Probabilistic Computation

Lecture 13

Understanding BPP

Recap

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- Probabilistic computation

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- NTM (on “random certificates”) for L:

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    - Pr[M(x)=yes]:

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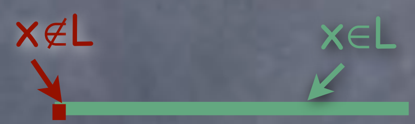
  - Pr[M(x)=yes]: 

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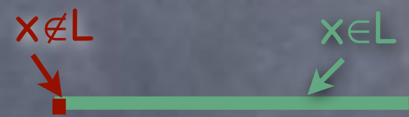
  - Pr[M(x)=yes]: 

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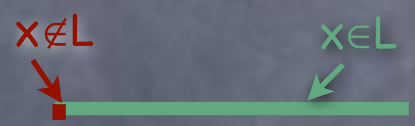

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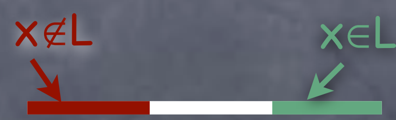

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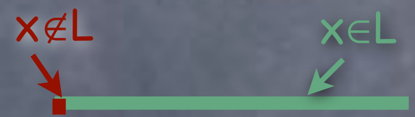
  - Pr[M(x)=yes]: 

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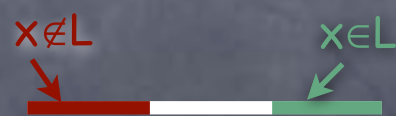
- BPTM for L: Pr[yes]: 

- RTM for L: Pr[yes]: 

Recap

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- PP, RP, co-RP, BPP

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  - PP too powerful: NP $\subseteq$ PP

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- PP, RP, co-RP, BPP
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- Today:

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- Today:
  - NP $\not\subseteq$ BPP, unless PH collapses
  - BPP $\subseteq \Sigma_2^P \cap \Pi_2^P$

BPP vs. NP

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- Can randomized algorithms efficiently decide all NP problems?

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  - Unlikely: NP $\subseteq$ BPP $\Rightarrow$ PH = Σ_2^P
  - Will show BPP $\subseteq$ P/poly
    - Then NP $\subseteq$ BPP $\Rightarrow$ NP $\subseteq$ P/poly
      - $\Rightarrow$ PH = Σ_2^P

$\text{BPP} \subseteq \text{P/poly}$

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r \ x						
	✓	✗	✗	✓	✓	✓
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	✓	✗	✗	✓	✓	✓
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  - Then, can give that random tape as advice
- One such random tape if average (over x) error probability is less than 2^{-n}

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BPP ⊆ P/poly

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  - BPP: can make worst error probability $< 2^{-n}$

r \ x						
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	✗	✗	✓	✓	✓	✓
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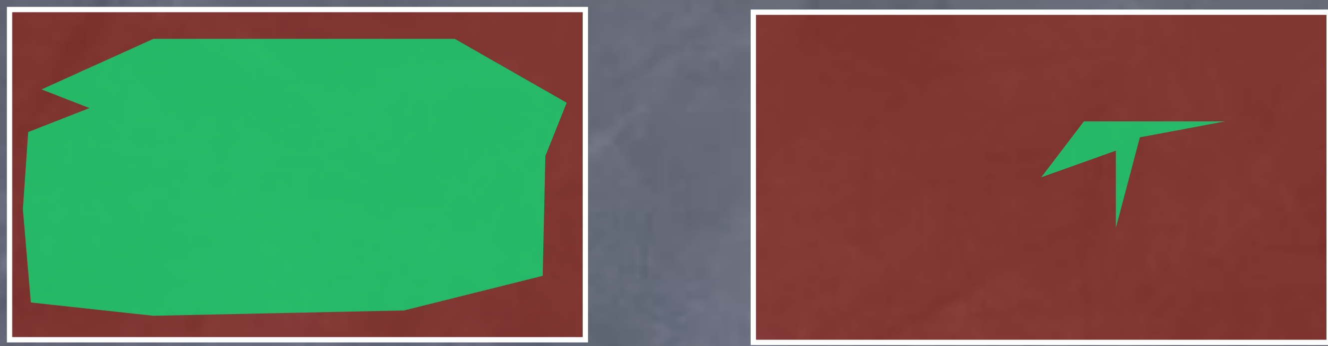
    - Note: Neighborhood of z is small (polynomially large), so can go through all of them in polynomial time

