Probabilistic Computation

Lecture 15

Computing with Less Randomness, or with Imperfect Randomness

Recall

Soundness Amplification for BPP

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Soundness Amplification for BPP

- Repeat M(x) t times and take majority
  - i.e. estimate Pr[M(x)=yes] and check if it is > 1/2
  - Error only if | estimate-real | ≥ gap/2
  - Estimation error goes down exponentially with t: Chernoff bound
    - Pr[|estimate - real| ≥ δ/2] ≤ $2^{-\Omega(t.\delta^2)}$
      - $t = O(n^d/\delta^2)$ enough for Pr[error] ≤ 2^{-n^d}

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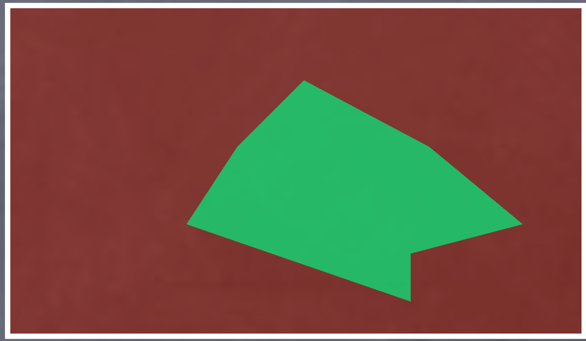

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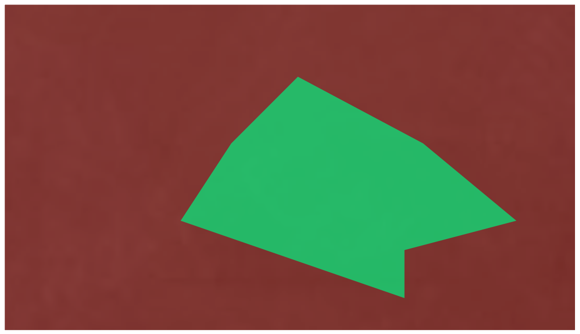

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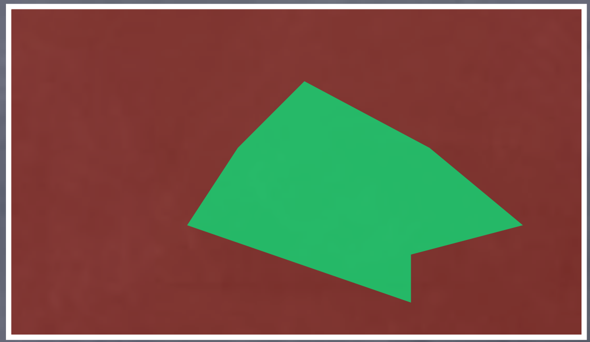

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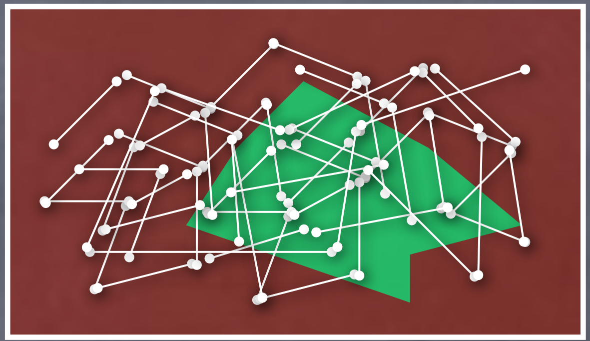

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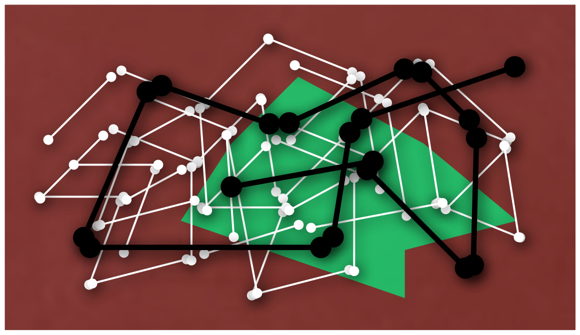

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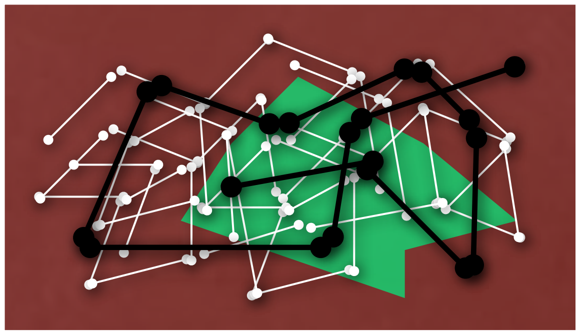

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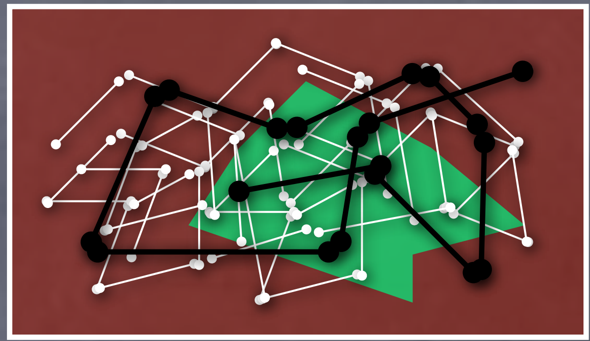

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        - Can we work with imperfect random sources?

Philosophical Issues with Randomness/Probability

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- Slightly imperfect
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  - Sufficient independence
  - Don't know the exact distribution, but belongs to a known class of distributions

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- Weaker guarantees: e.g. Block source

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      - Wt. of t strings from a $\delta < 1/m$ bias SV source $< e$. (Wt of t strings under uniform distribution)

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      $(1+x)^{1/x} \leq e$
      - Wt. of t strings from a $\delta < 1/m$ bias SV source $< e$. (Wt of t strings under uniform distribution)
  - If on perfect randomness, Pr[error] < $1/(e2^n)$, then on imperfect randomness with bias < 1/m, Pr[error] < $1/2^n$

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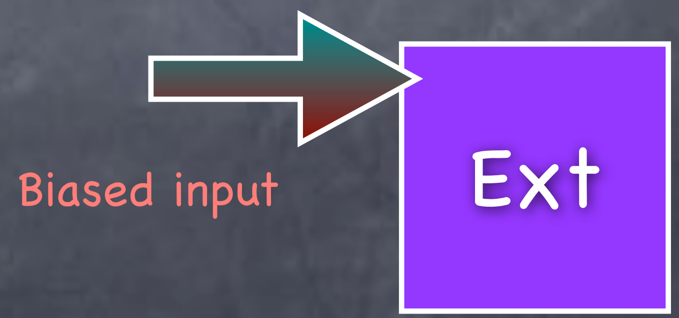

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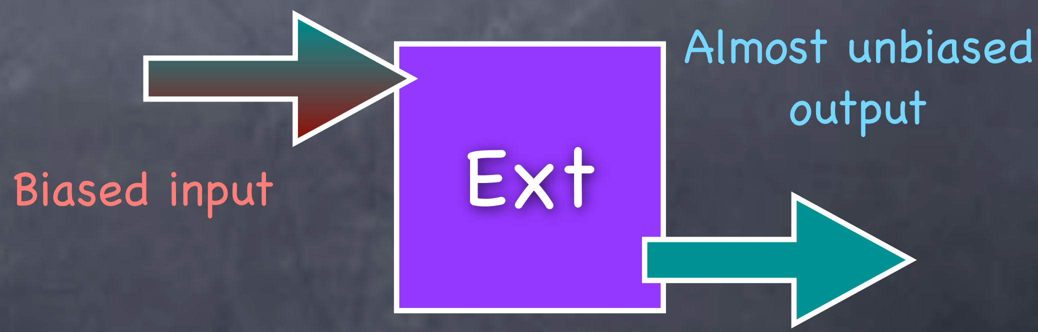

