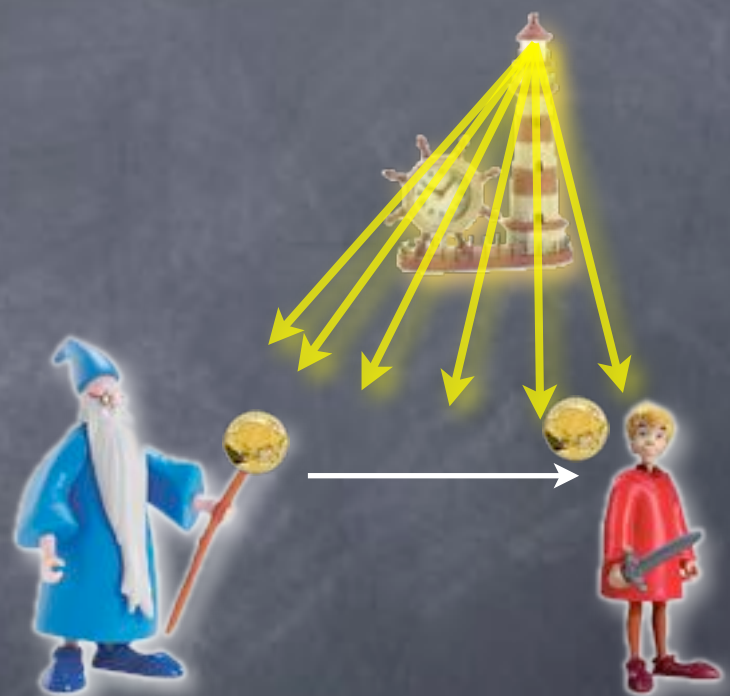

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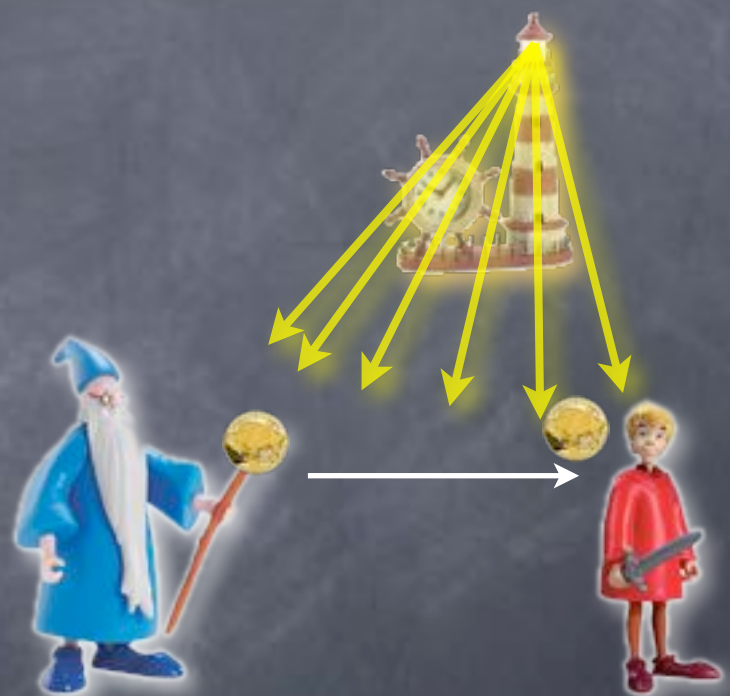

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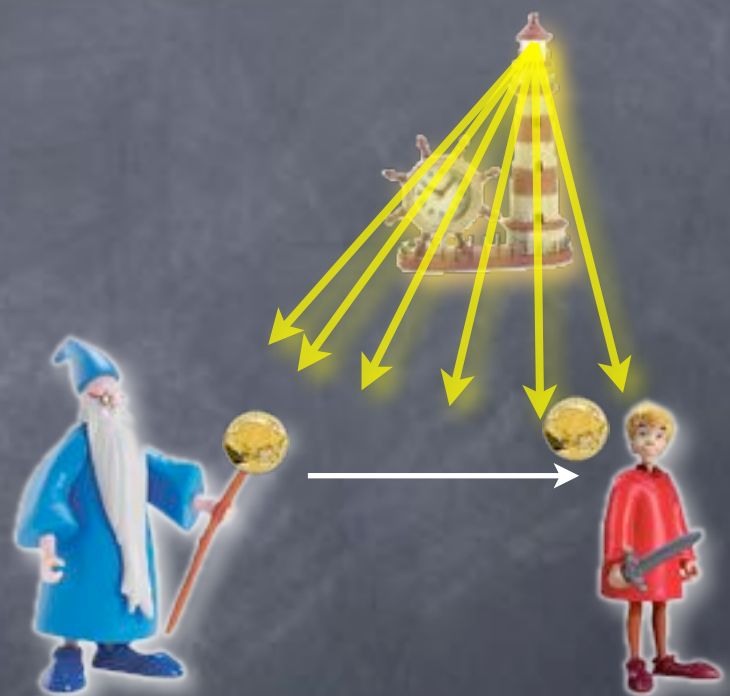

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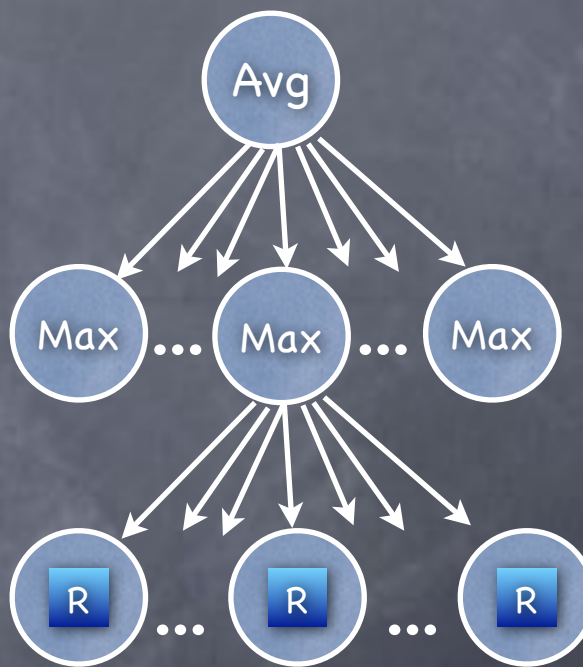