$BPP \subseteq \Sigma_2^P$

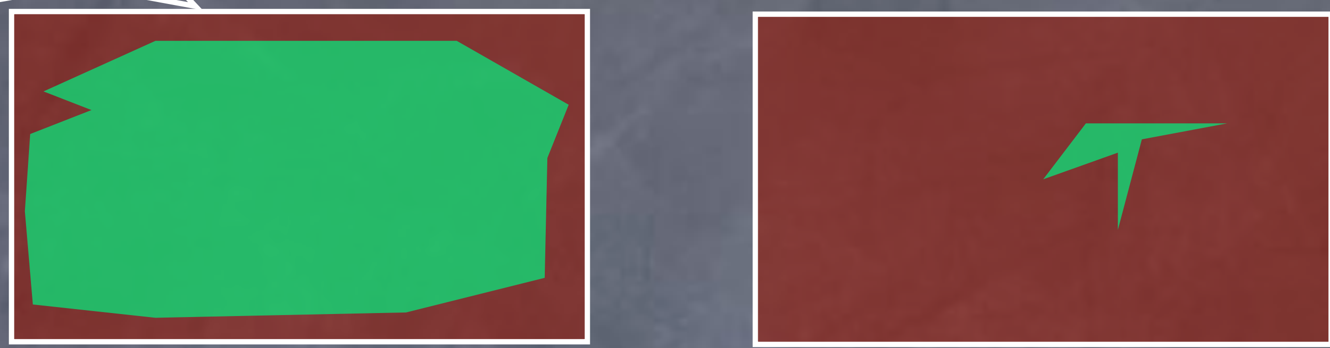


Space of random tapes = $\{0,1\}^m$

$Yes_x = \{r \mid M(x,r)=yes\}$

BPP $\subseteq \Sigma_2^P$

$x \in L$: $|Yes_x| > (1-2^{-n})2^m$



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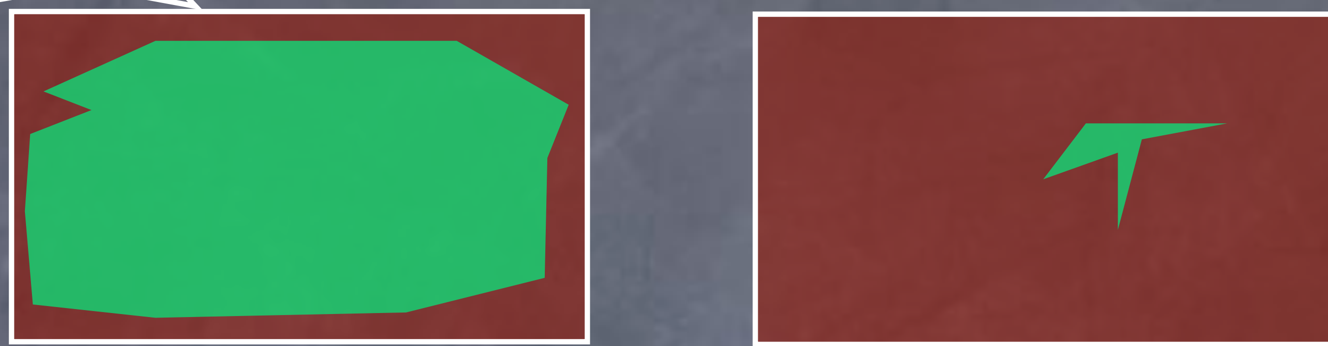

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- $x \notin L$: Yes_x very small, so its few shifts cover only a small region

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    - In fact, most P work (if k big enough)!

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  - Define a probability distribution over all candidate P's and prove that the property holds with positive probability (often even close to one)

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- Probabilistic Method (finding hay in haystack)
  - To prove ∃P with some property
  - Define a probability distribution over all candidate P's and prove that the property holds with positive probability (often even close to one)
    - Distribution s.t. easy to prove positive probability of property holding

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  - $Pr_{(\text{over } P)}[P(S) \neq \{0,1\}^m] = Pr_{(\text{over } P)}[\exists z\ z \notin P(S)]$

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$BPP \subseteq \Sigma_2^P$

- Probabilistic method to find $P = \{u_1,u_2,...,u_k\}$, s.t. for all large S, $P(S) = \{0,1\}^m$
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    - Pick each u_i independently, and uniformly at random from $\{0,1\}^m$
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    $= \Sigma_z \Pi_i\ (|S^c|/2^m)$

$\text{BPP} \subseteq \Sigma_2^P$

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    $= \Sigma_z \Pi_i (|S^c|/2^m) \quad < \Sigma_z \Pi_i\ 2^{-n}$

$BPP \subseteq \Sigma_2^P$

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    $= \Sigma_z\, \Pi_i\ (|S^c|/2^m) \quad < \Sigma_z\, \Pi_i\ 2^{-n} \quad = 2^m.(2^{-n})^k = 1$

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    $= \Sigma_z \Pi_i \Pr_{(\text{over } ui)}[z \oplus u_i \notin S] = \Sigma_z \Pi_i \Pr_{(\text{over } ui)}[u_i \notin z \oplus S]$

    $= \Sigma_z \Pi_i (|S^c|/2^m) \quad < \Sigma_z \Pi_i\ 2^{-n} \quad = 2^m.(2^{-n})^k = 1$
  - So (with $|S|>(1-2^{-n})2^m$ and $k=m/n$), $\exists P$, $P(S) = \{0,1\}^m$