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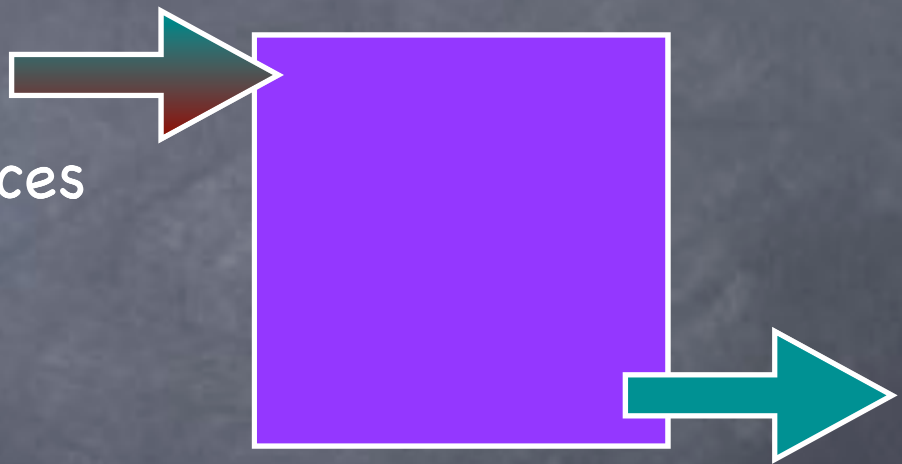

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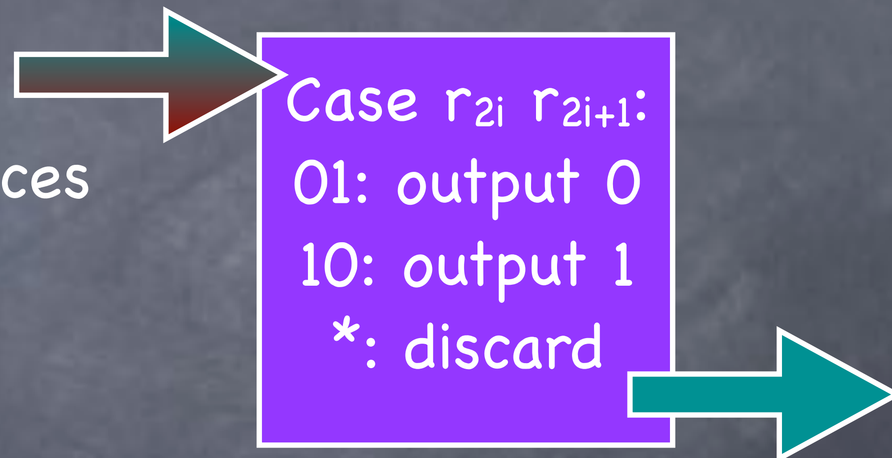

Simple extractor for von Neumann Sources

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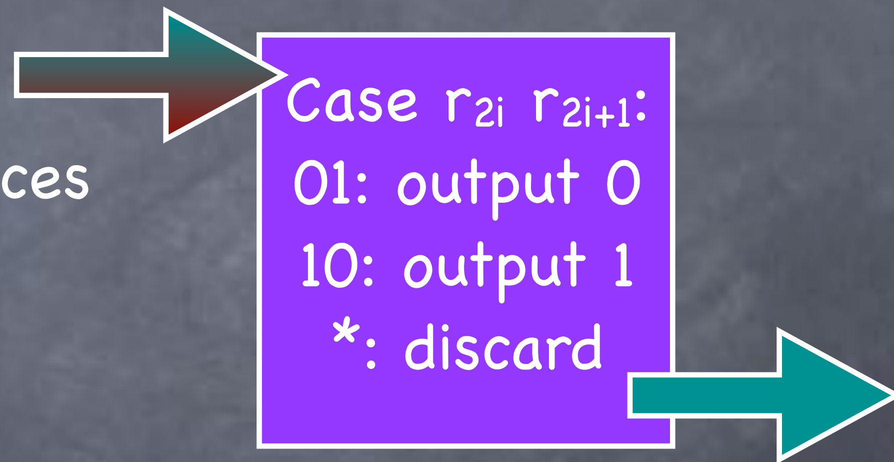
  - Perfectly random output



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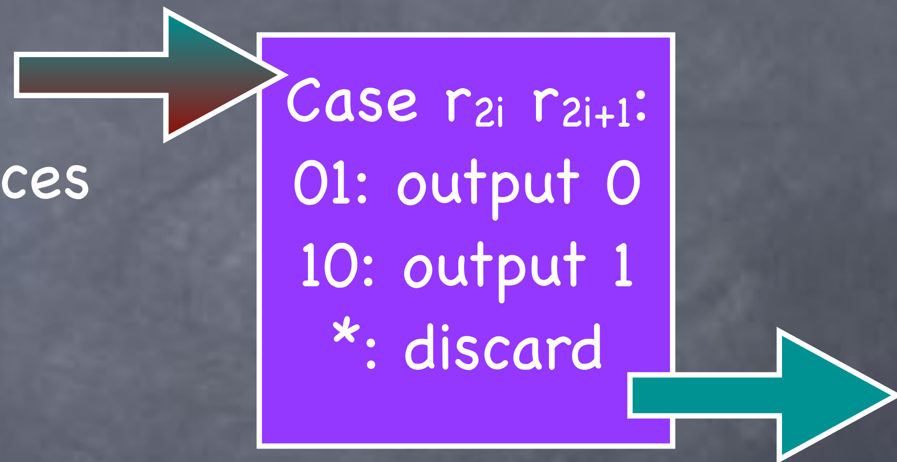

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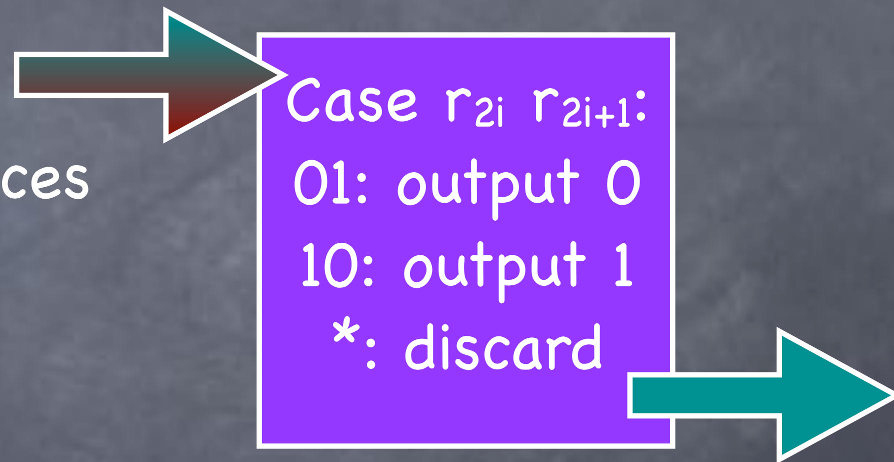

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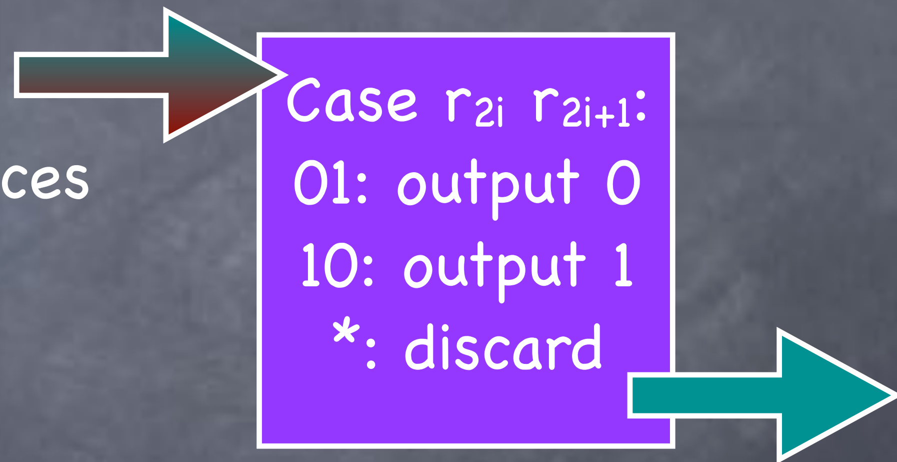

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  - Running time (per bit): constant number of tries, expected
- Can be generalized to sources which are (hidden) Markov chains

