Probabilistic Computation

Lecture 12

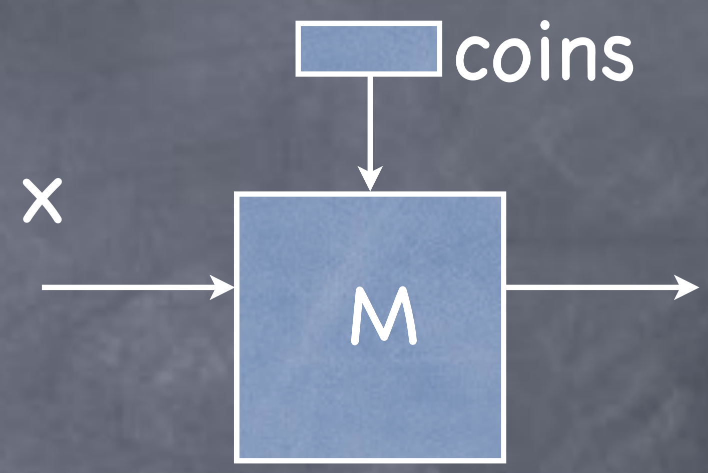
Flipping coins, taking chances

PP, BPP

Probabilistic Computation

Probabilistic Computation

- Output depends not only on x, but also on random "coin flips"



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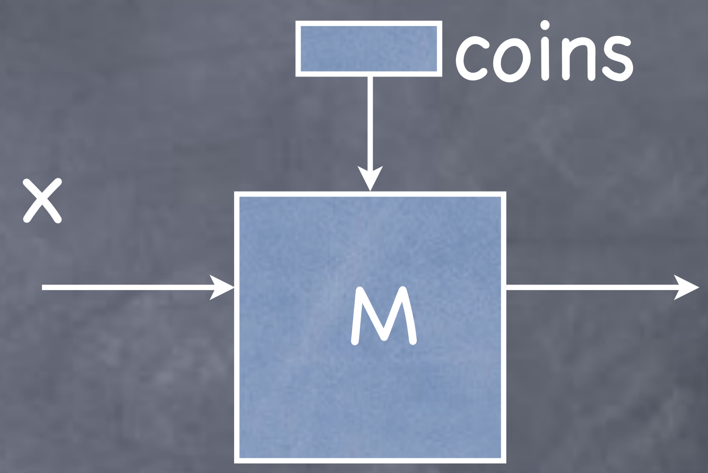
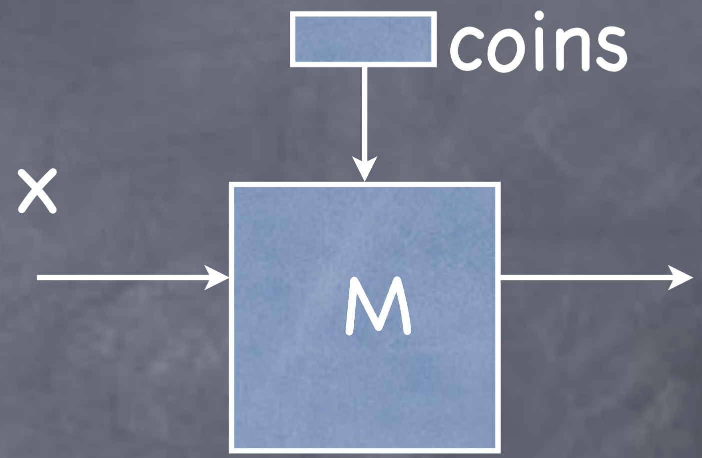
- M,x define a probability distribution over outcomes



Probabilistic Computation

- Output depends not only on x, but also on random "coin flips"
- M,x define a probability distribution over outcomes
- If for all x, M(x) equals f(x) with very high probability, could be used as f(x)



Language Decided by a Probabilistic Computation

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    - Last two: If on any x neither, M doesn't decide a language!
      - When M does decide, much better than random guess

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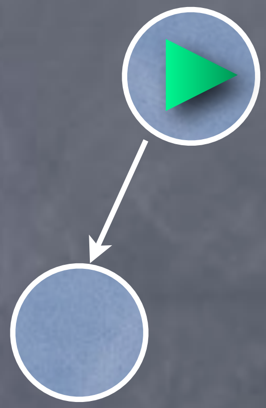

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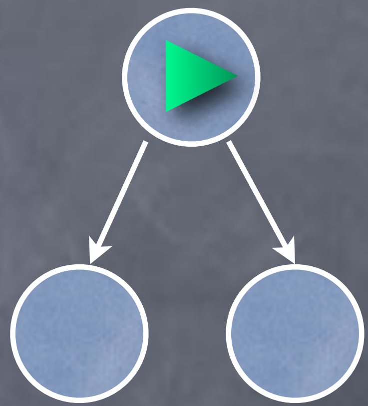

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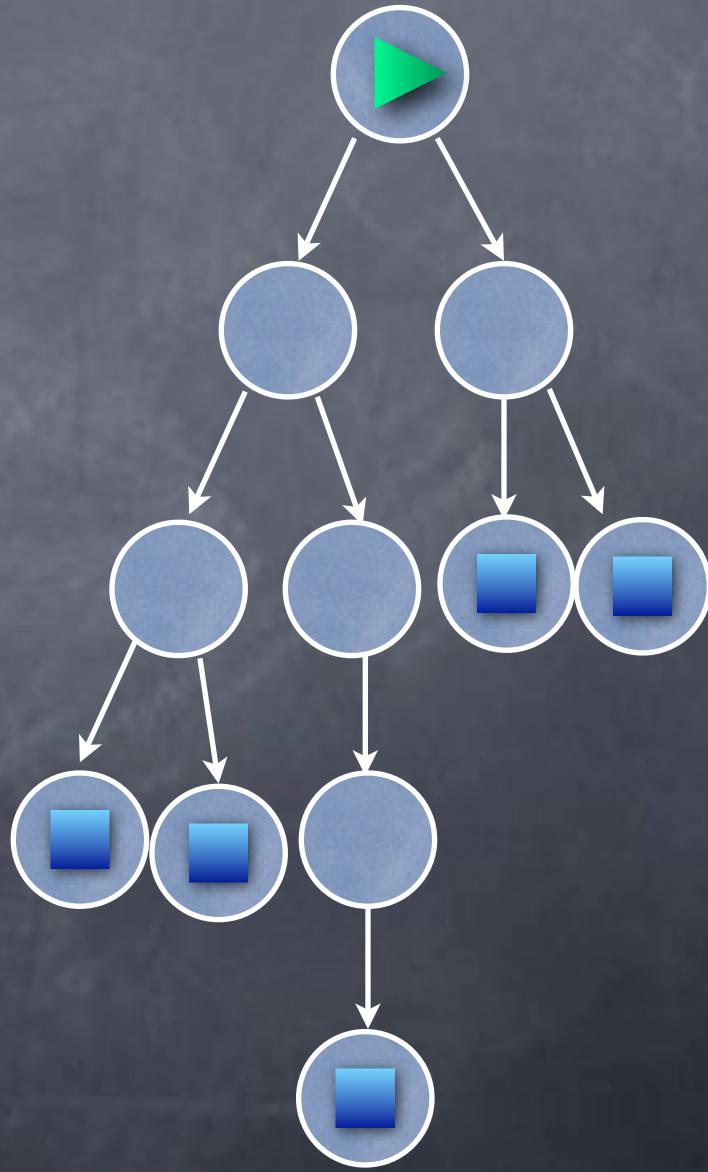

