Interactive Proofs

Lecture 17

IP = PSPACE

So far

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- IP

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- IP
- AM, MA

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- GNI $\in$ IP

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  - Using AM protocol for set lower-bound

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  - Using AM protocol for set lower-bound
  - In fact, IP[k] in AM[k+2]

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  - Even though prover unbounded, cannot convince poly time verifier of everything
- PSPACE $\subseteq$ IP
  - Prover can convince verifier of high complexity statements

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  - Could then use the "fact" that IP[poly]=AM[poly]
    - Or modify the proof (as we'll do)

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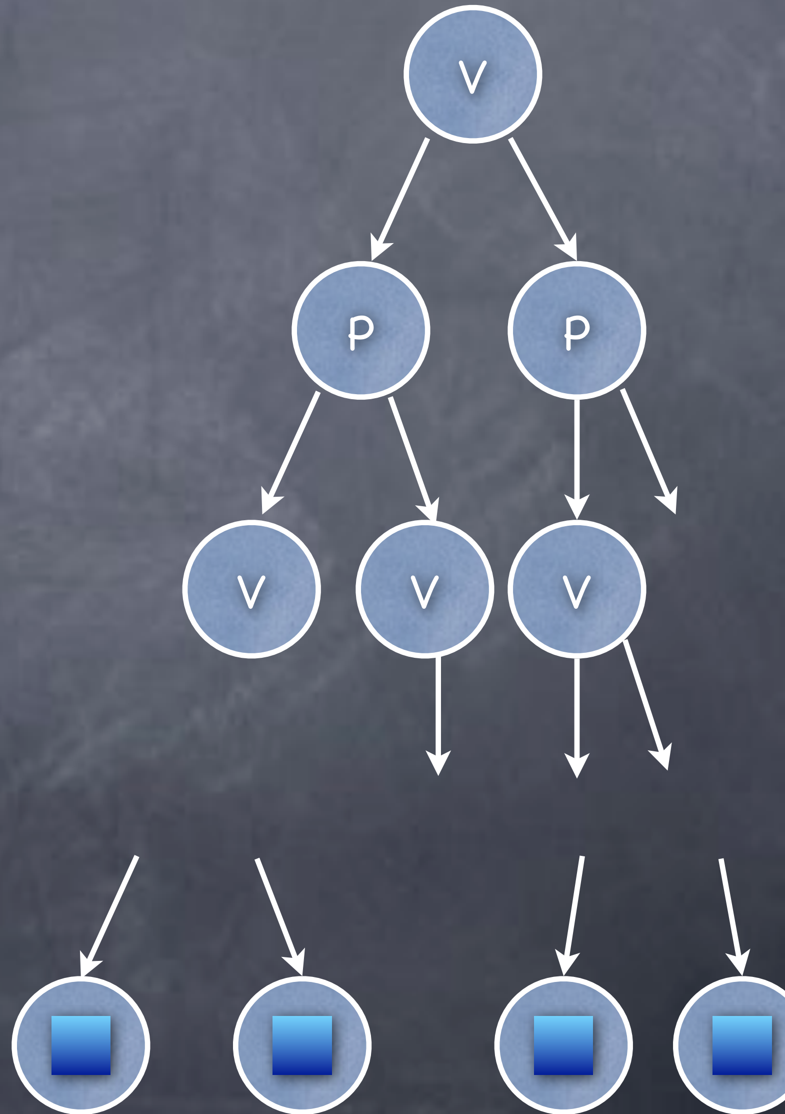

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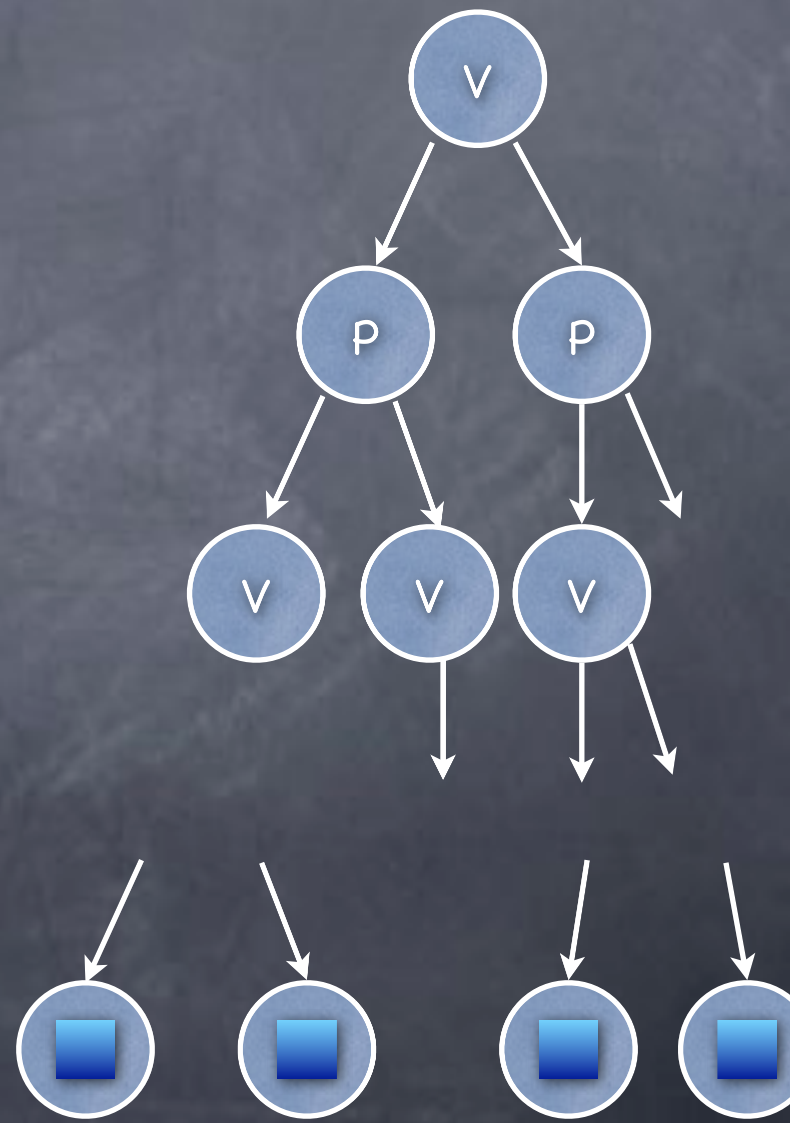