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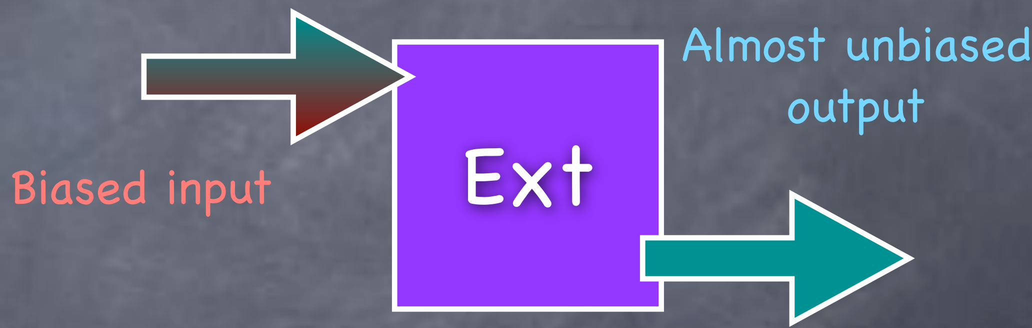

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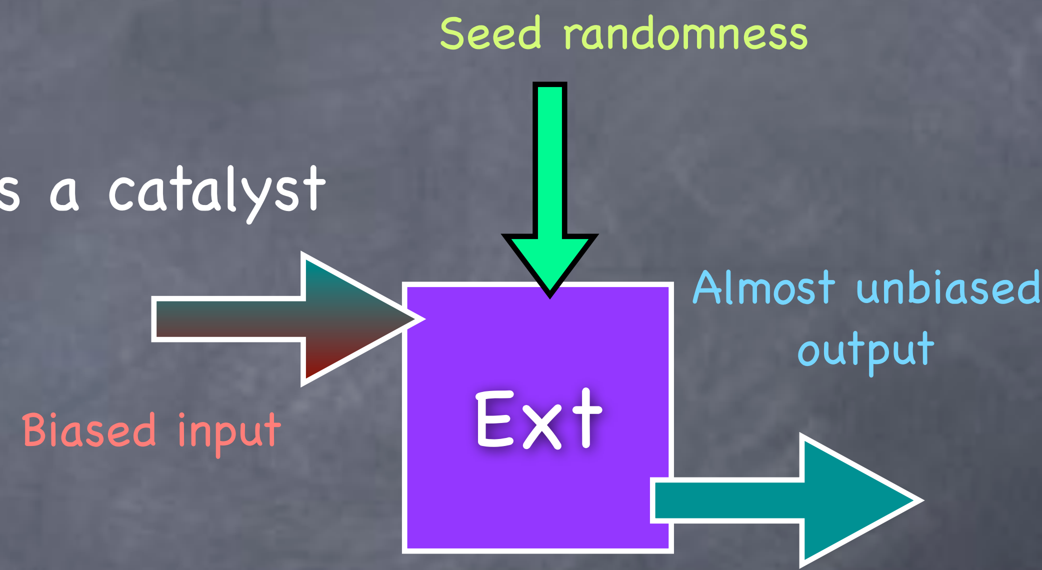

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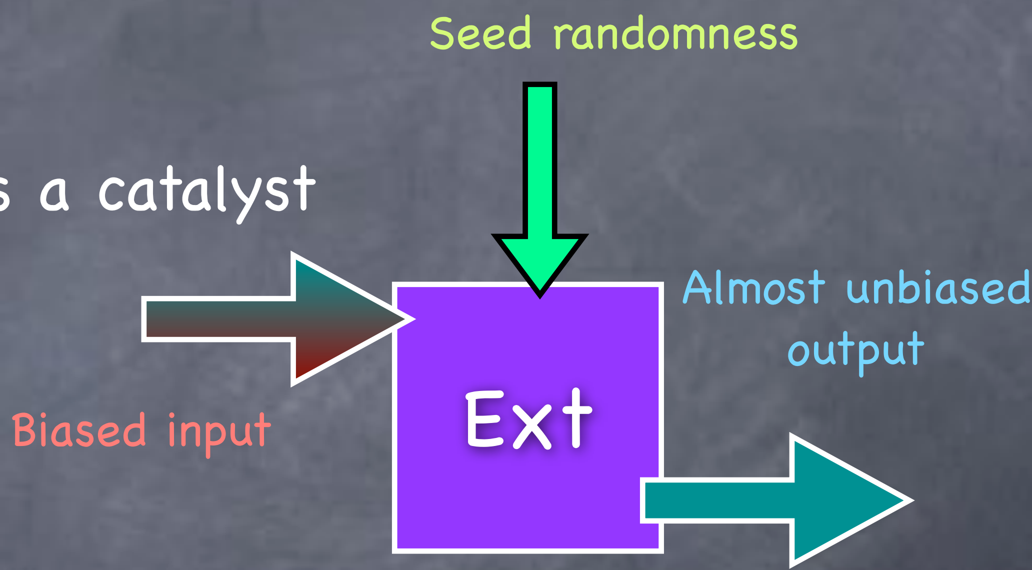

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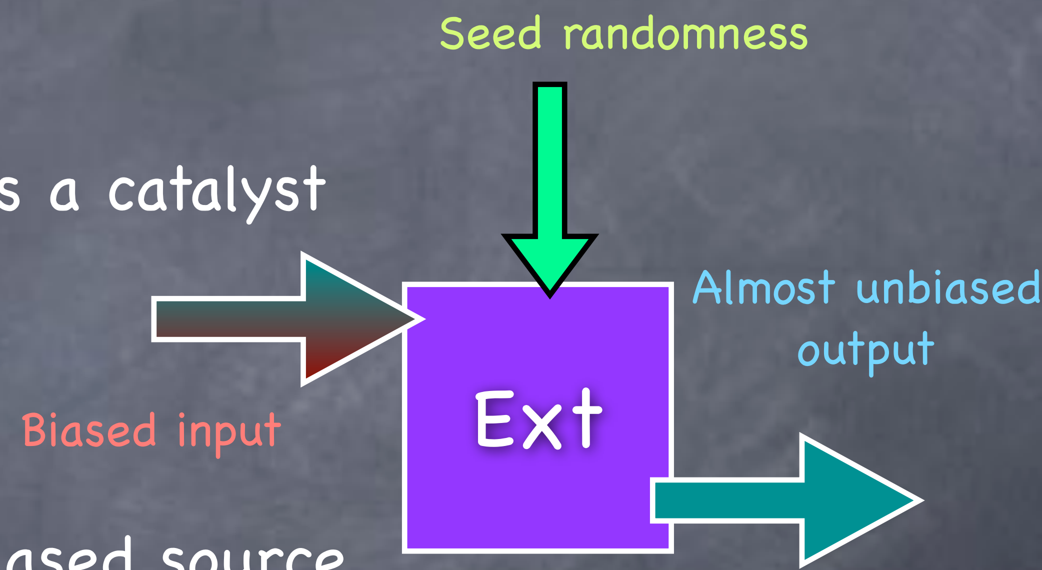

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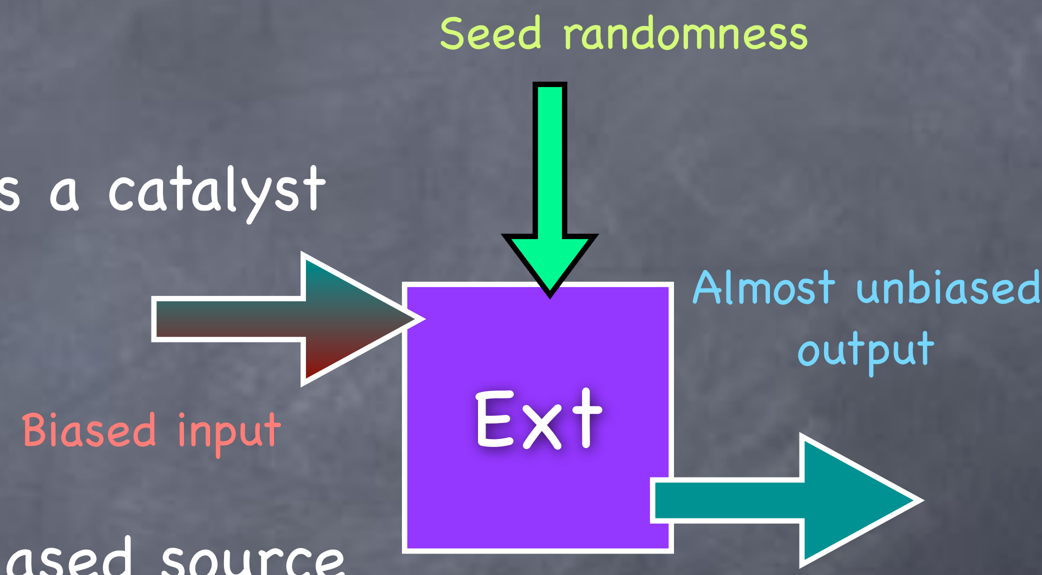