Probabilistic TM

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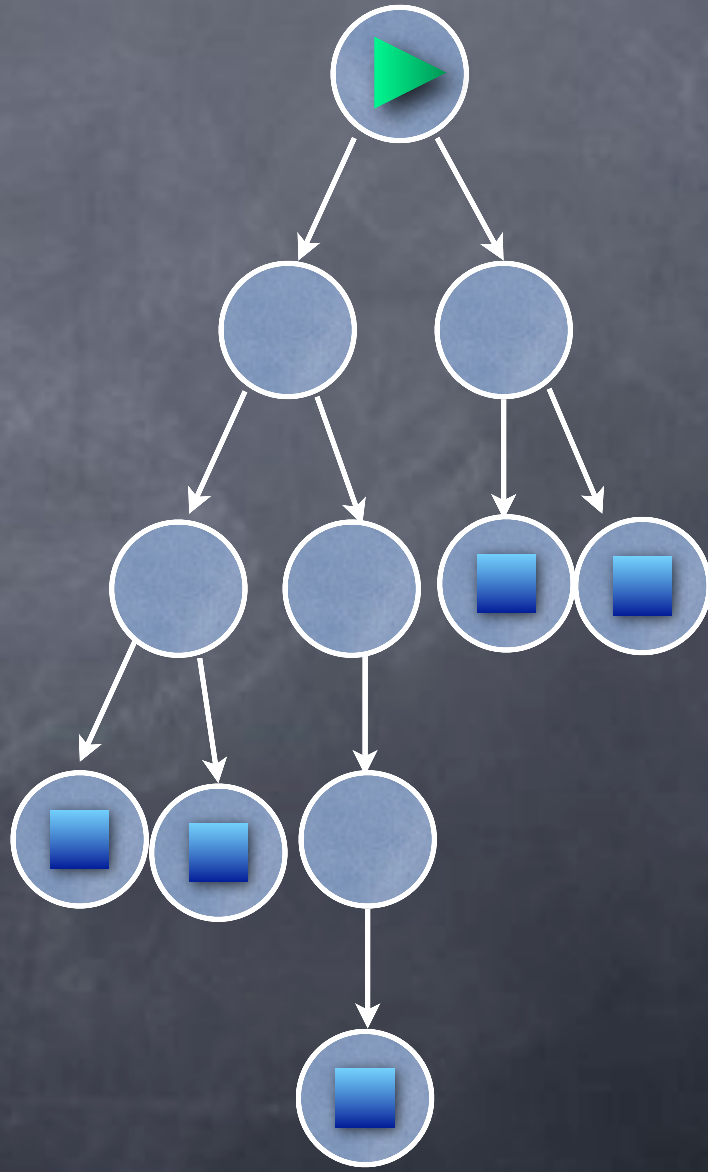
  - Defines a probability with which an input is accepted or rejected