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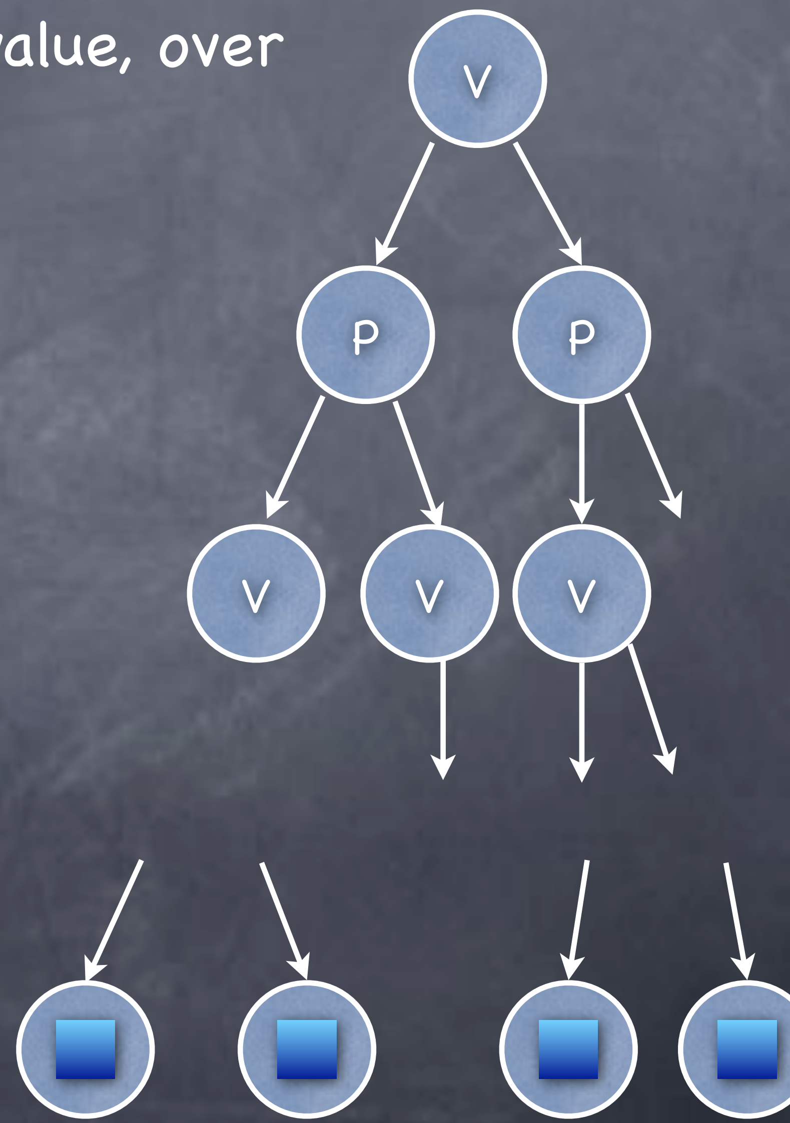

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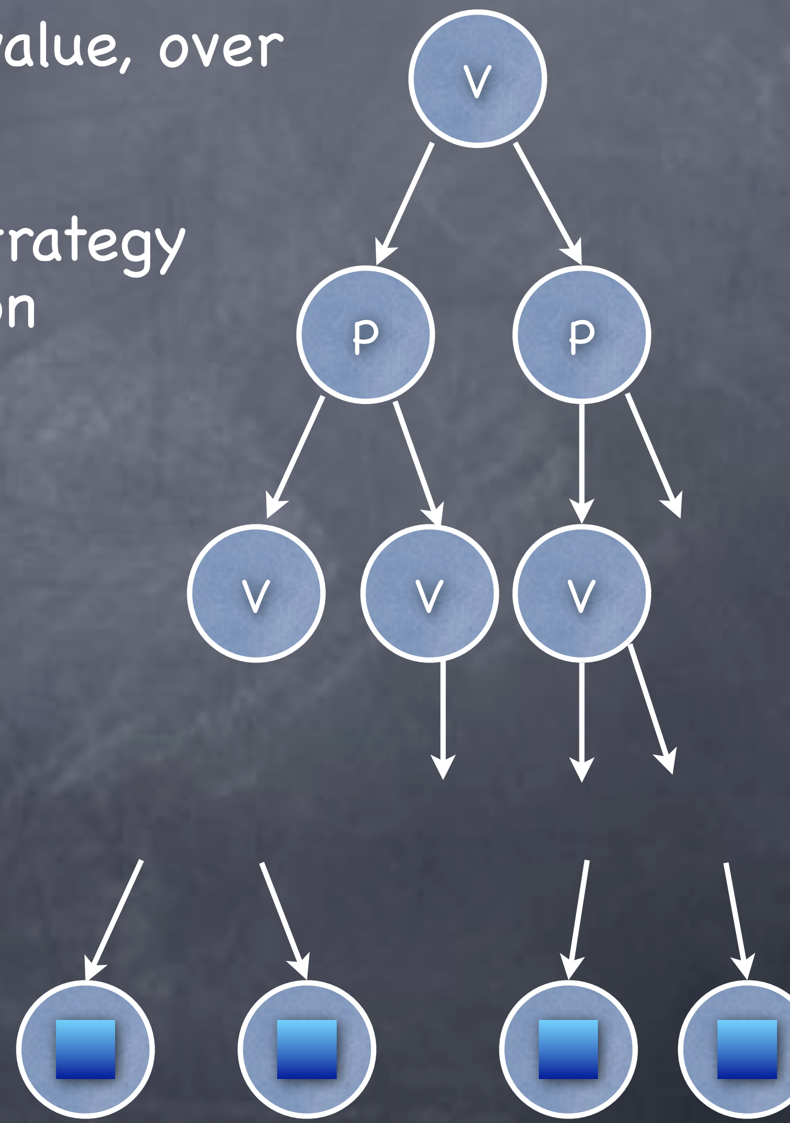