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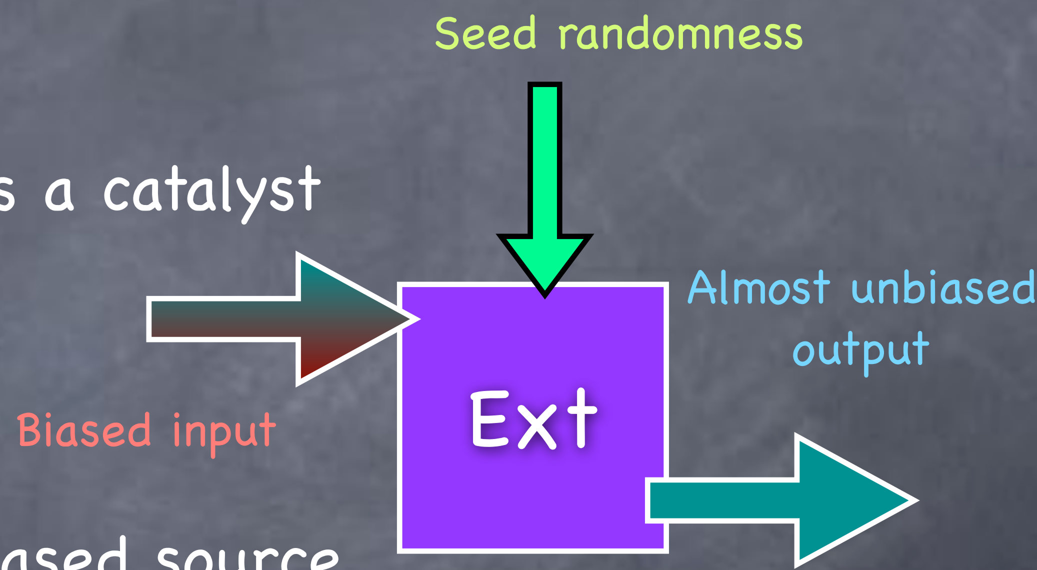
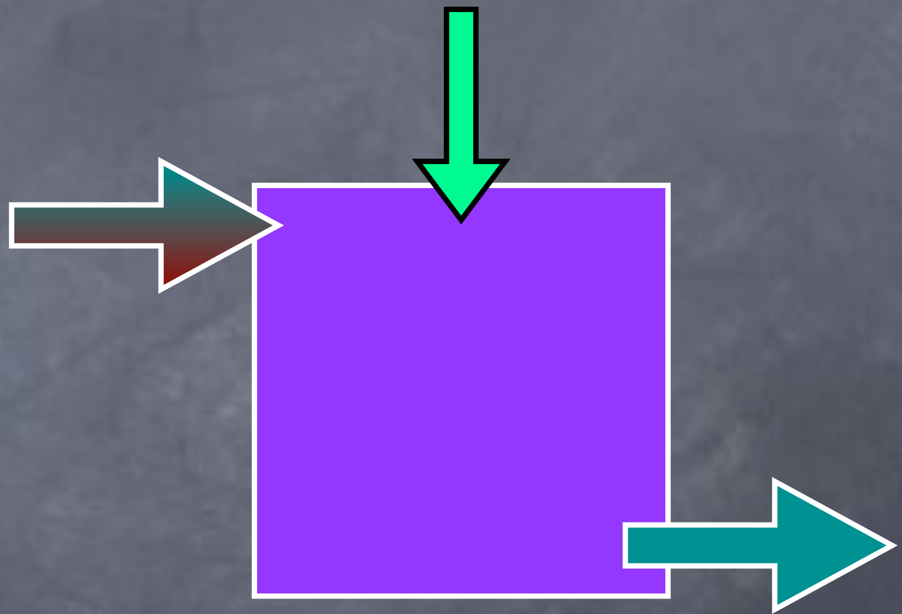
    - Error probability remains bounded [Exercise]



Extractor for SV sources



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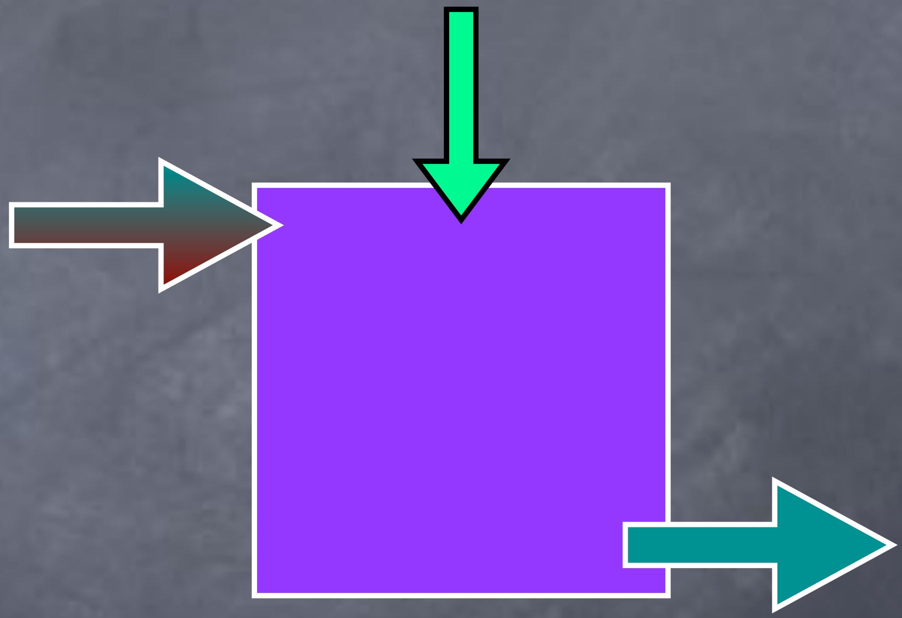

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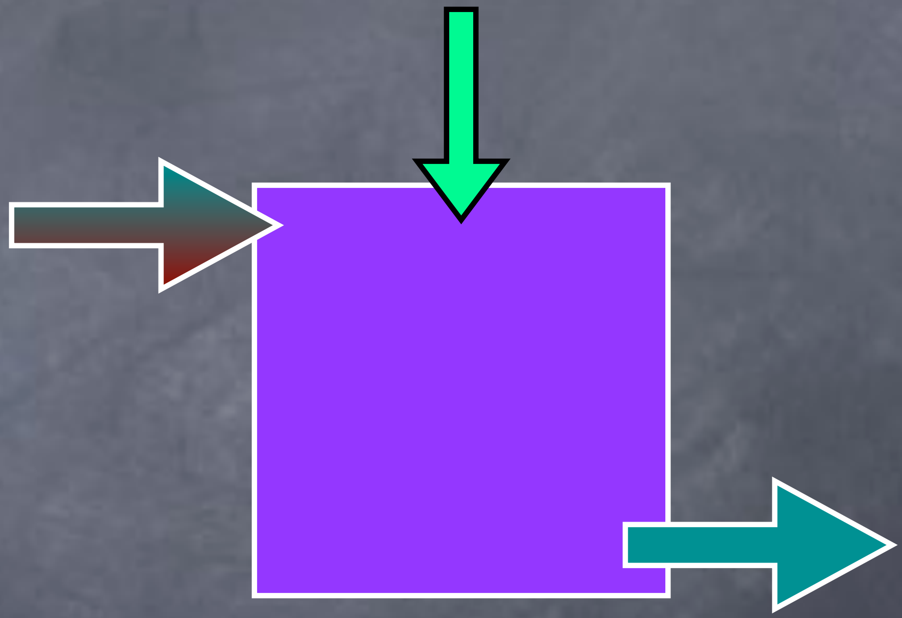