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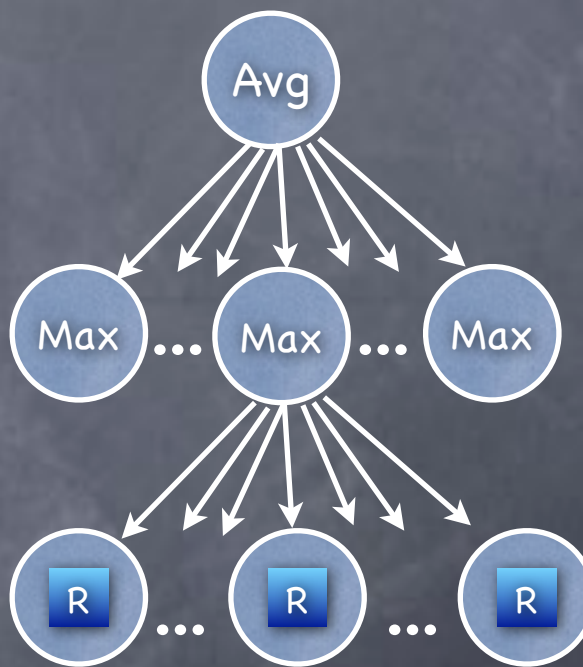

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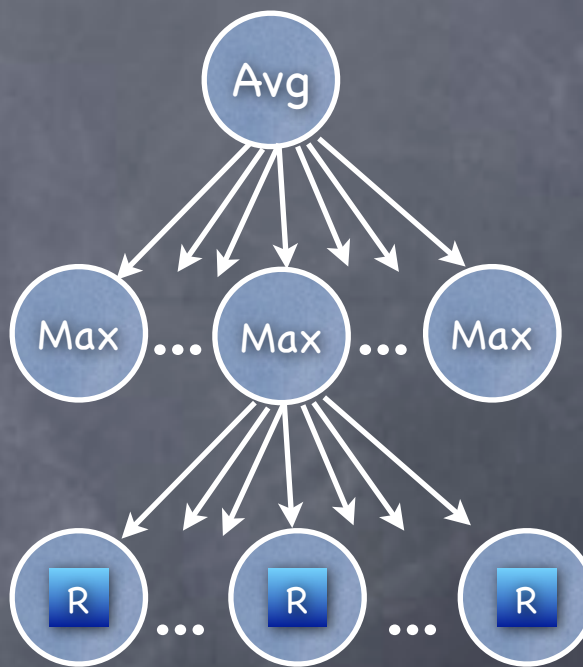

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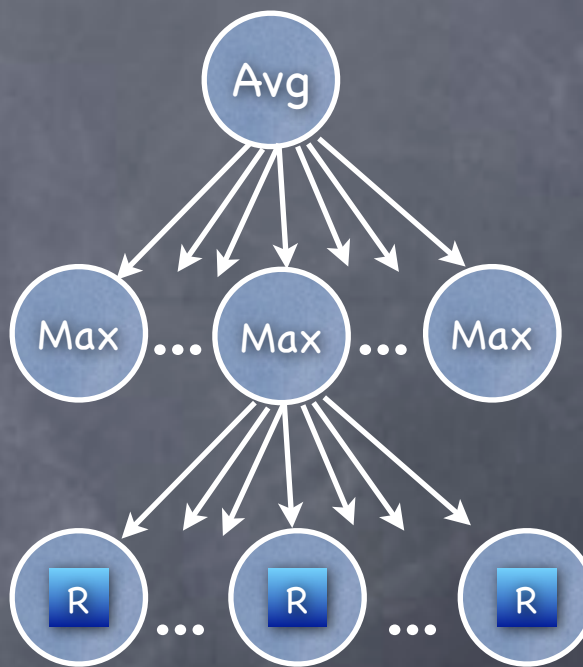

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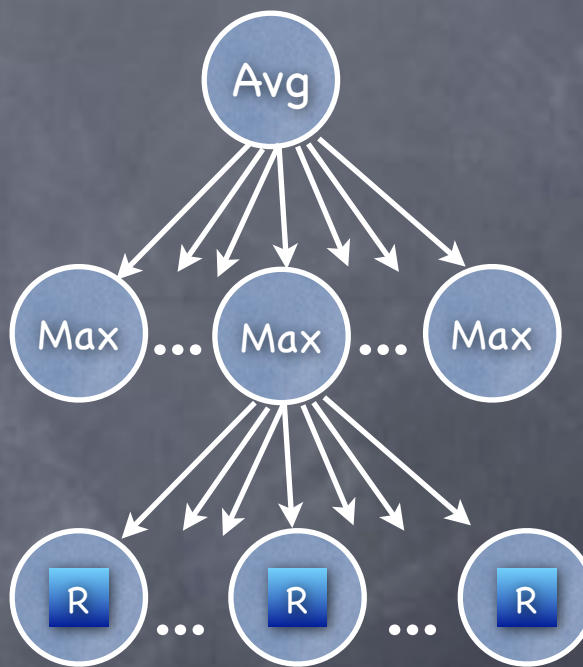
  - Extends to AM[k], with k alternating levels