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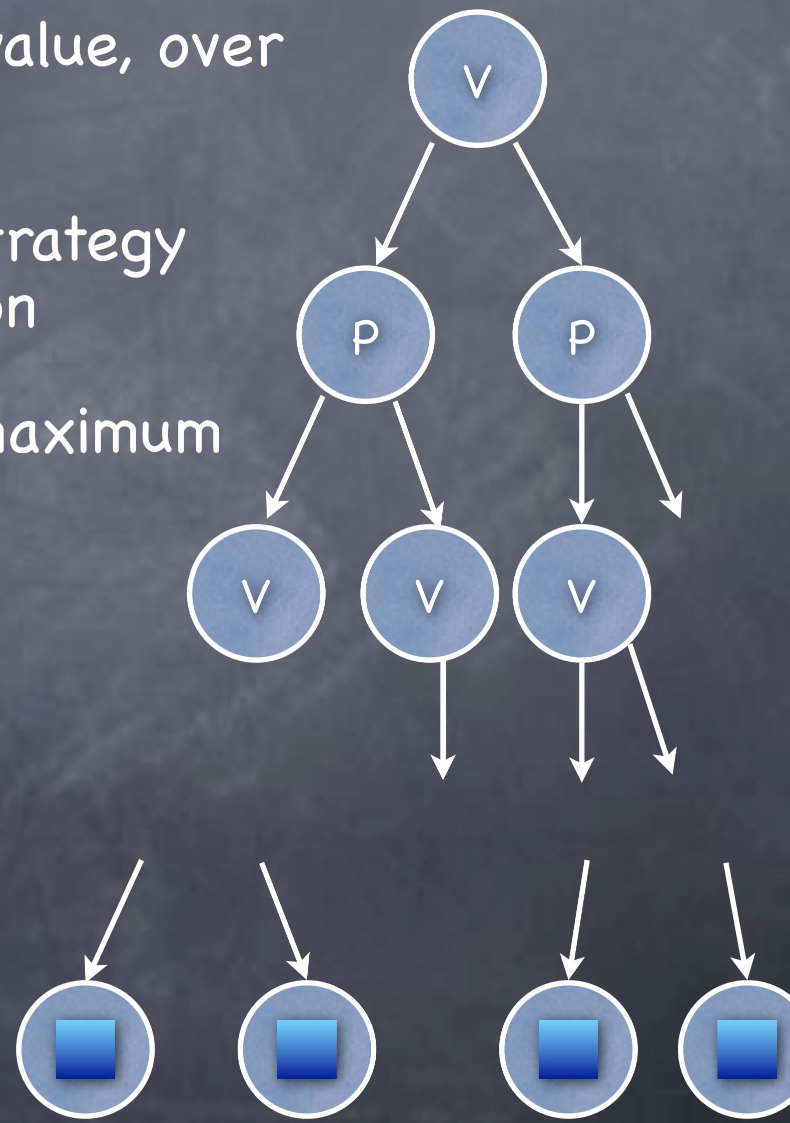

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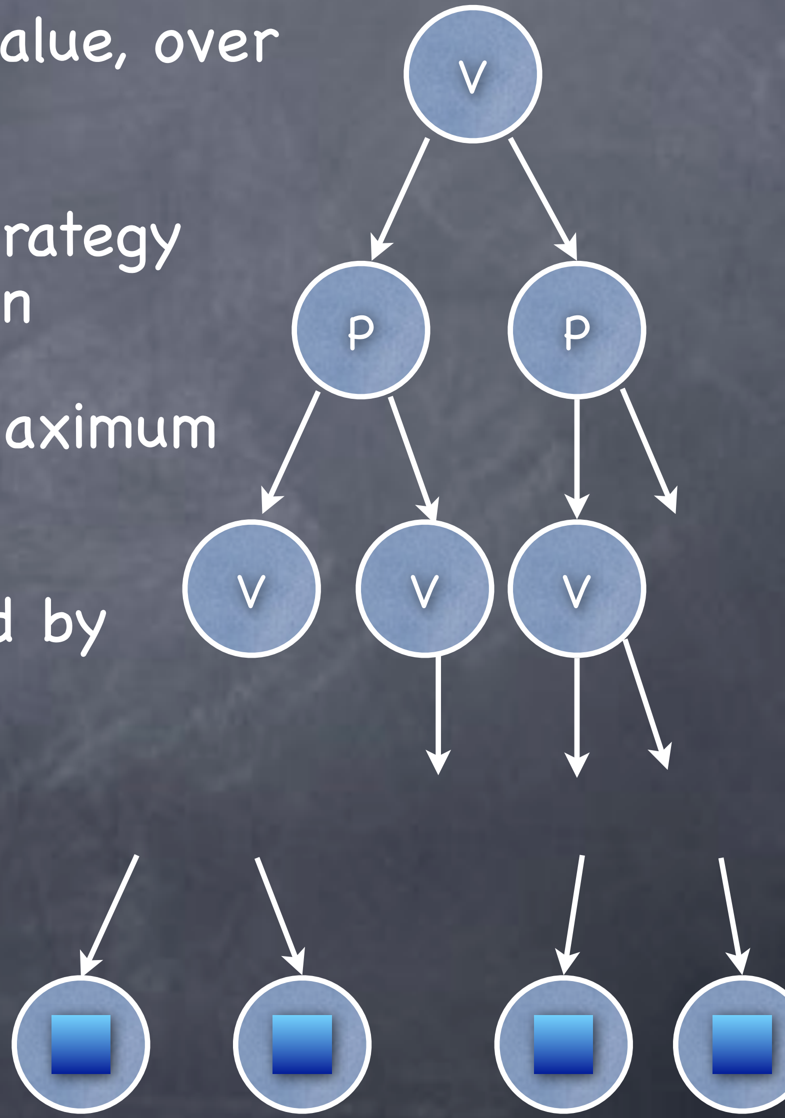