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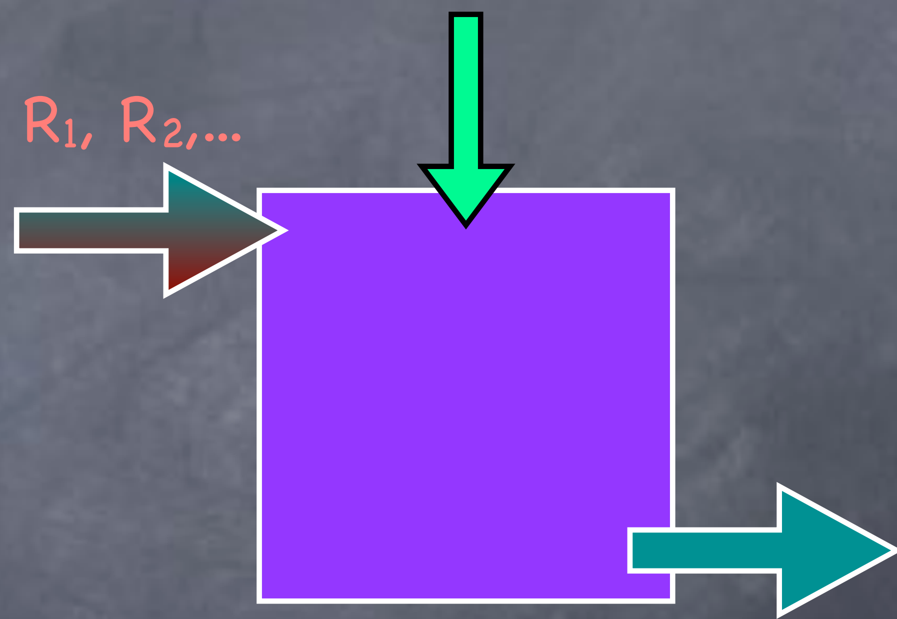

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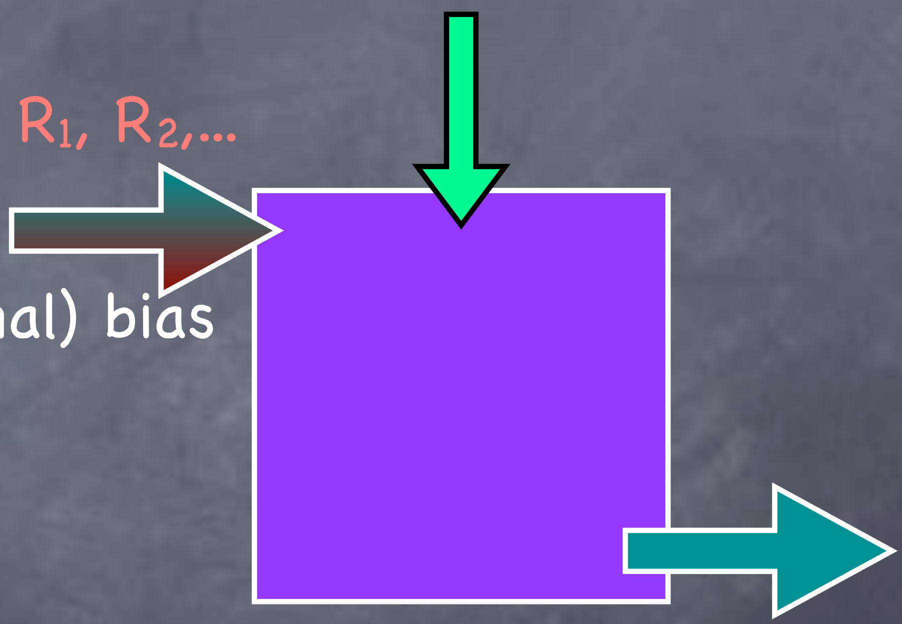

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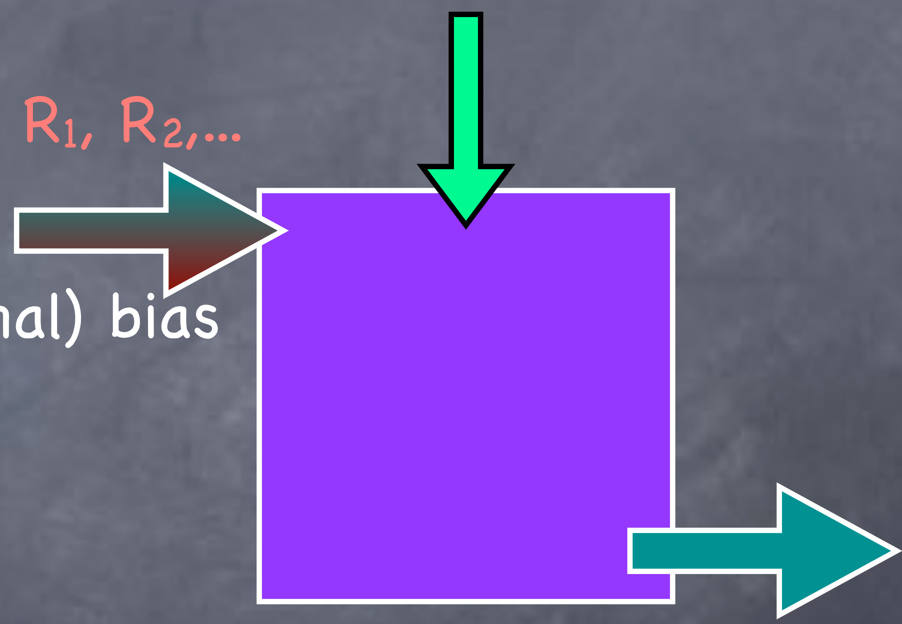

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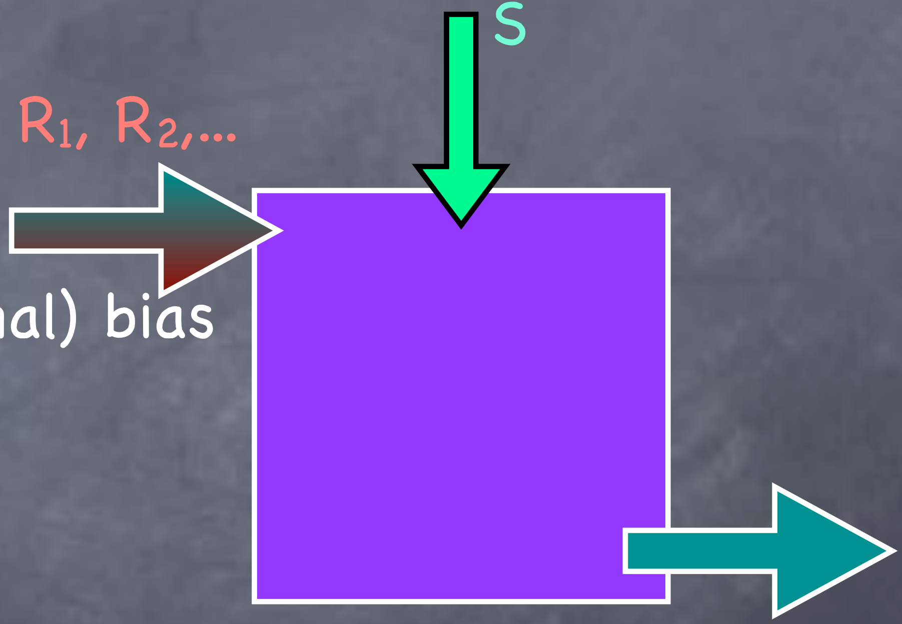