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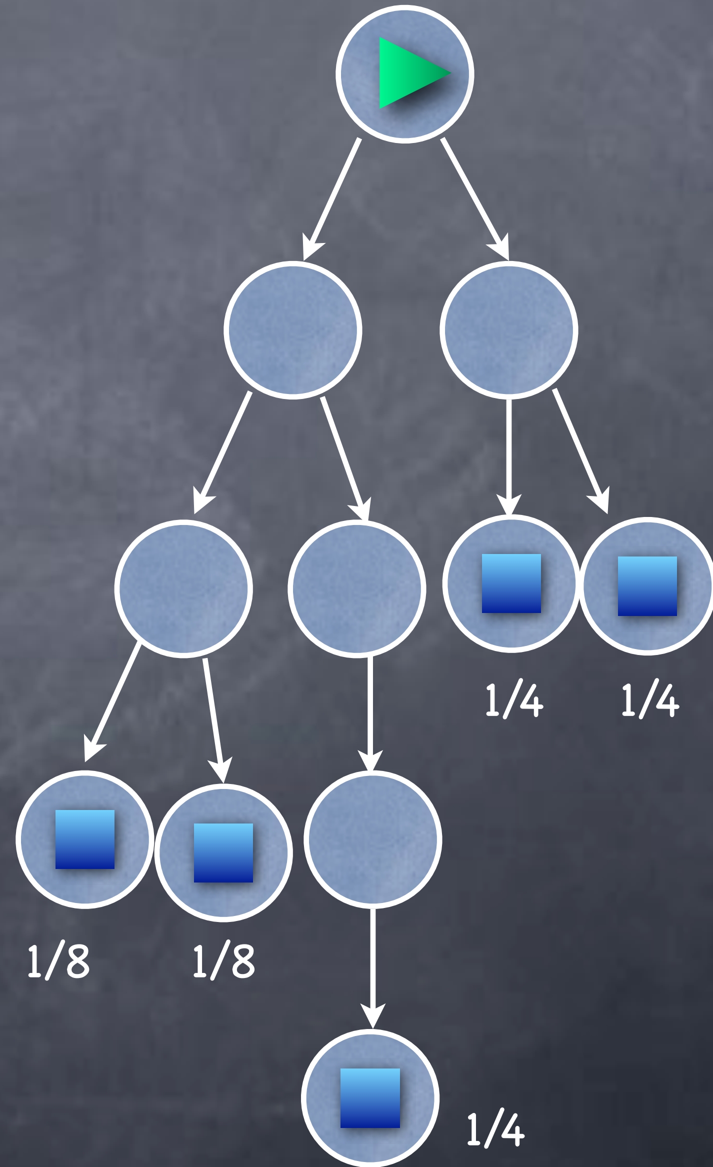
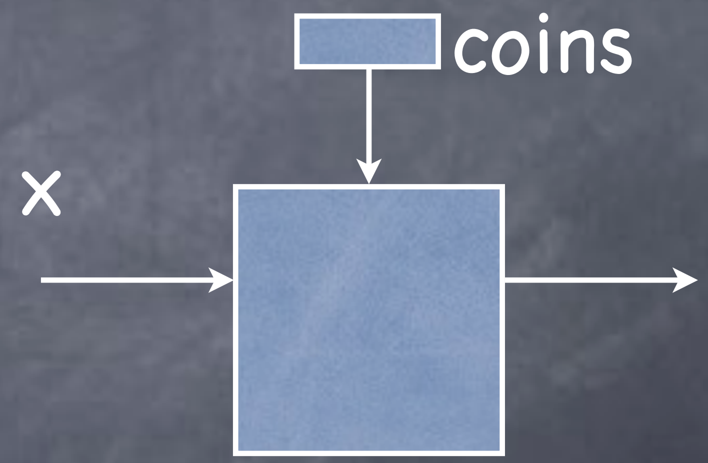
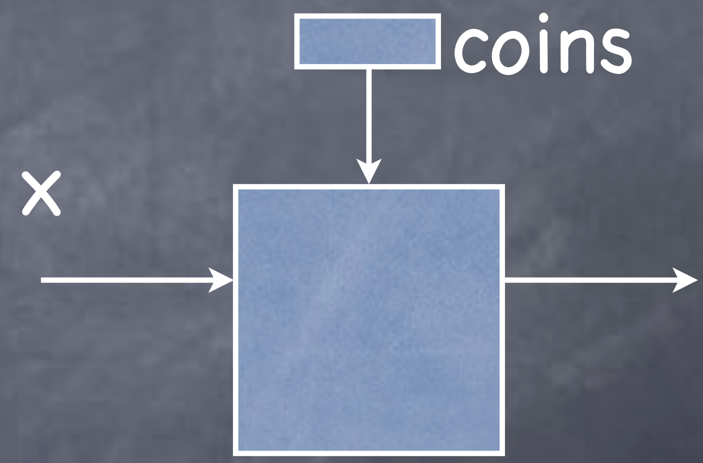


Random Tape



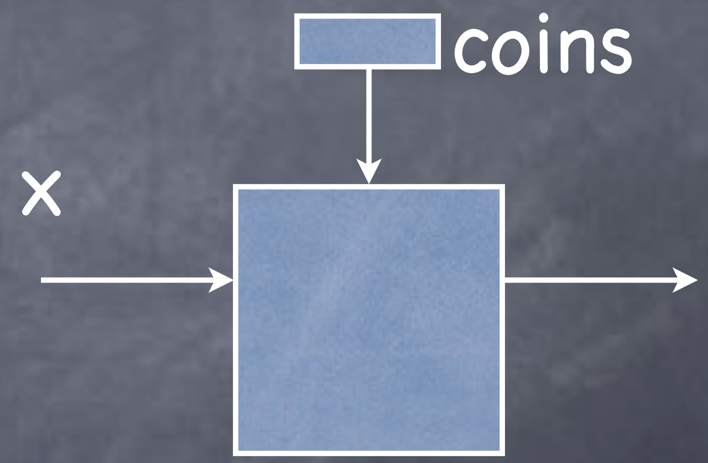
Random Tape

- Random choice: flipping a fair coin



Random Tape

- Random choice: flipping a fair coin
  - Coin flip is written on a read-once "random tape"