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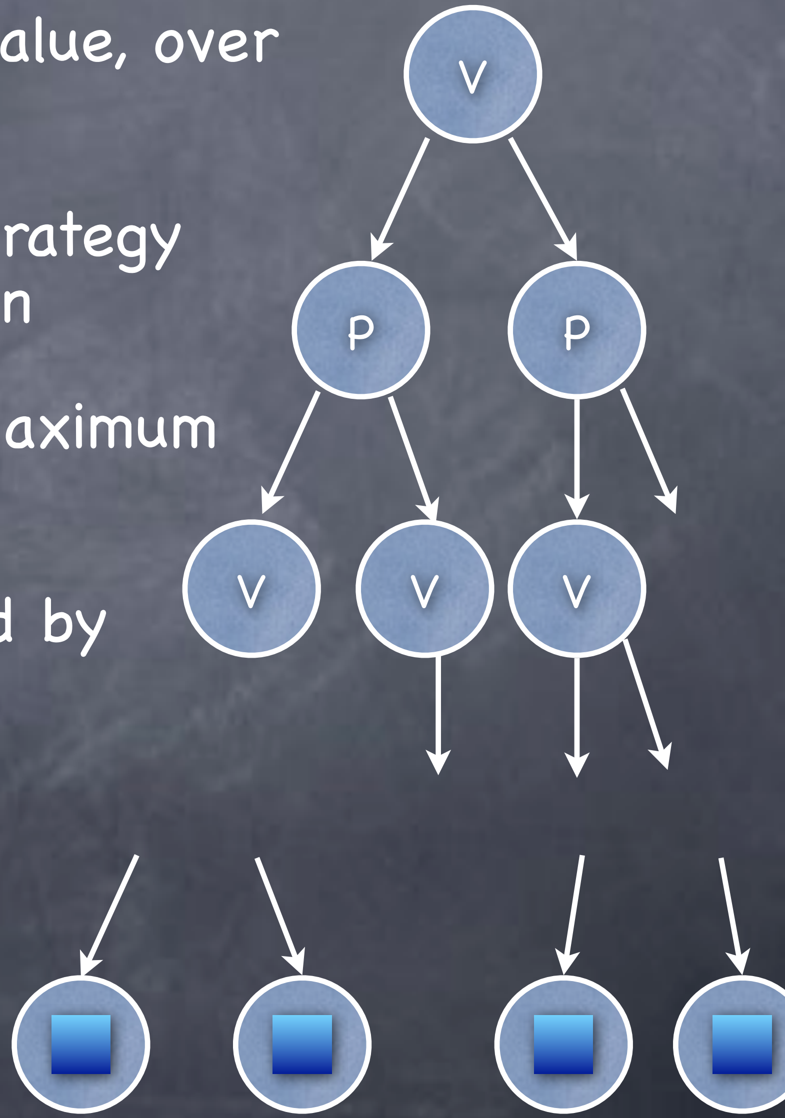

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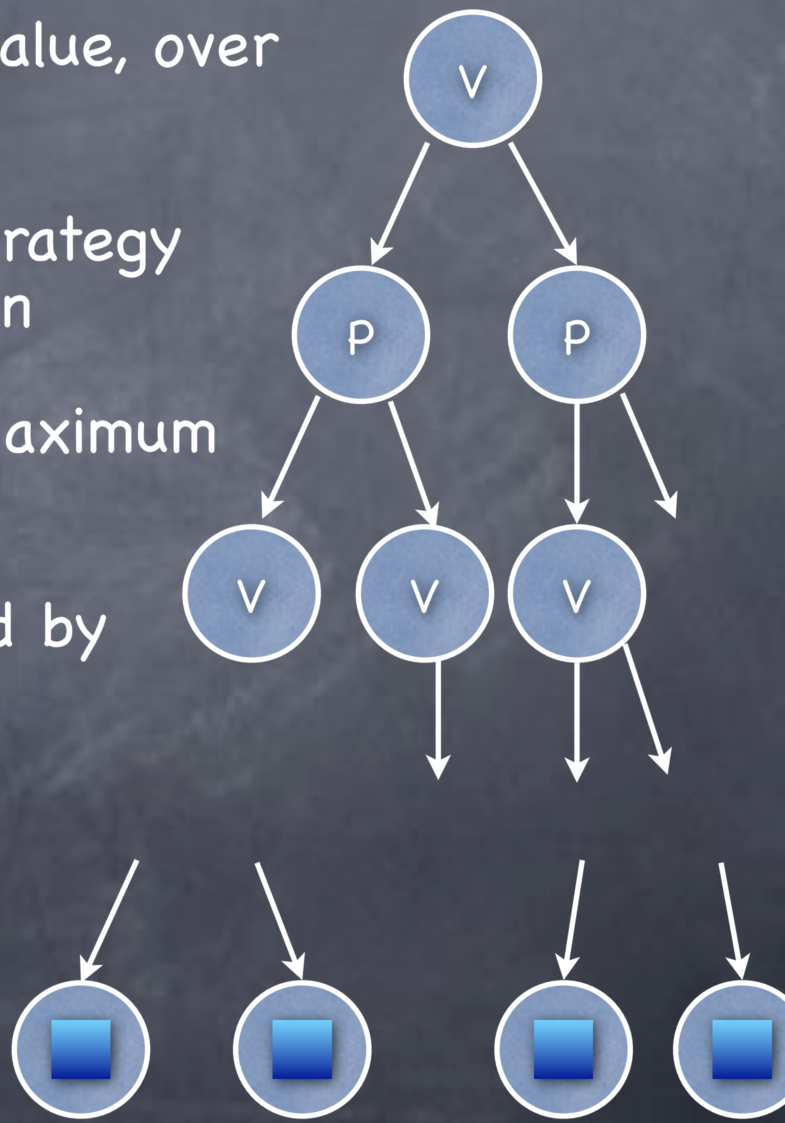