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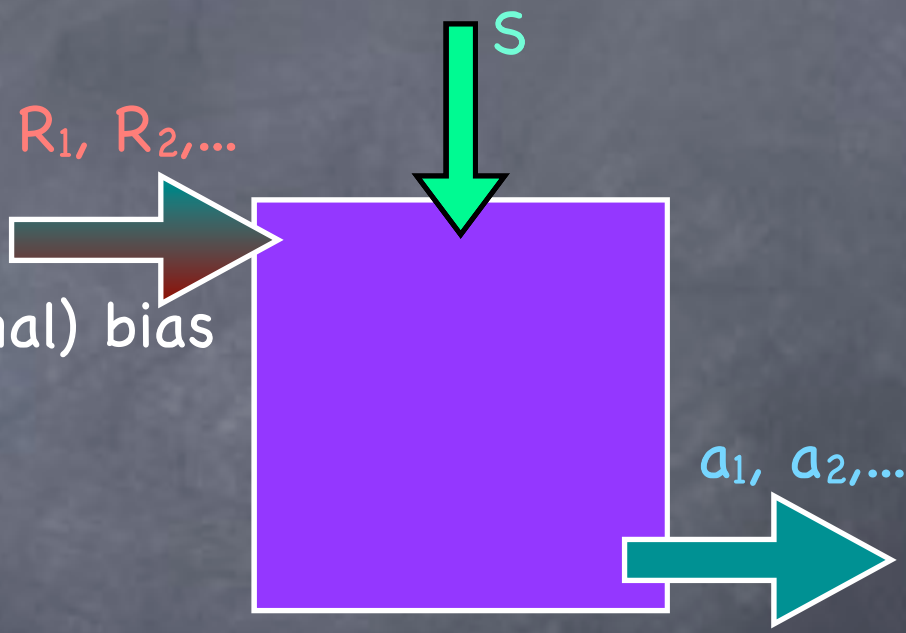

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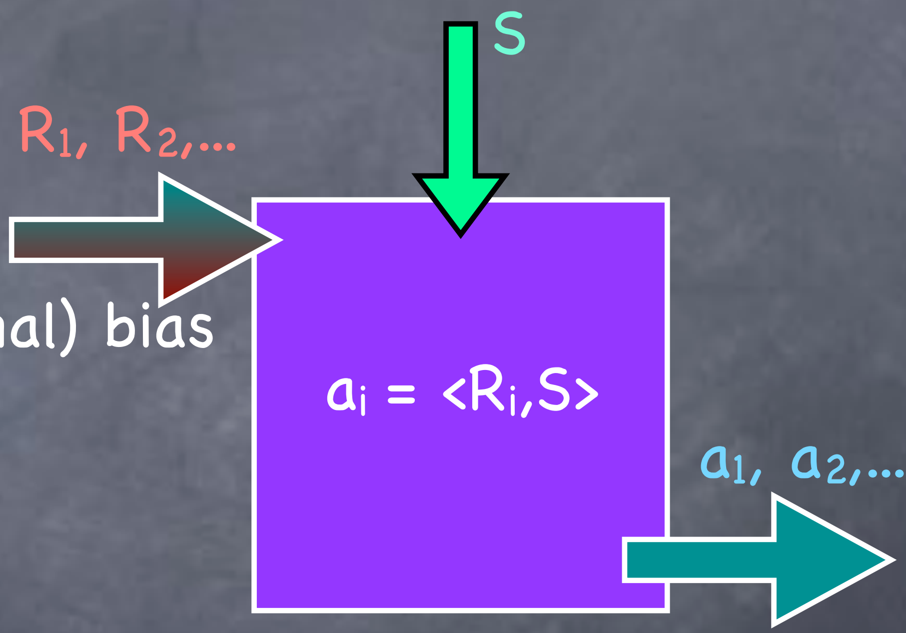

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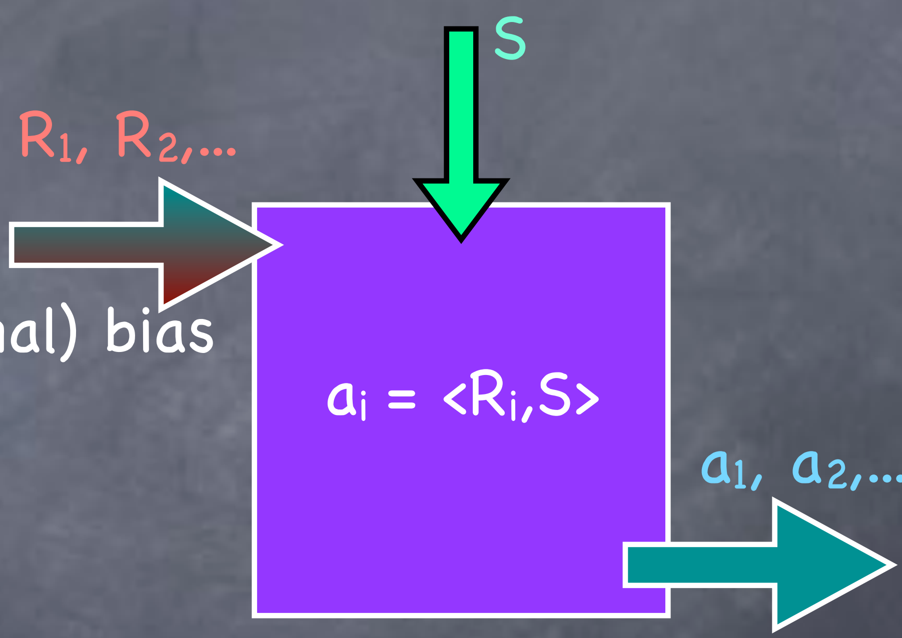
  - Input: SV(δ) for a constant δ<1
  - Plan: to get to a small (conditional) bias (O(1/m)) for each output bit.
    - Weak extraction
  - Using seed-length d = O(log m)



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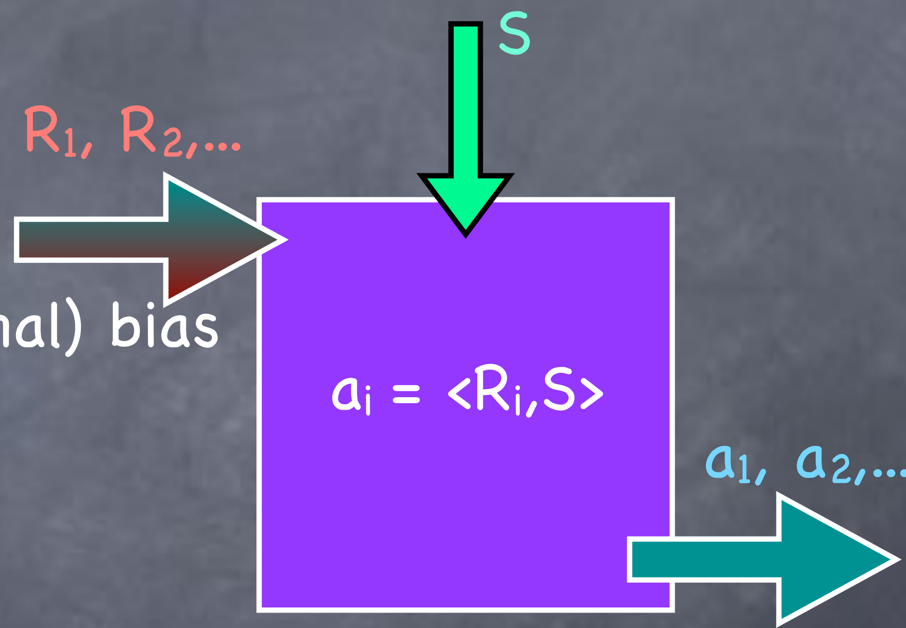