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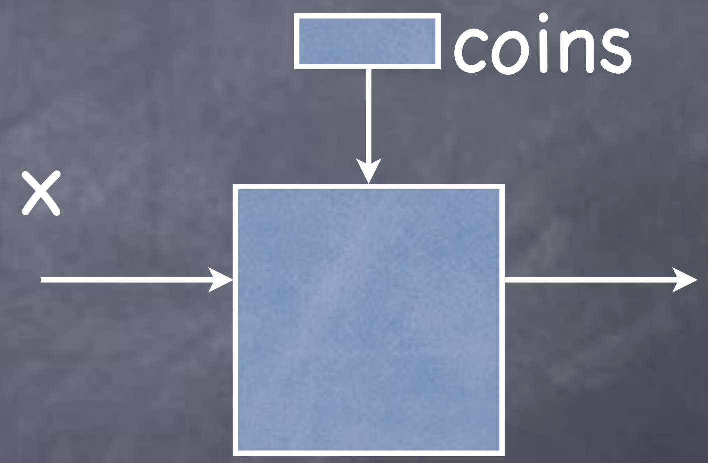
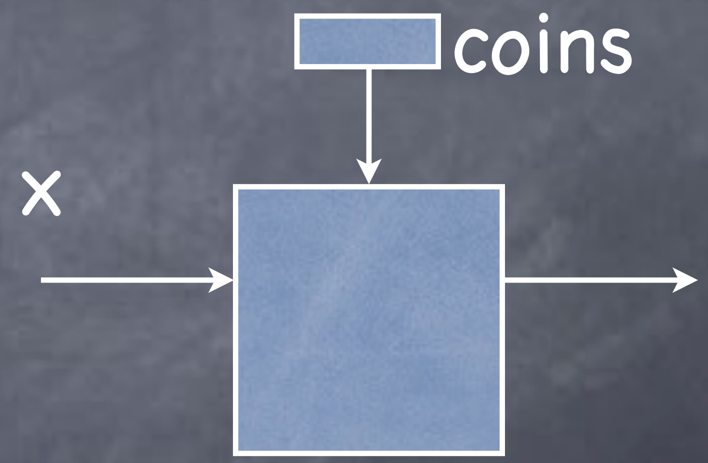
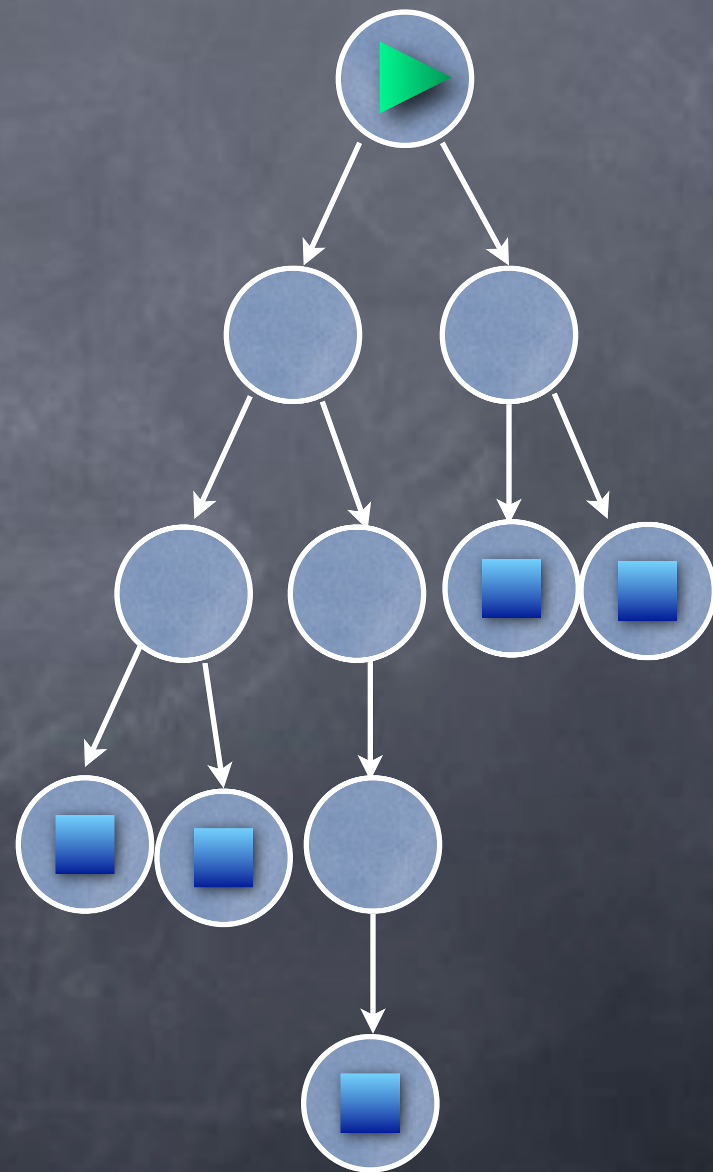


Random Tape

- Random choice: flipping a fair coin
  - Coin flip is written on a read-once "random tape"
  - Enough coin flips made and written on the tape first, then start execution
    - When considering bounded time TMs, length of random tape (max coins used) also bounded