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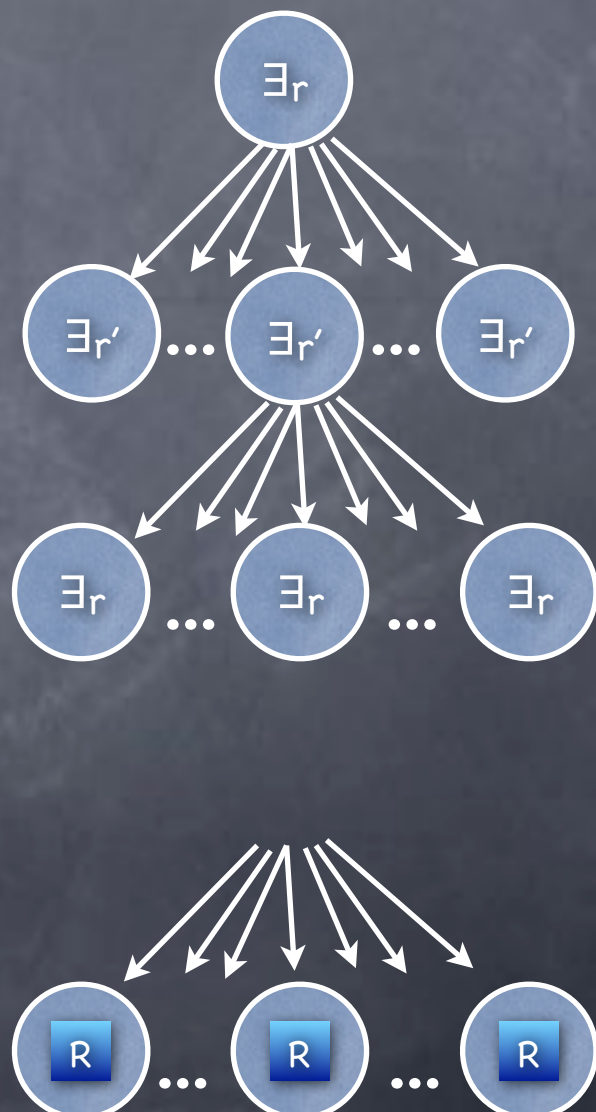

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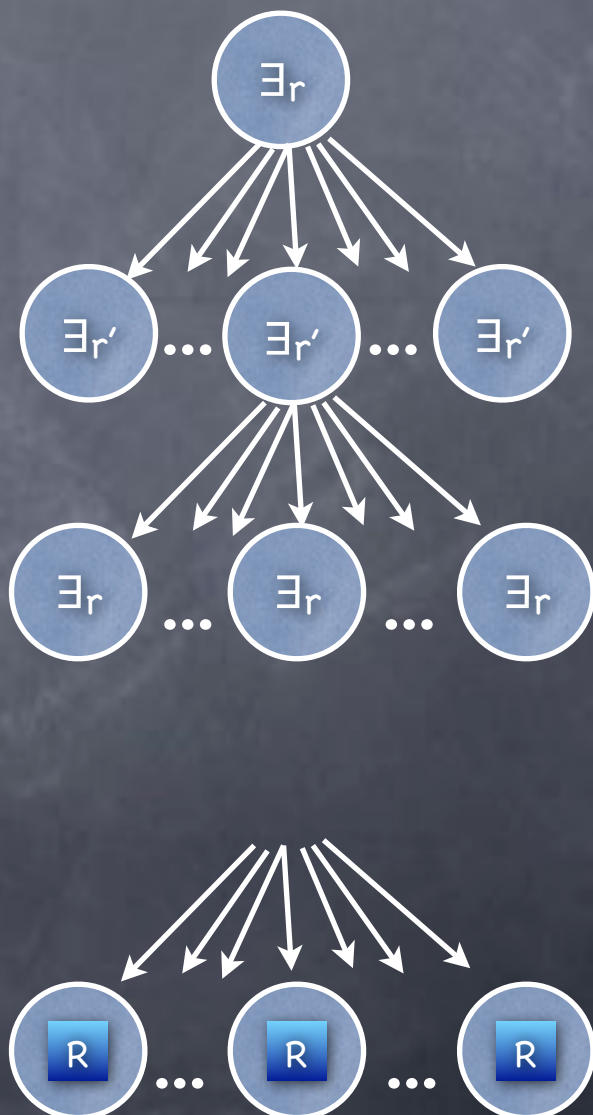

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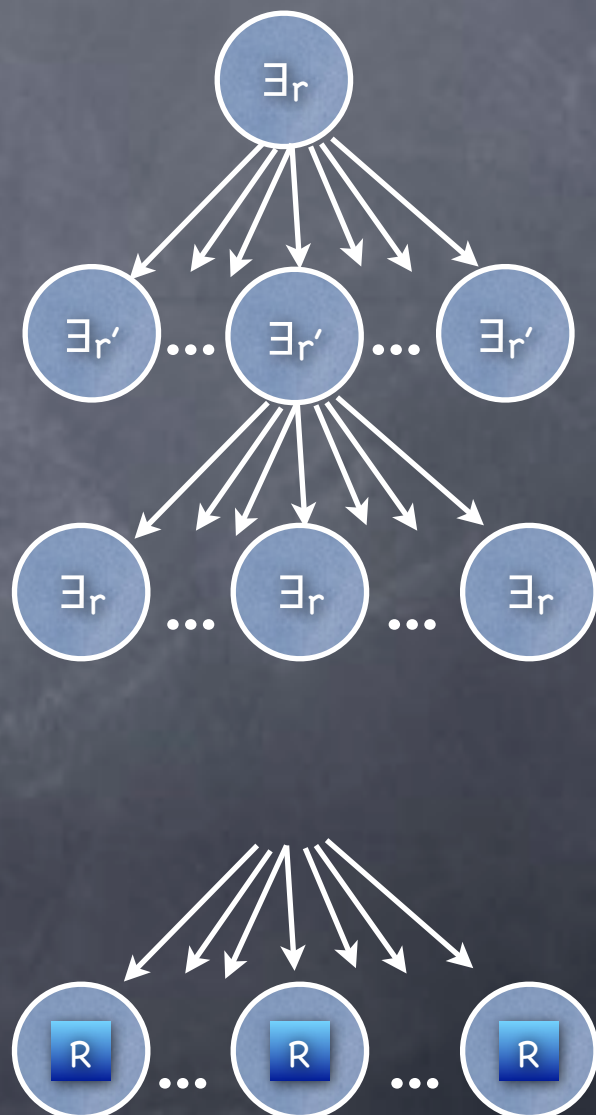