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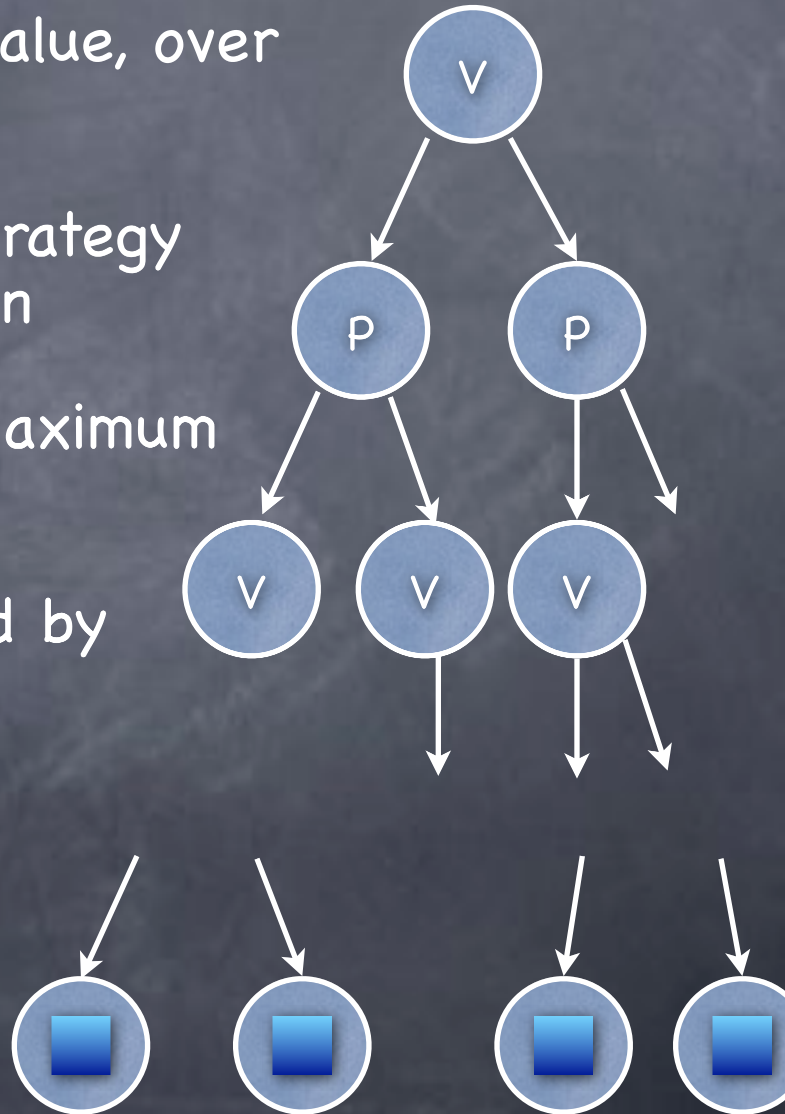
    - In PSPACE: depth polynomial



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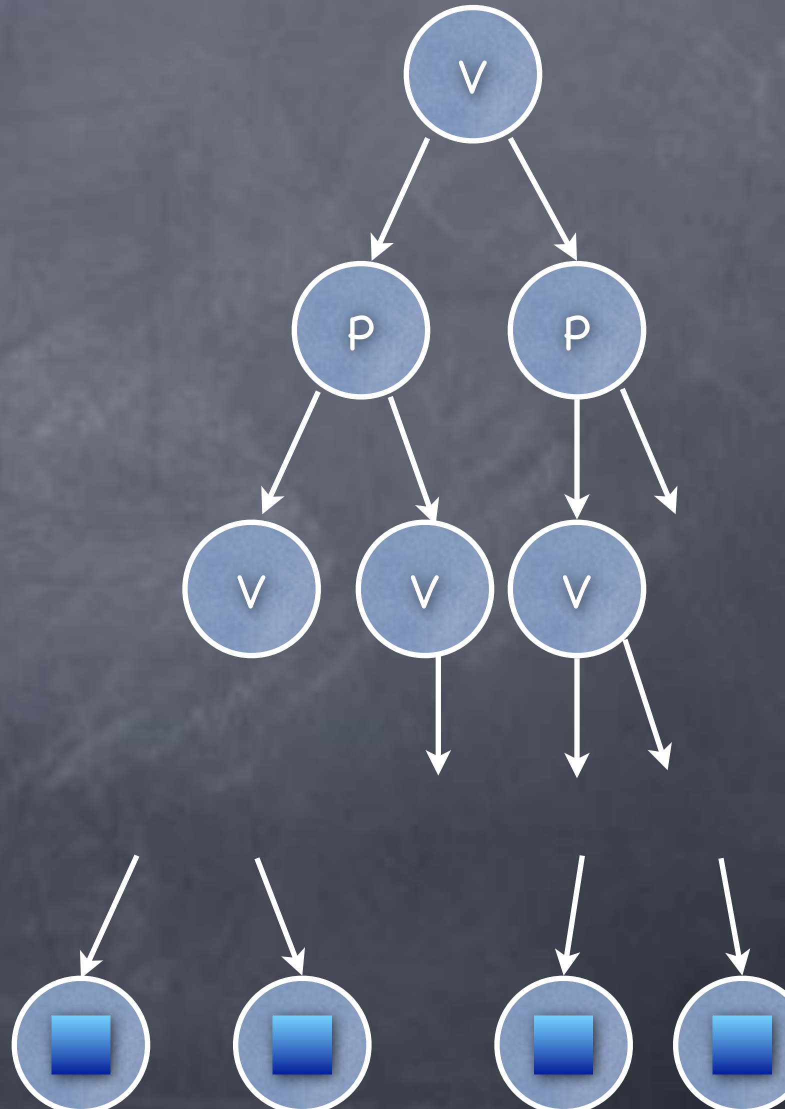