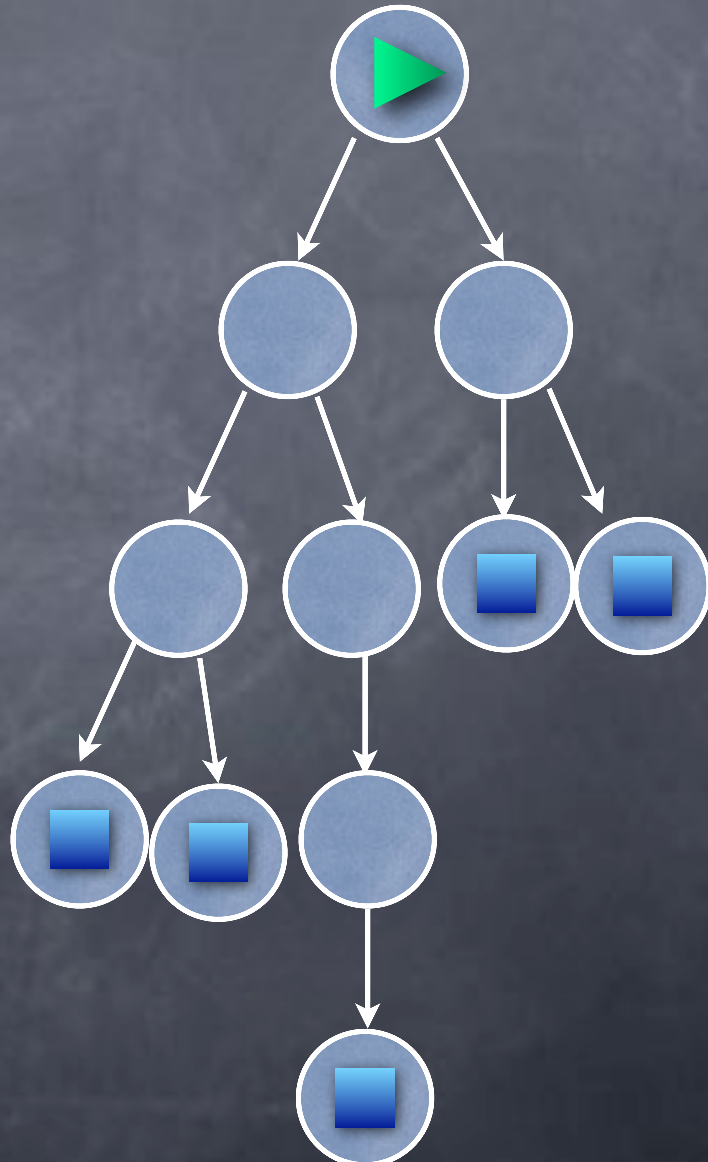


Random Tape



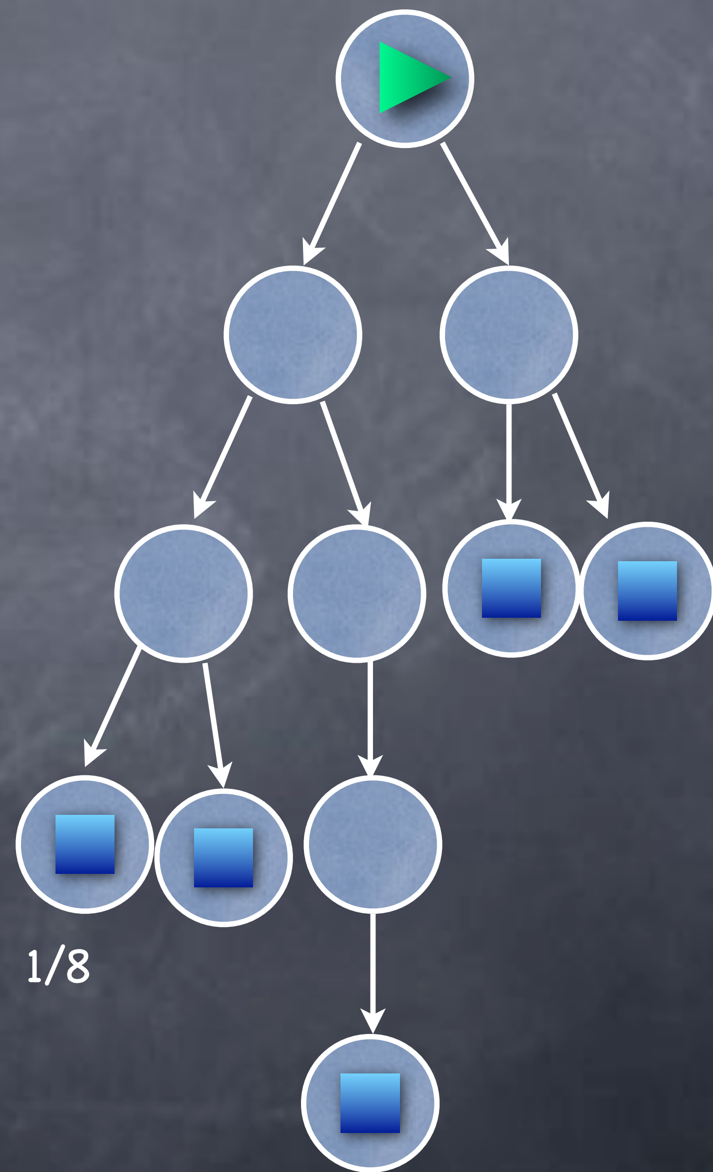
Random Tape

0	0	0



Random Tape

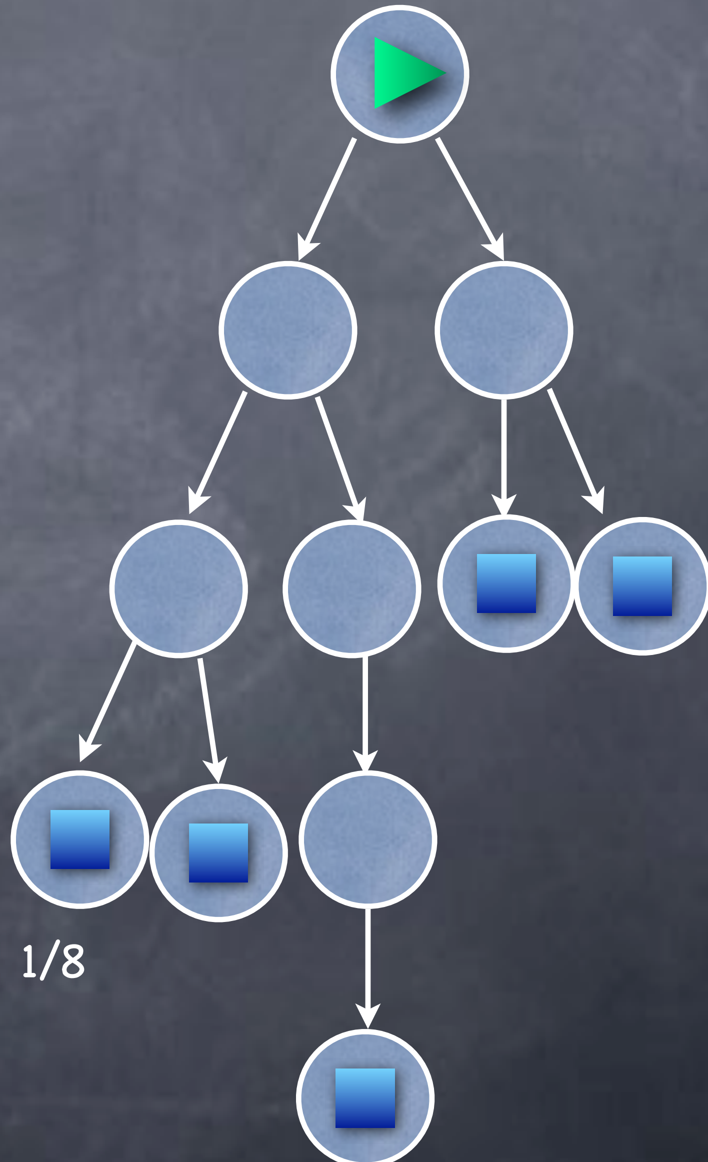
0	0	0



Random Tape

0	0	0