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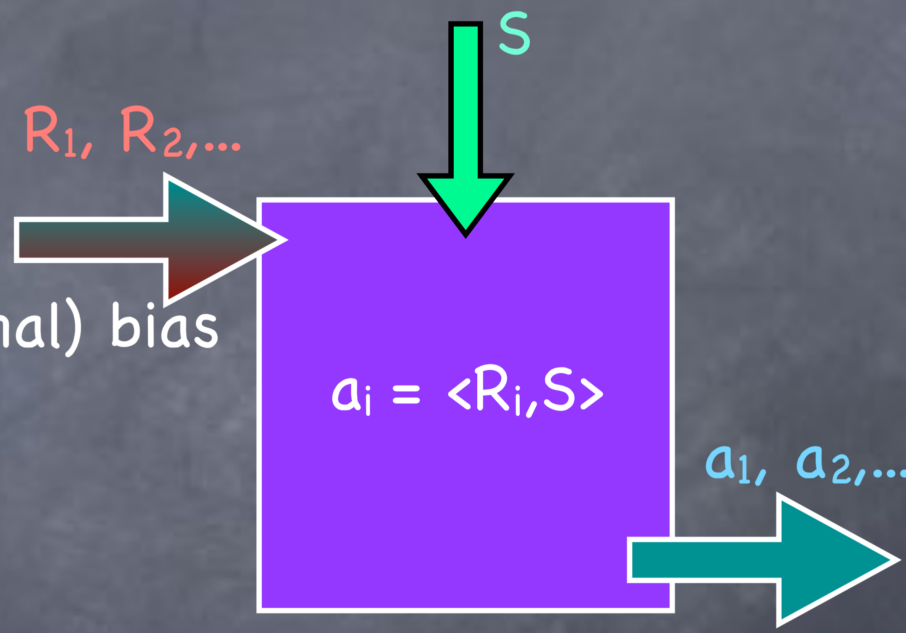
  - Collision prob $\leq$ max prob $\leq (1/2 + \delta/2)^d = 1/\text{poly}(m)$



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- Bottom line: Can efficiently run BPP algorithms using very general classes of sources of randomness

Extracting from independent sources

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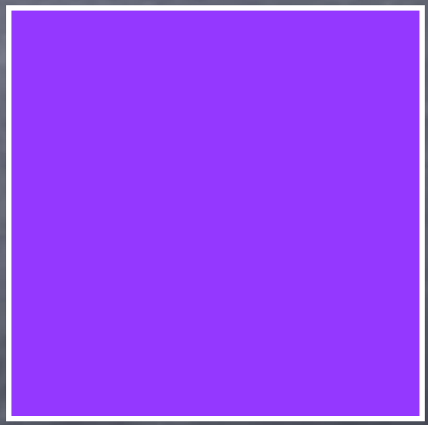

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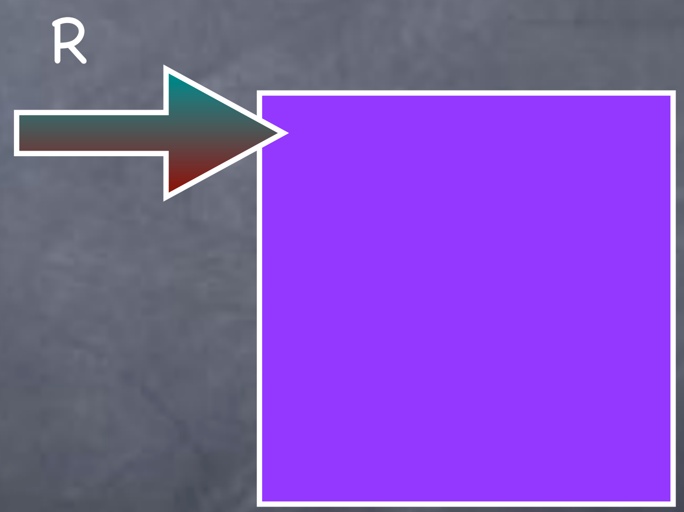

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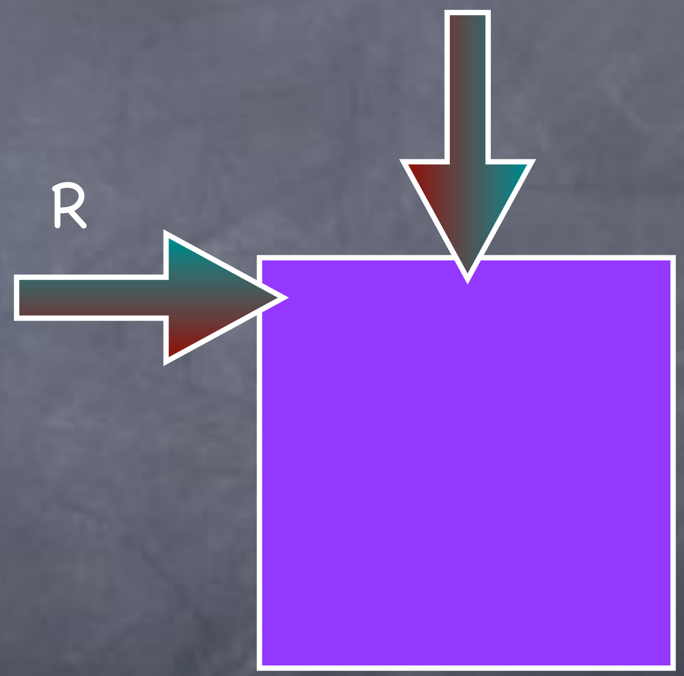

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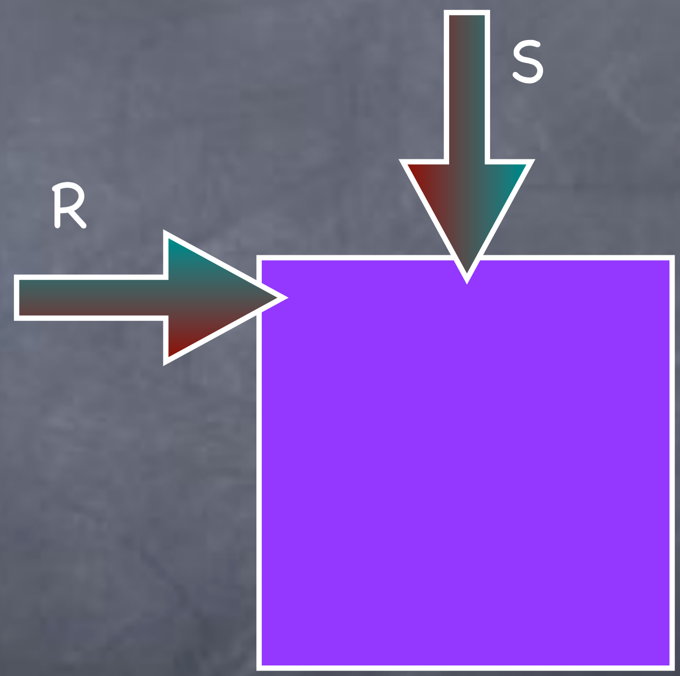

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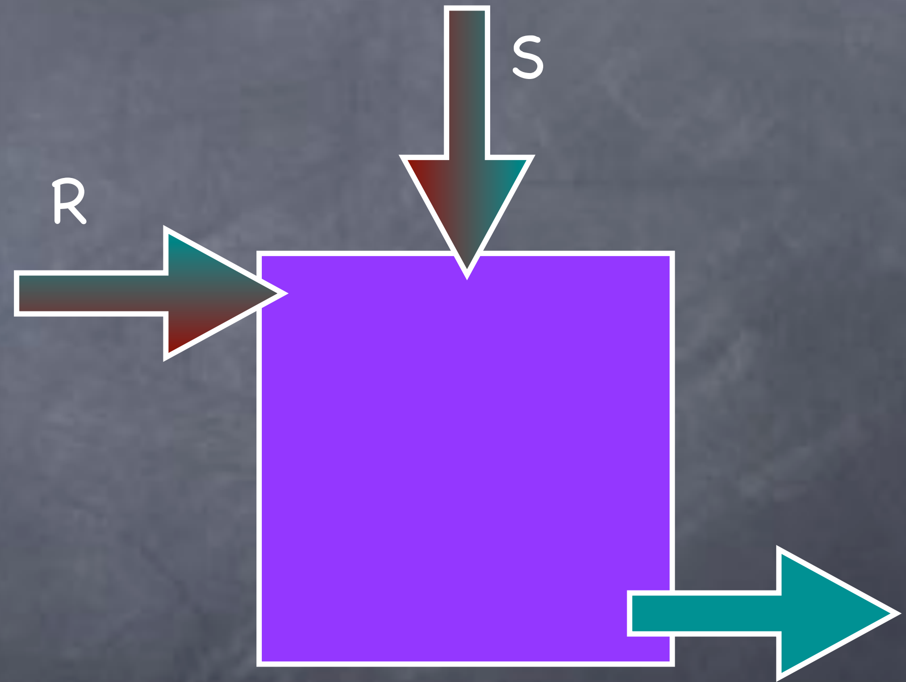