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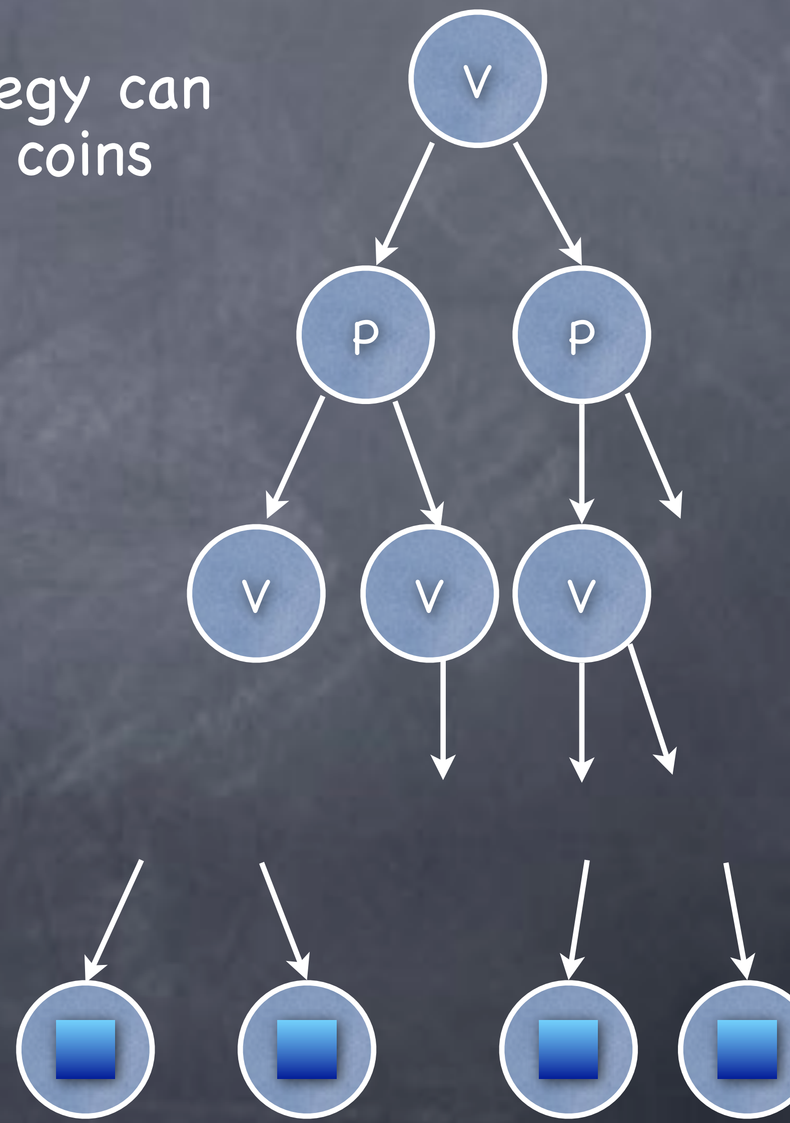

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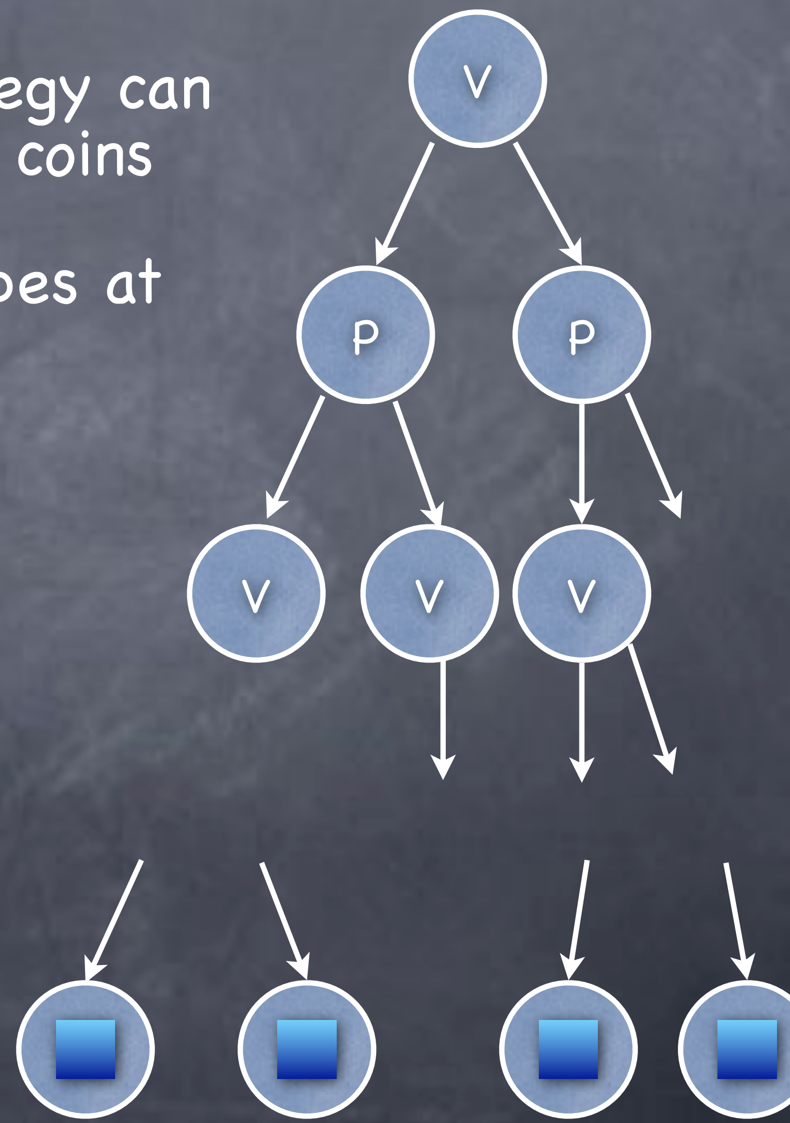