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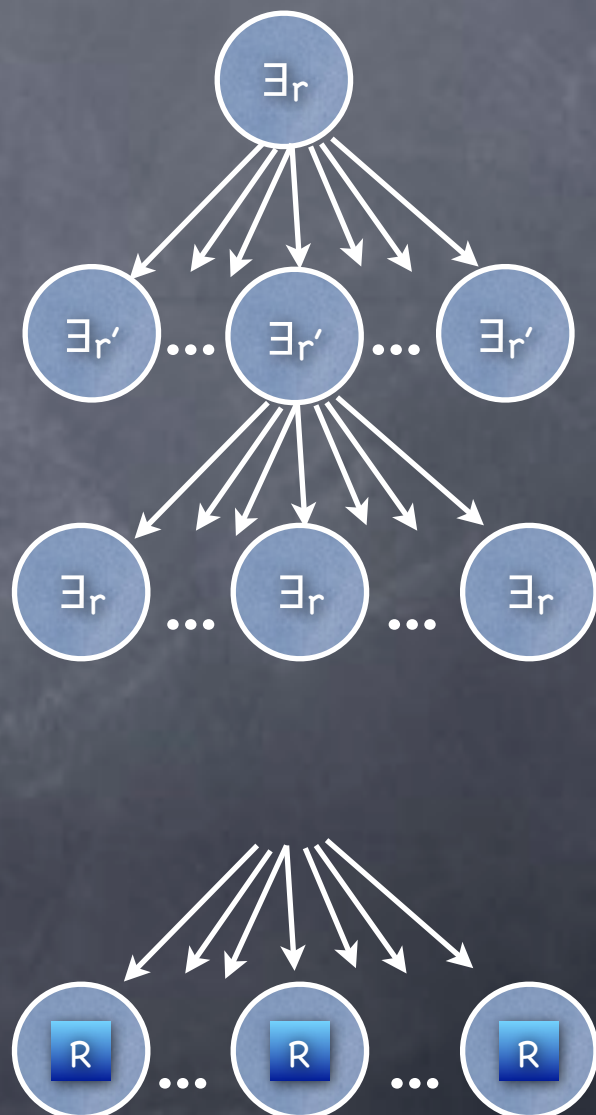

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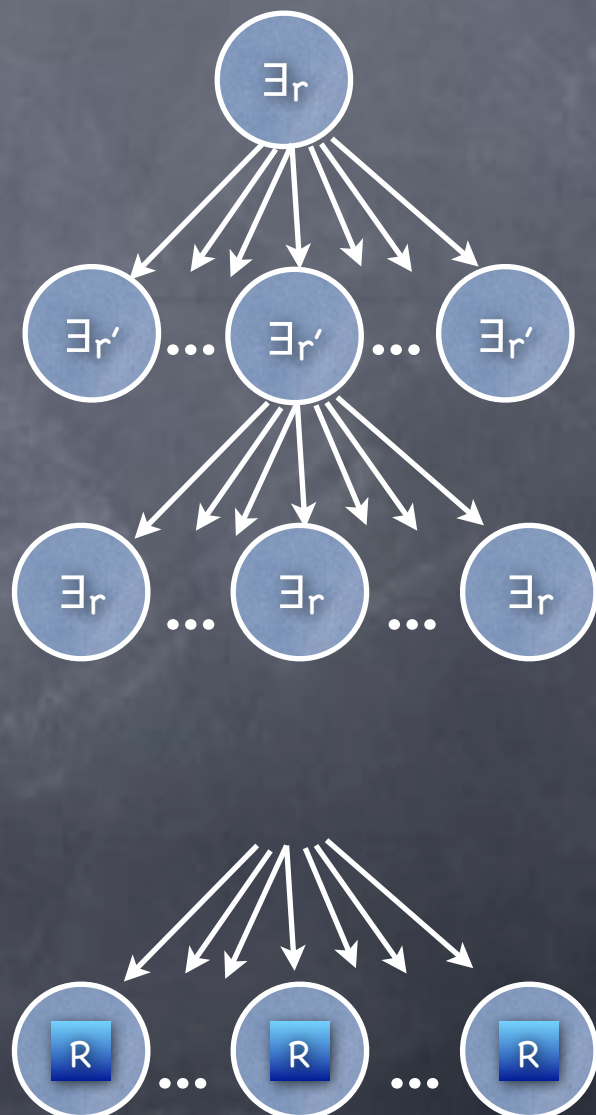