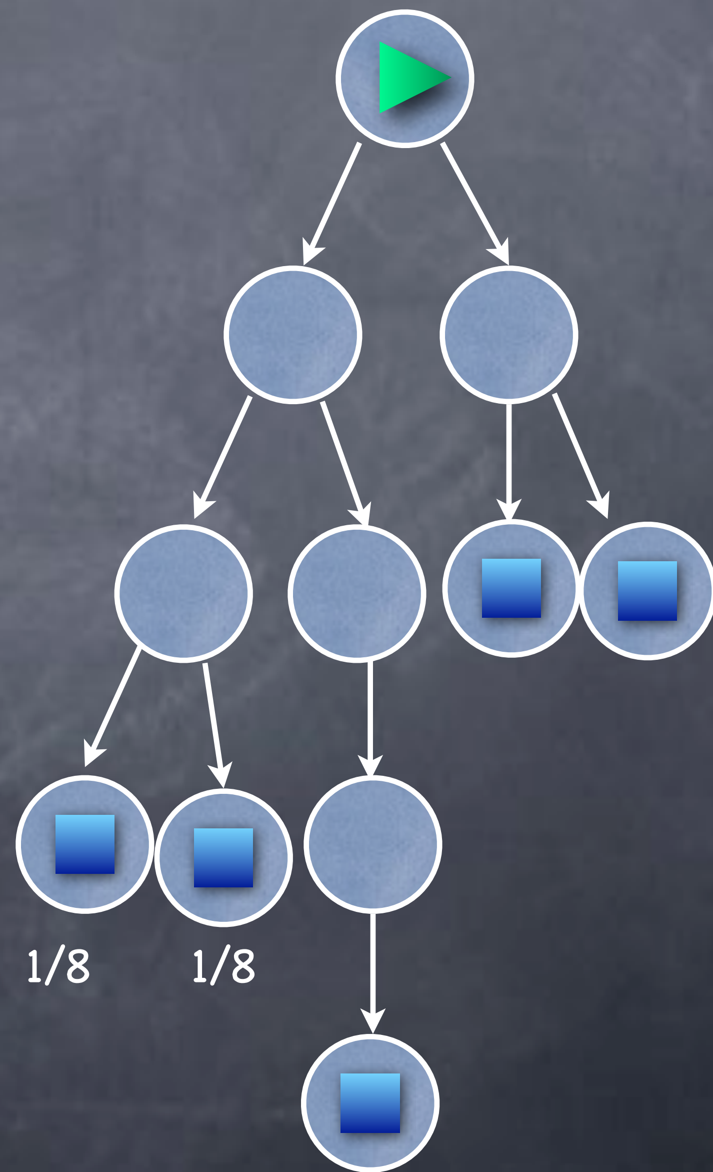
0	0	1



Random Tape

0	0	0

0	0	1

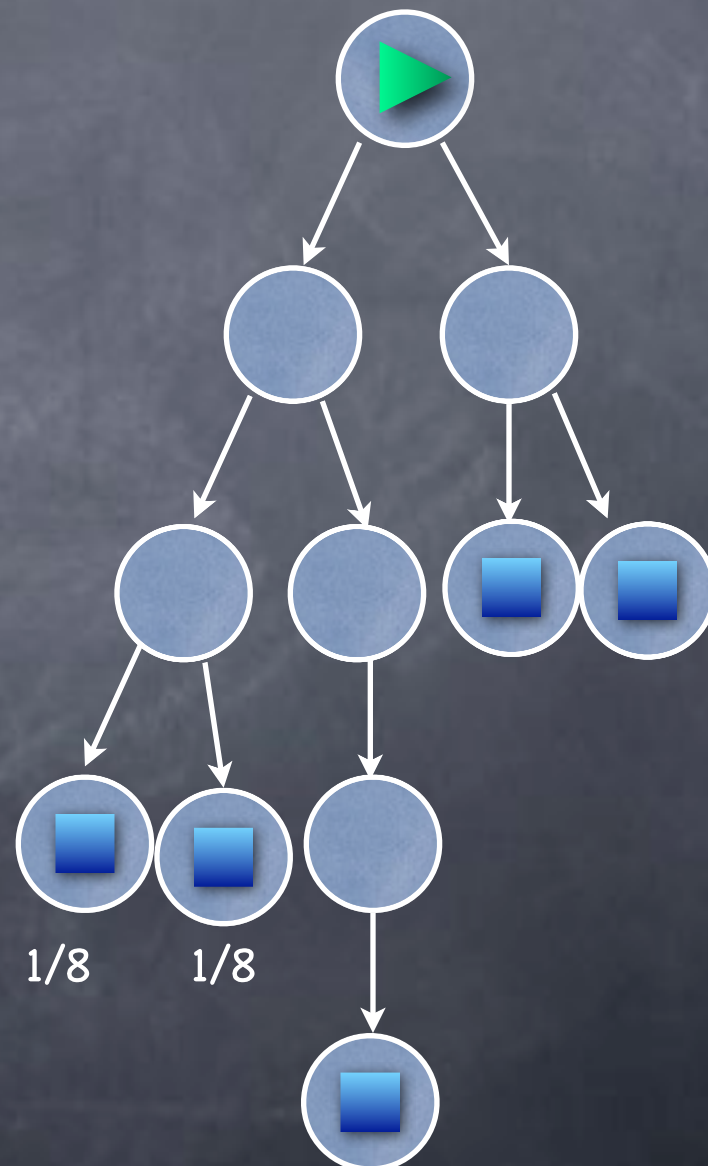


Random Tape

0	0	0

0	0	1

0	1

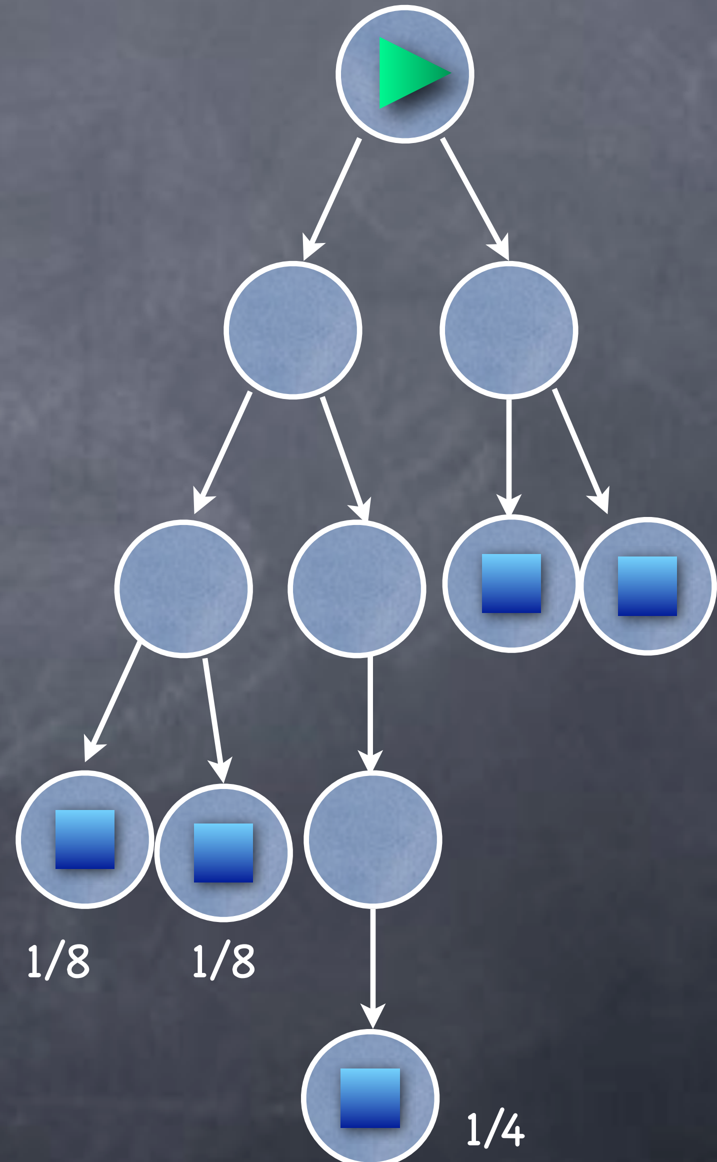


Random Tape