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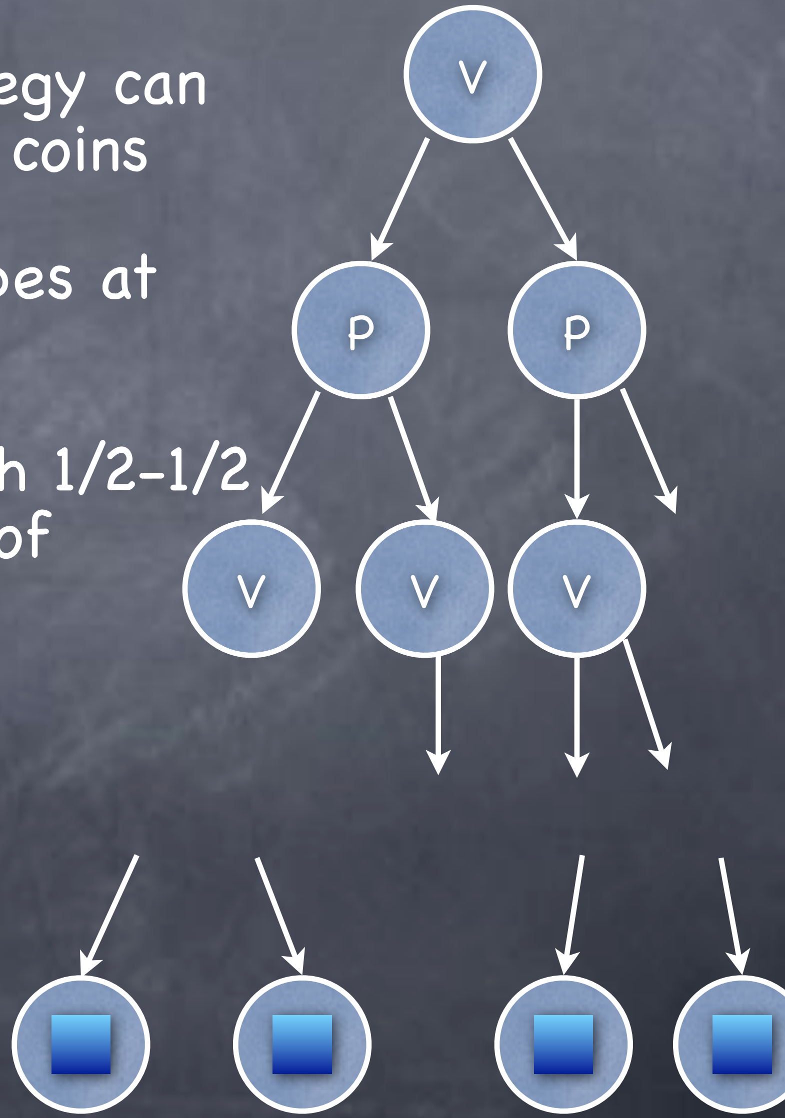

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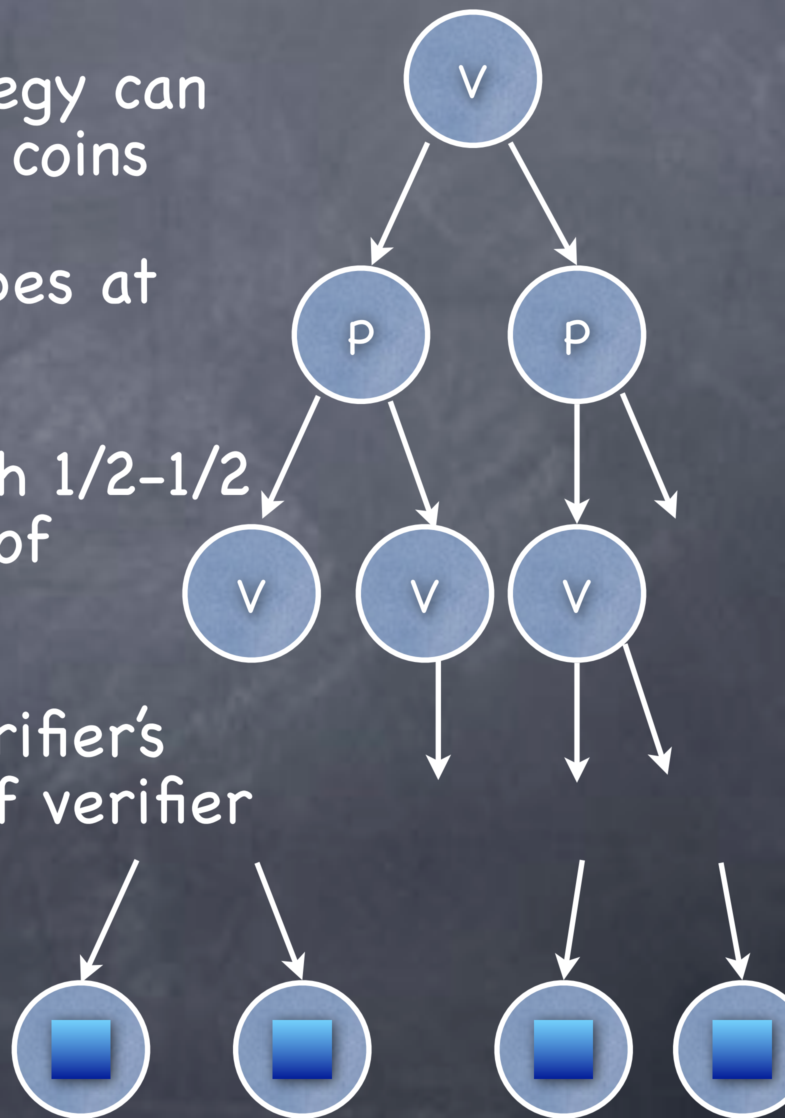