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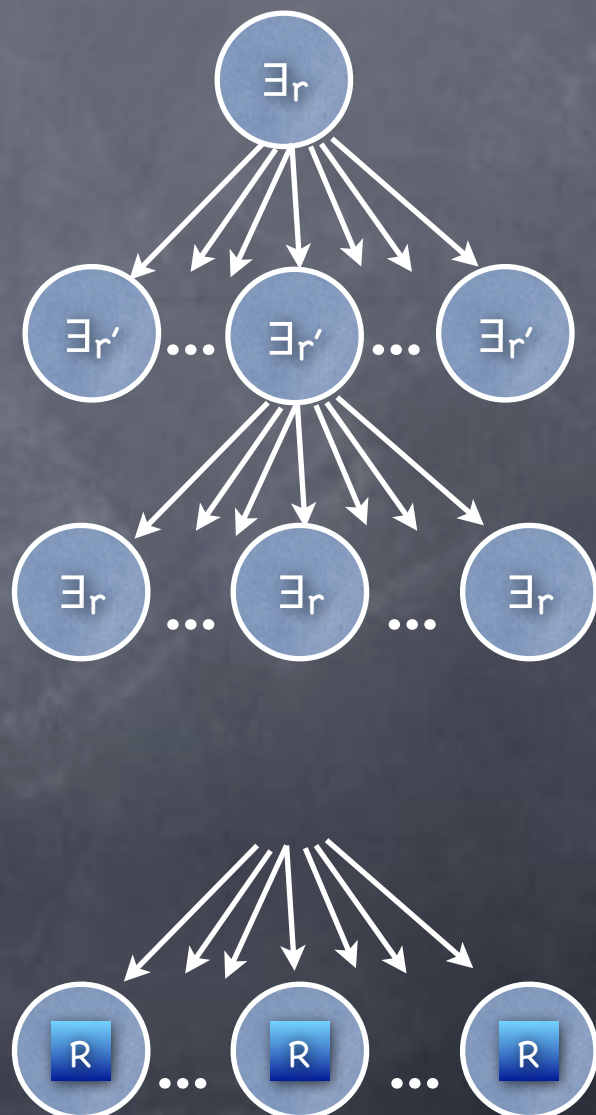

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  - k unchanged, size increases by a polynomial factor

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  - If r > 1-r, M_- stricter than M: $\{x|\ M_-(x)=1\} \subseteq \{x|\ M(x)=1\}$

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  - By parallel repetition, can increase threshold to exponentially close to 1, starting from $1/2 + c$

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- A language L has an AM[k,r] protocol **iff** L has a pair of complementary (k,r) ATTMs for r>1/2+c

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  - Enough, because we can reduce error (increase thresholds) for both AM protocols and ATTMs

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  - Repeat ~k/2 times to reduce to AM[2]

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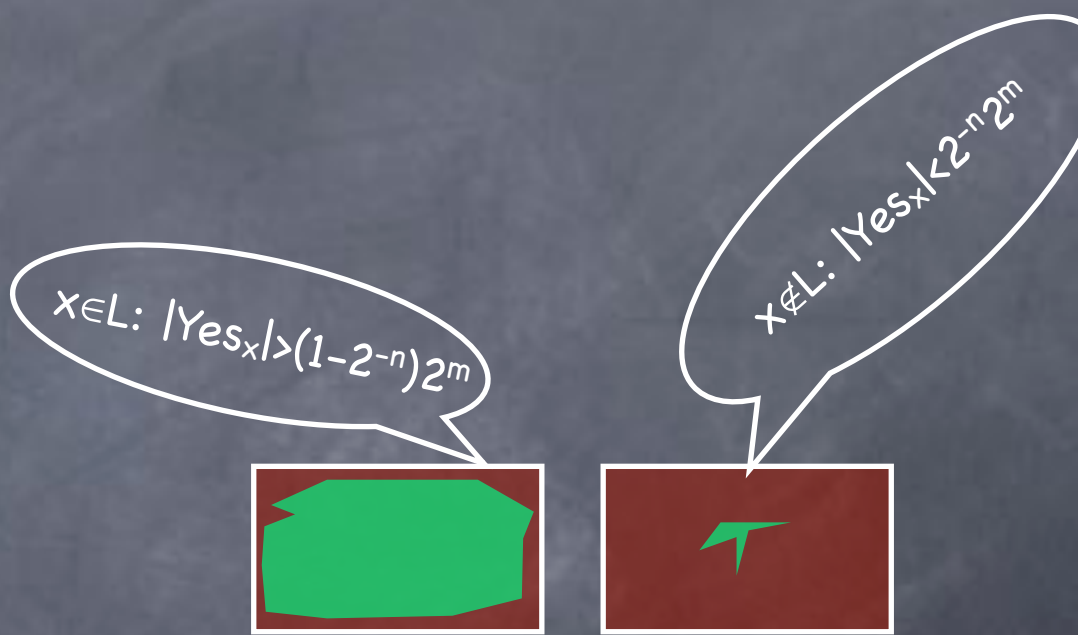

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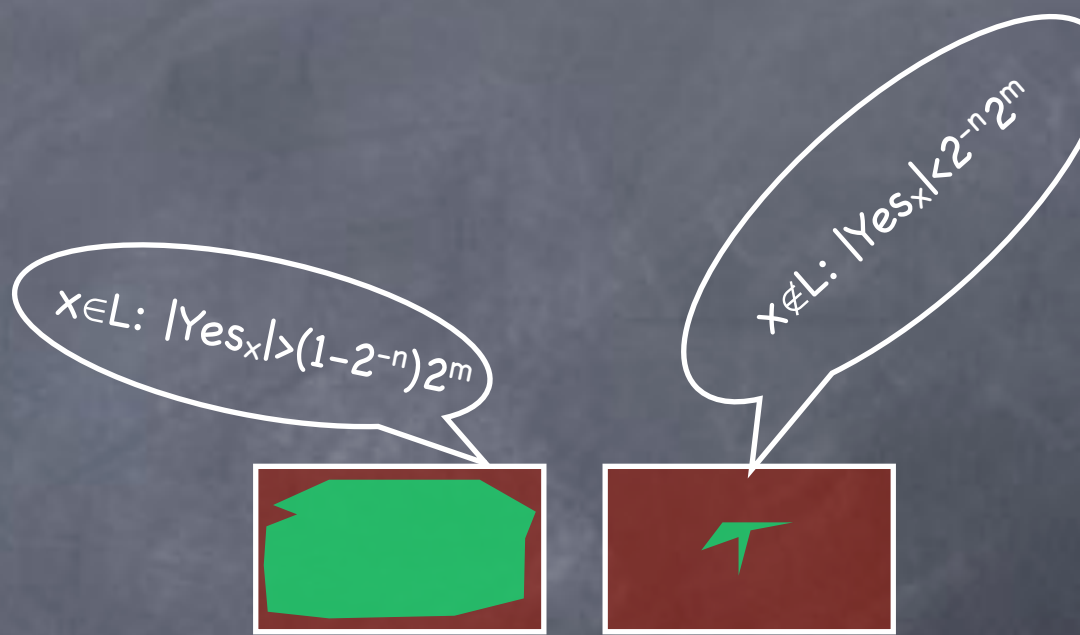