0	0	0

0	0	1

0	1



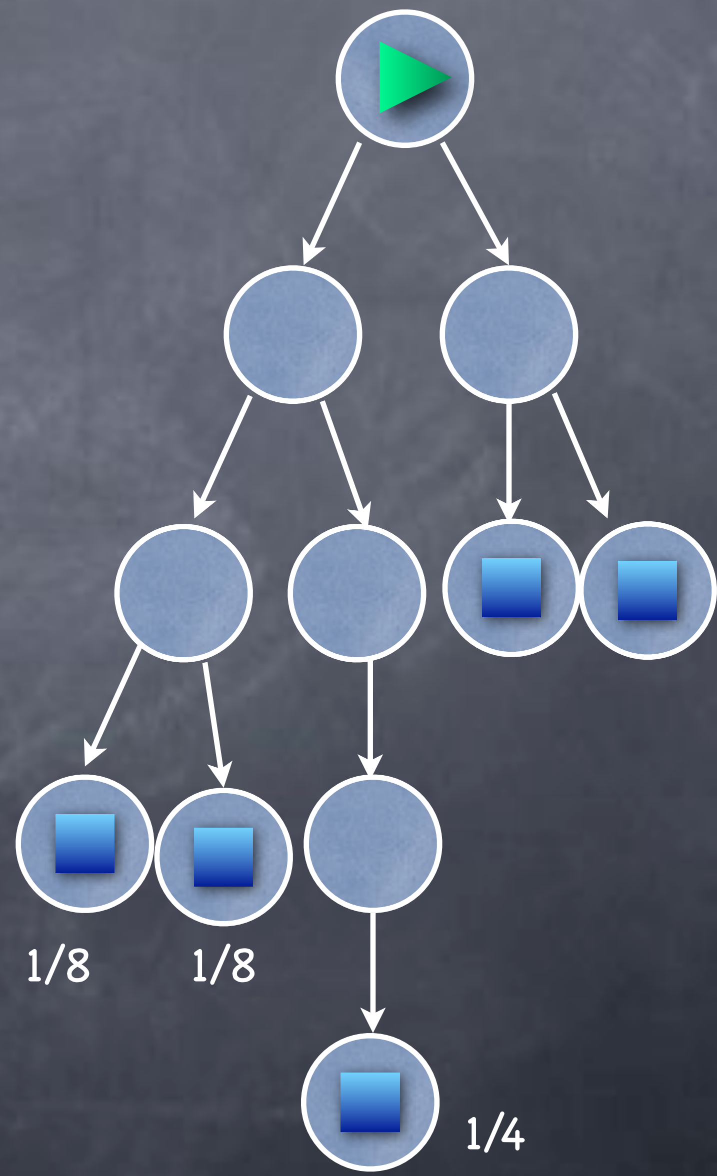
Random Tape

0	0	0

0	0	1

0	1

1	0



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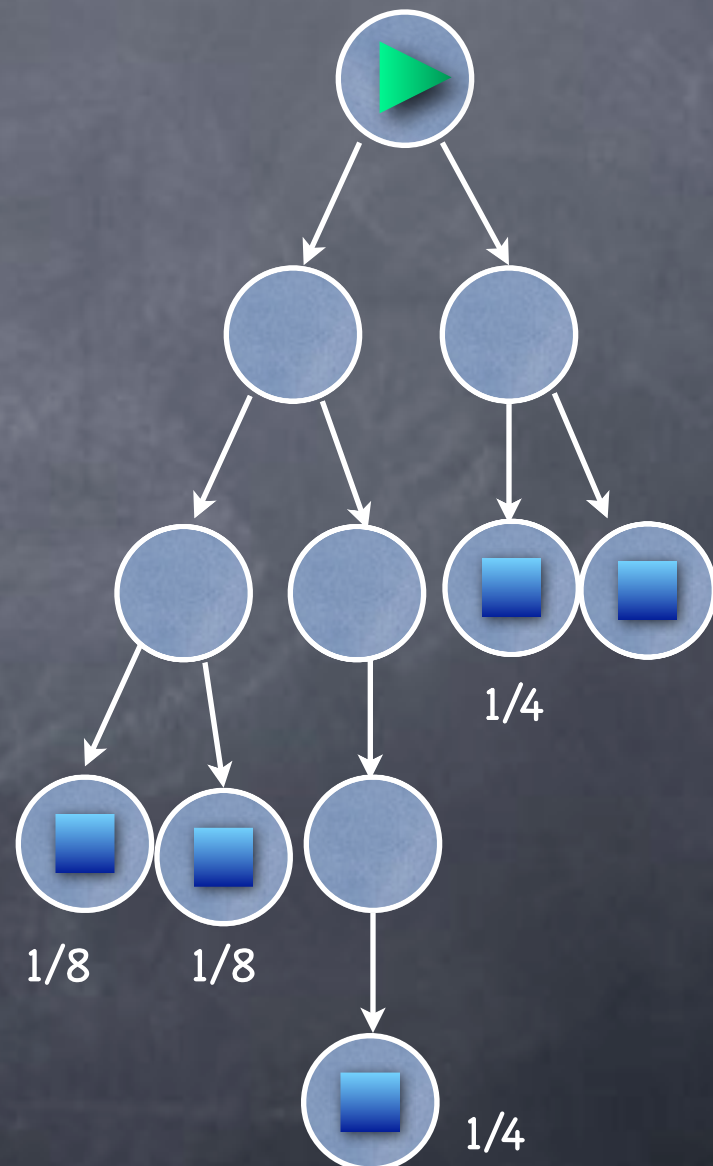
0	0	0