BPP $\subseteq \Sigma_2^P$

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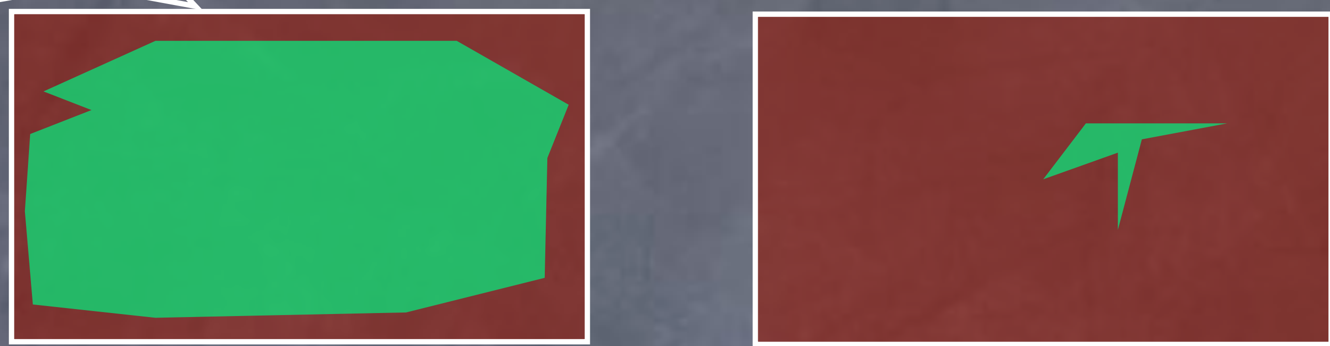

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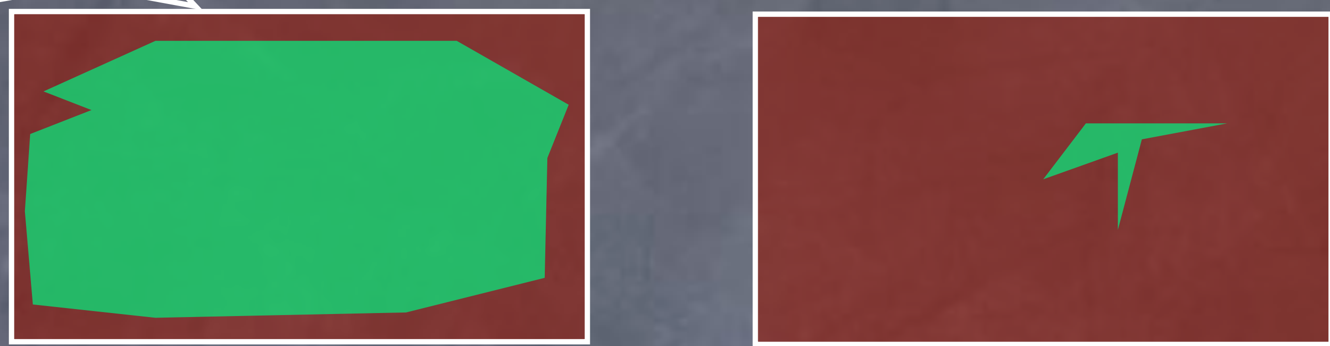

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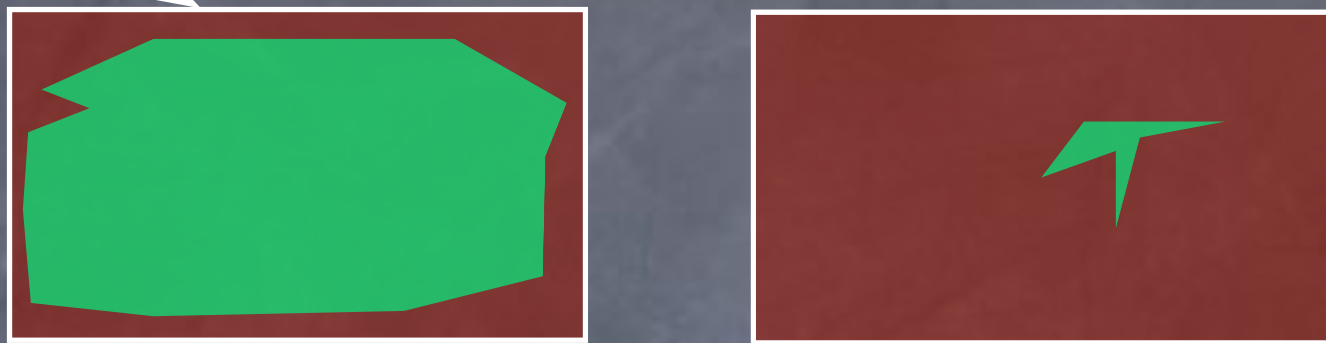

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- $L = \{ x \mid \exists P\ \forall z \text{ for some } r \in P^{-1}(z)\ M(x,r)=yes \}$

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    - Just run M(x) for t steps and accept if it accepts?
      - If $(M,x,1^t)$ in L, we will indeed accept with prob. > 2/3
      - But M may not have a bounded gap. Then, if $(M,x,1^t)$ not in L, we may accept with prob. very close to 2/3.

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