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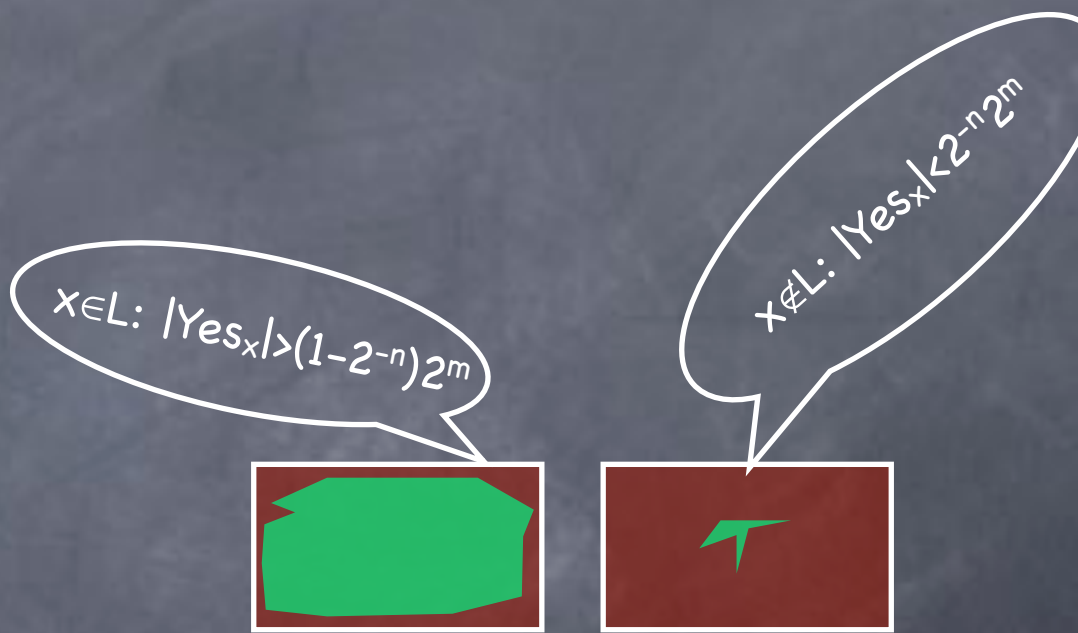

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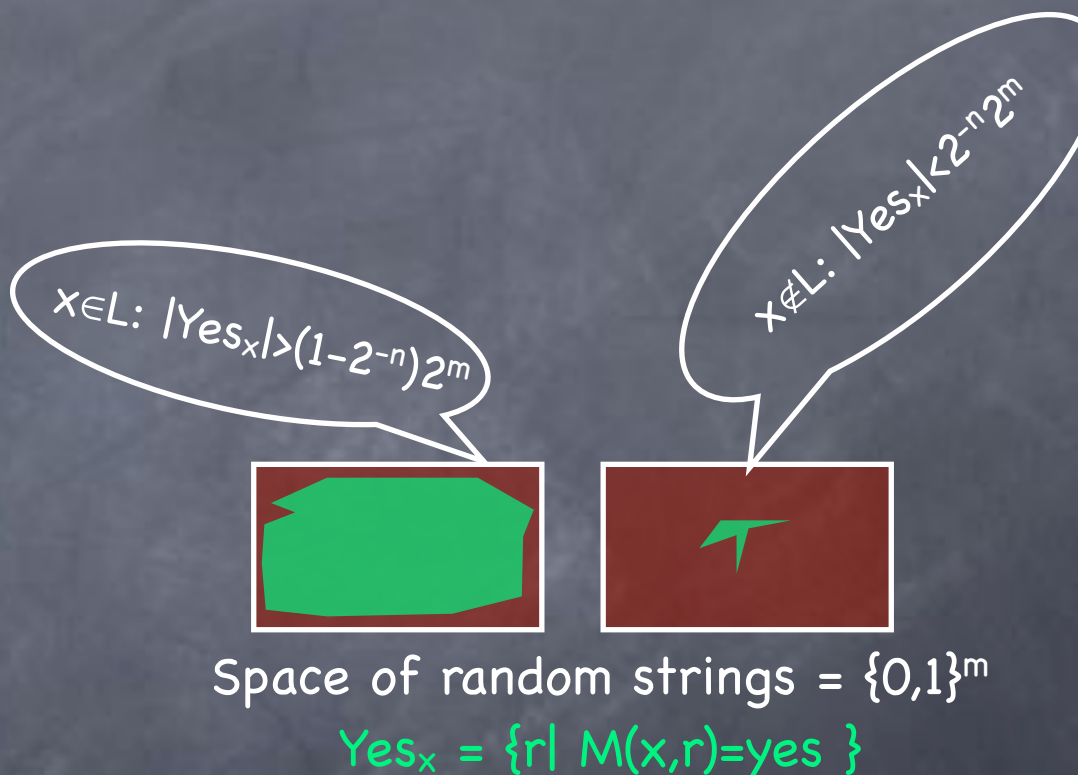