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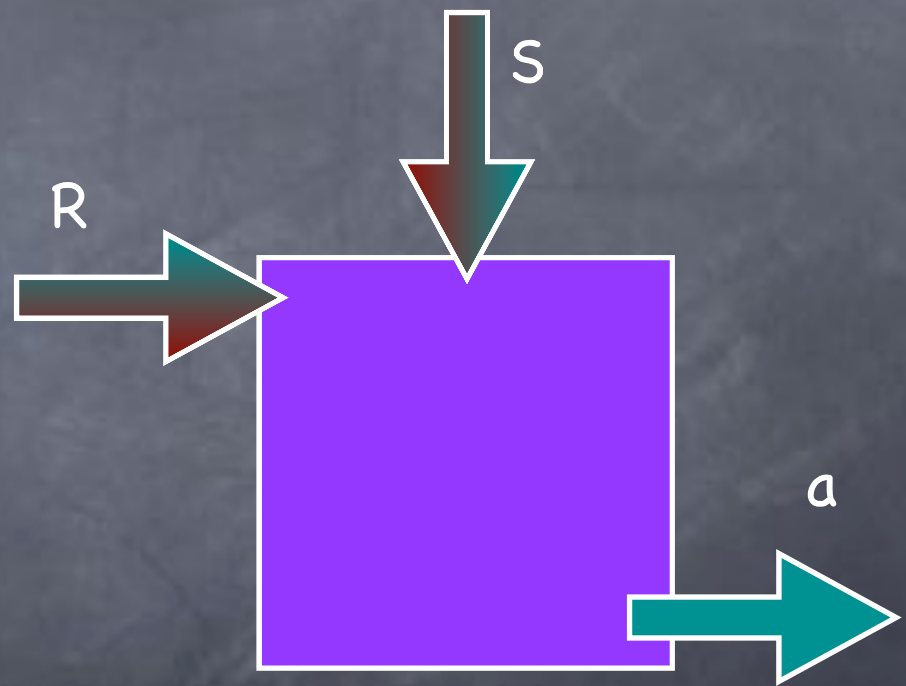

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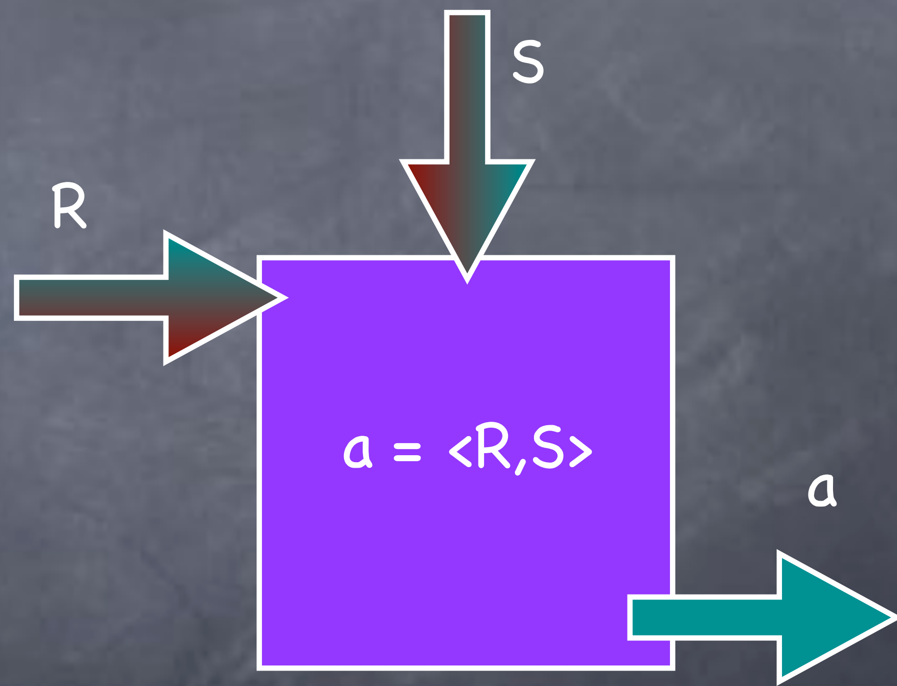

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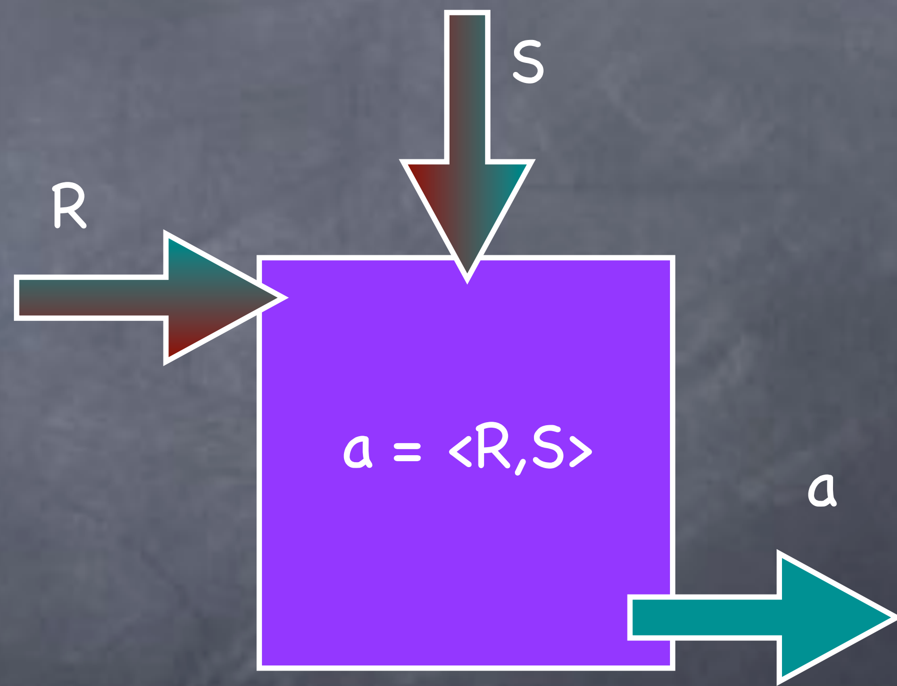

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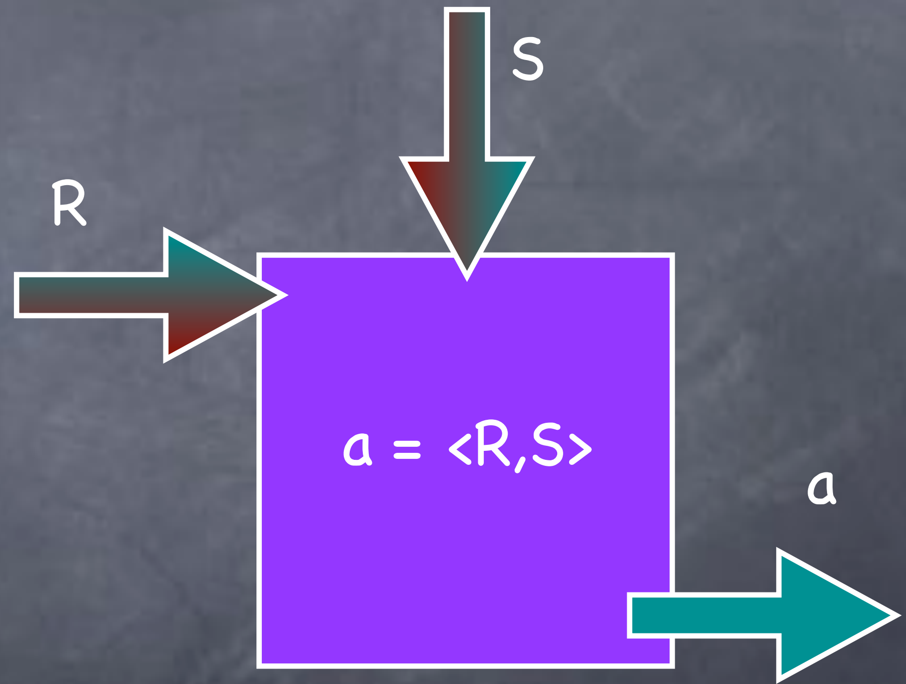
  - Known, with a few more sources



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