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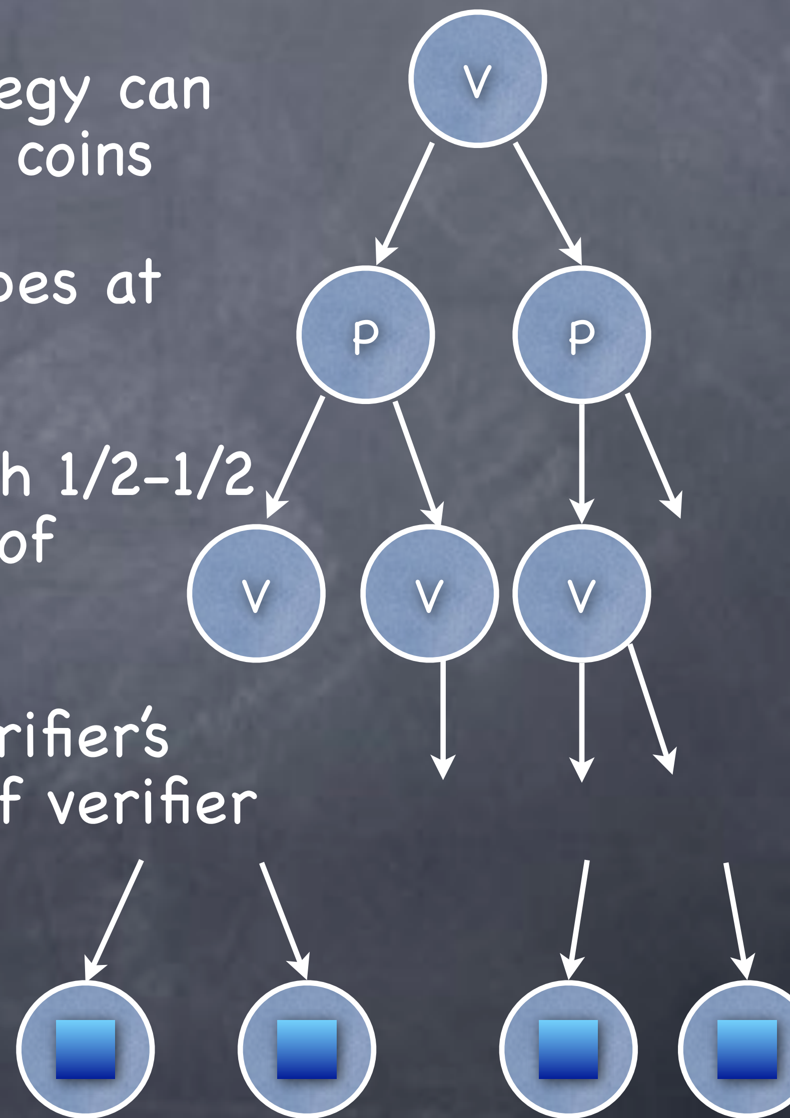

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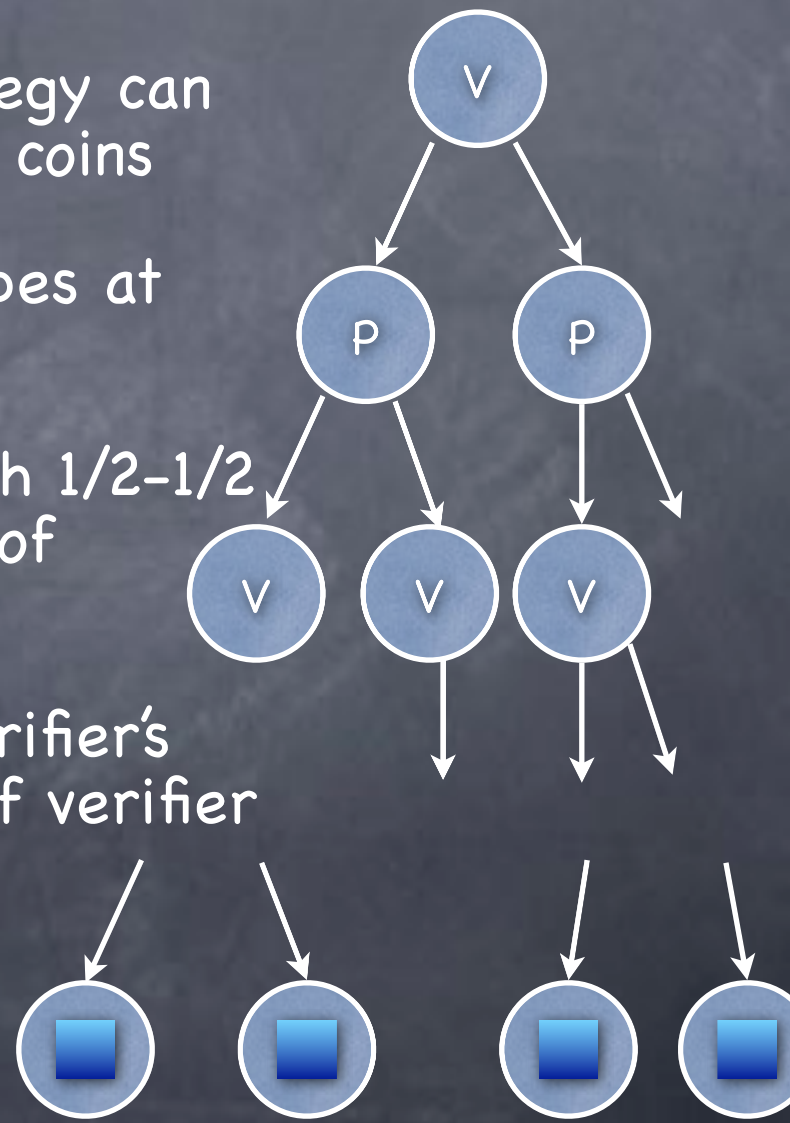

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  - Decide whether a QBF is true or not

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- Recall TQBF
  - Decide whether a QBF is true or not
  - QBF: $Q_1x_1\ Q_2x_2\ \dots\ Q_nx_n\ F(x_1,\dots,x_n)$ for quantifiers Q_i and a formula F on boolean variables

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      - Can always use a polynomial linear in each variable since $x^n=x$ for x=0 and x=1

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    - Consider proving = K (will be useful in the general case)

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Needs degree to be small

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  - Base case: n=0. Verifier will simply use oracle access to P.
  - For n>0: Let $R(X) := \Sigma_{x2}...\Sigma_{xn} P(X,x_2,...,x_n)$
    - $\Sigma_{x1}...\Sigma_{xn} P(x_1,...,x_n) = R(0) + R(1)$
    - R has only one variable and degree at most d

Only Σ, no Π

  - Prover sends T=R (as d+1 coefficients) to verifier

Needs degree to be small

  - Verifier checks K = T(0) + T(1). Still needs to check T=R

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    - Note: P_1 has degree at most d; verifier has oracle access to P_1 (as it knows a, and has oracle access to P)

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        - At most nd/p if n variables. Can take p exponential.

IP Protocol for TQBF

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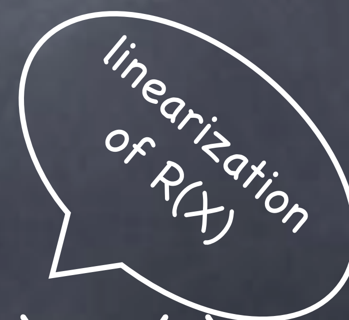

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  - Verifier checks (as appropriate) $L(1).L(0) = K$ or $L(1)+L(0) = K$

linearization of R(X)

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