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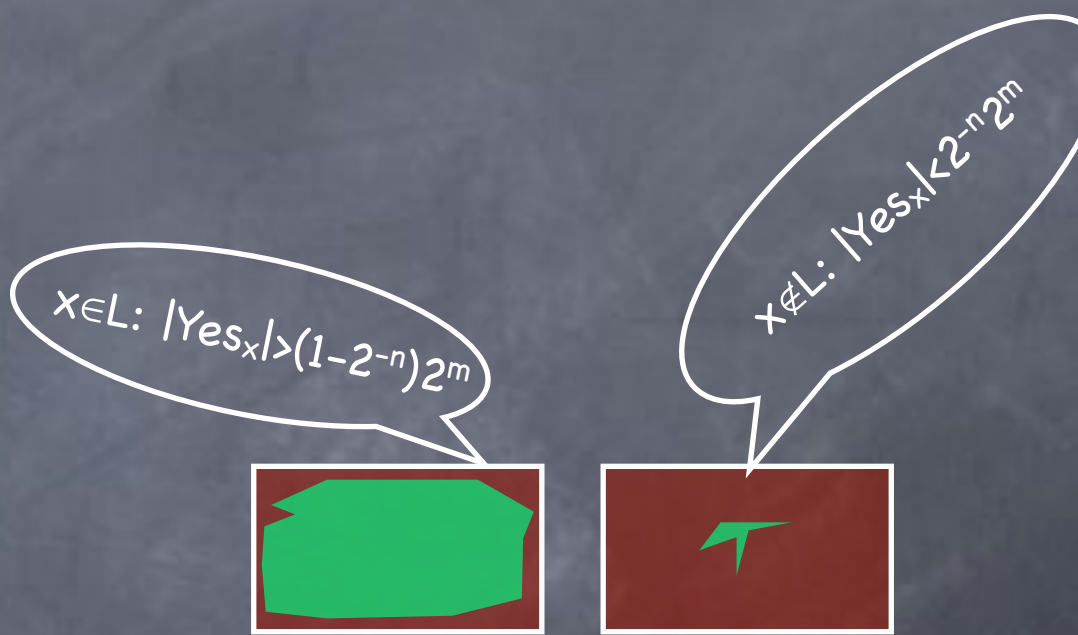

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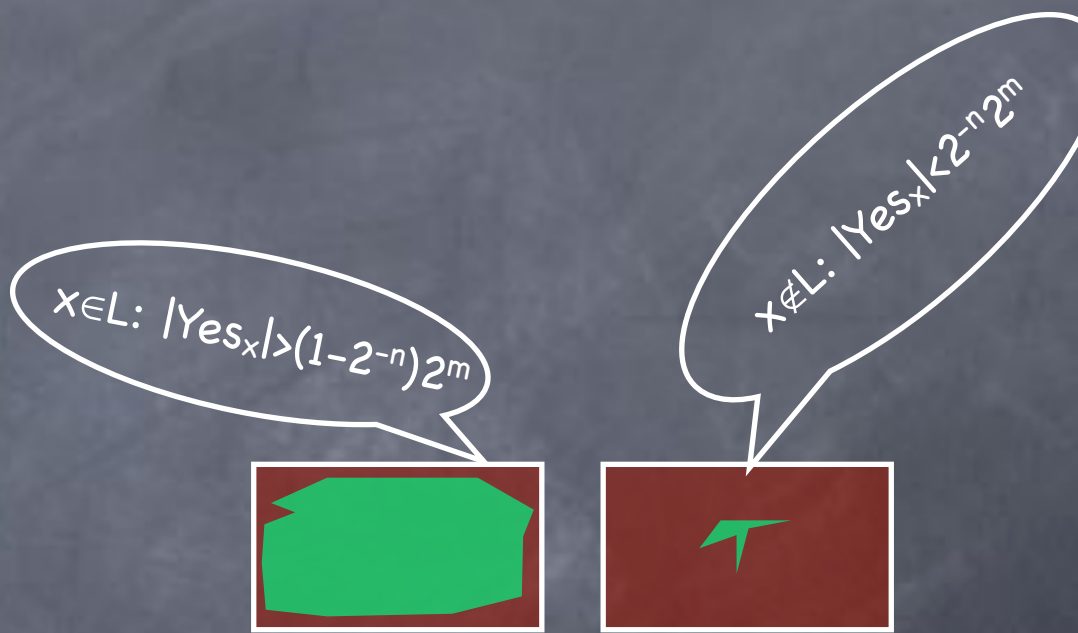

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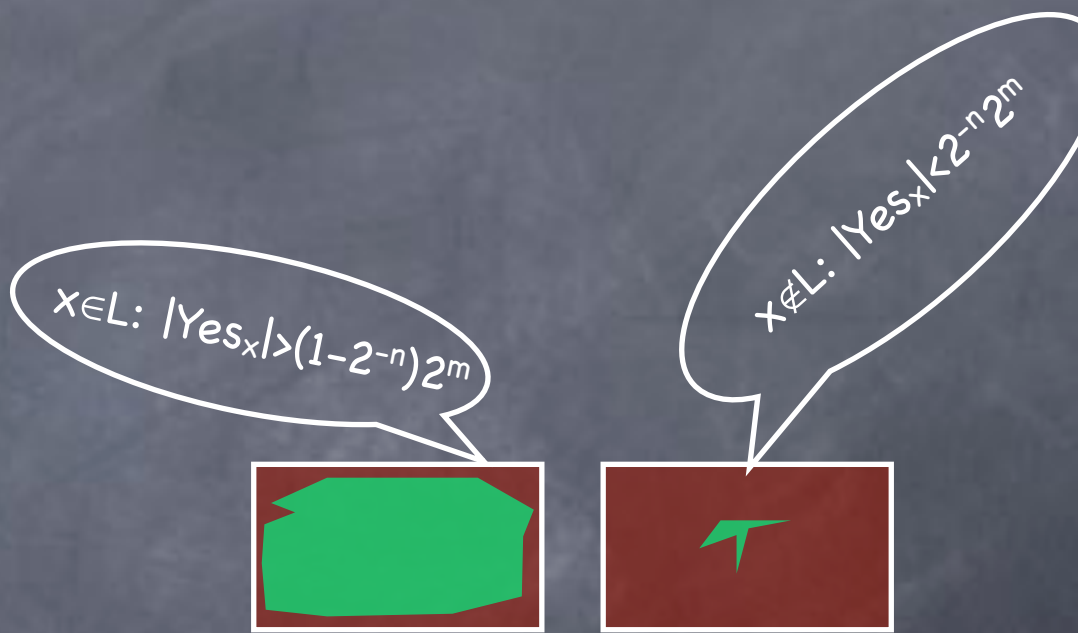

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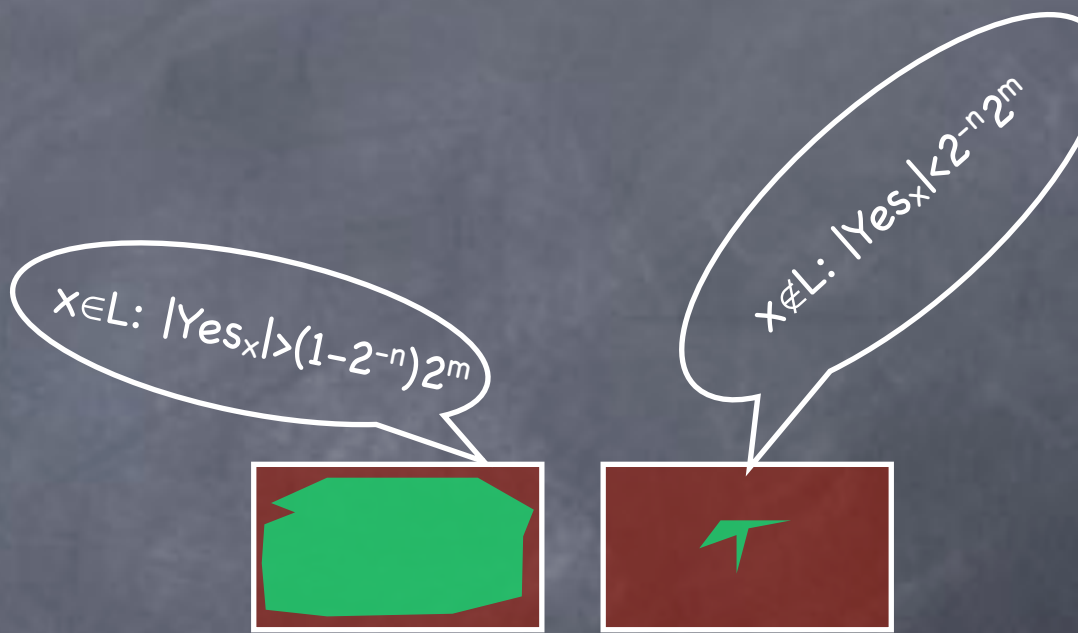

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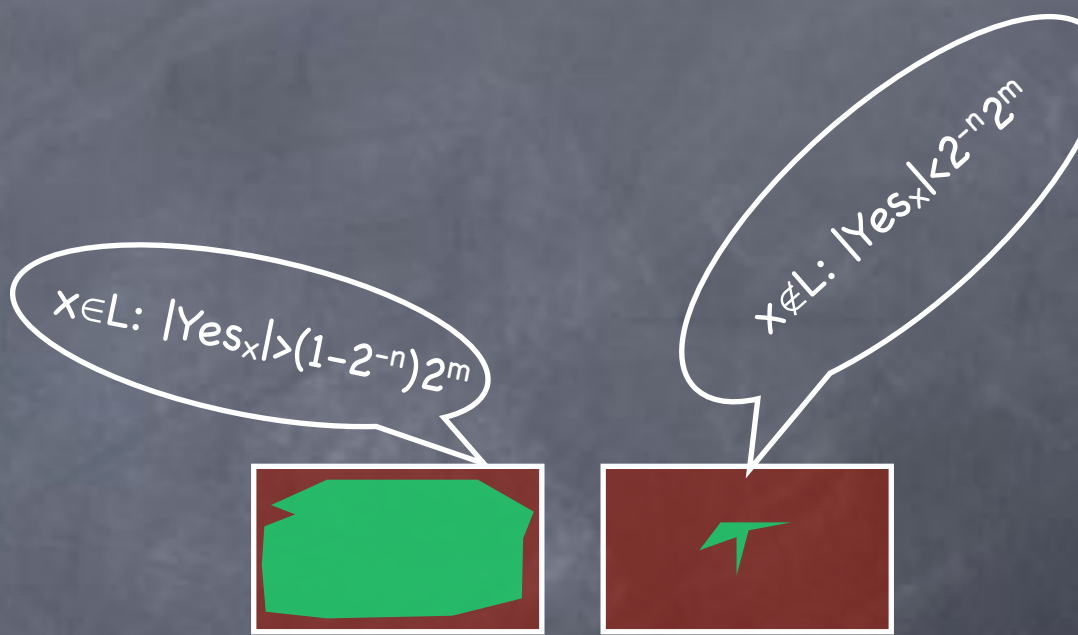

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