0	0	1

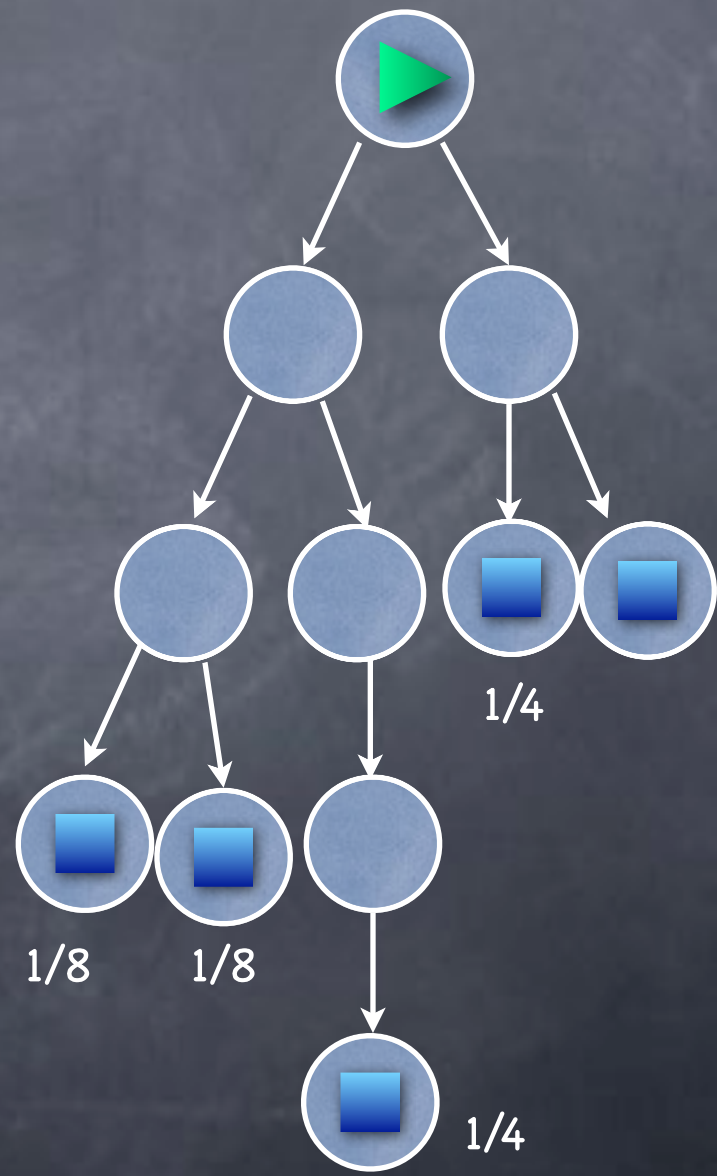
0	1

1	0



Random Tape

0	0	0
0	0	1
0	1	
1	0	
1	1	



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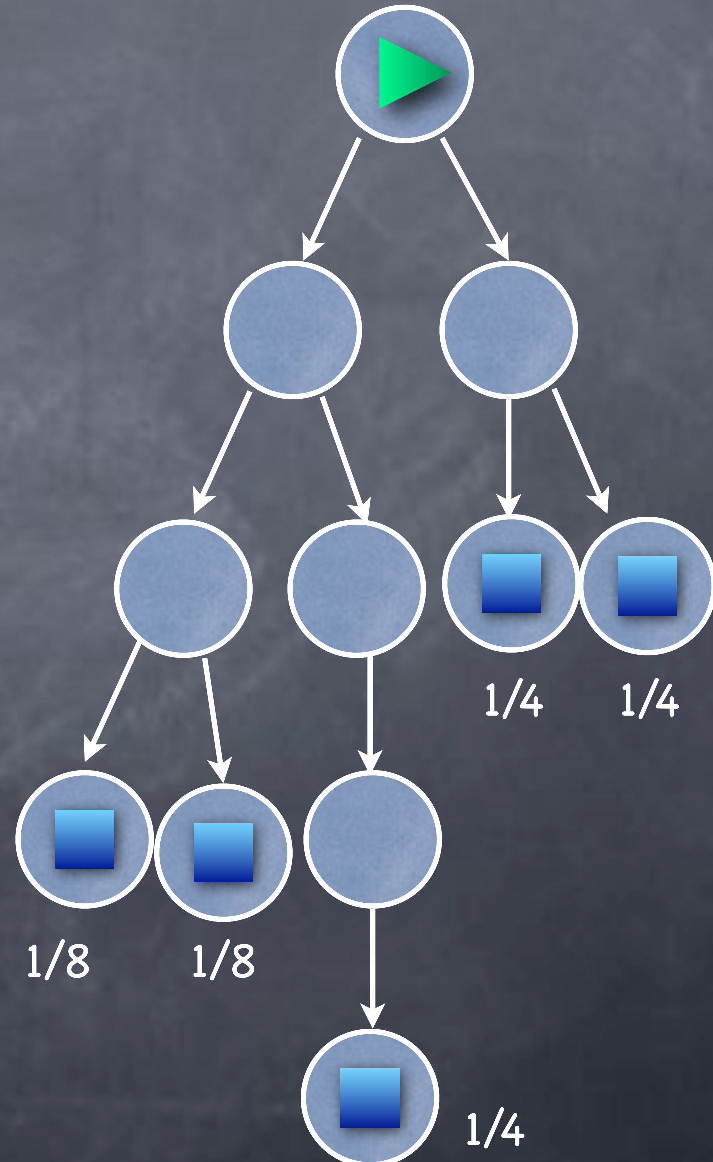
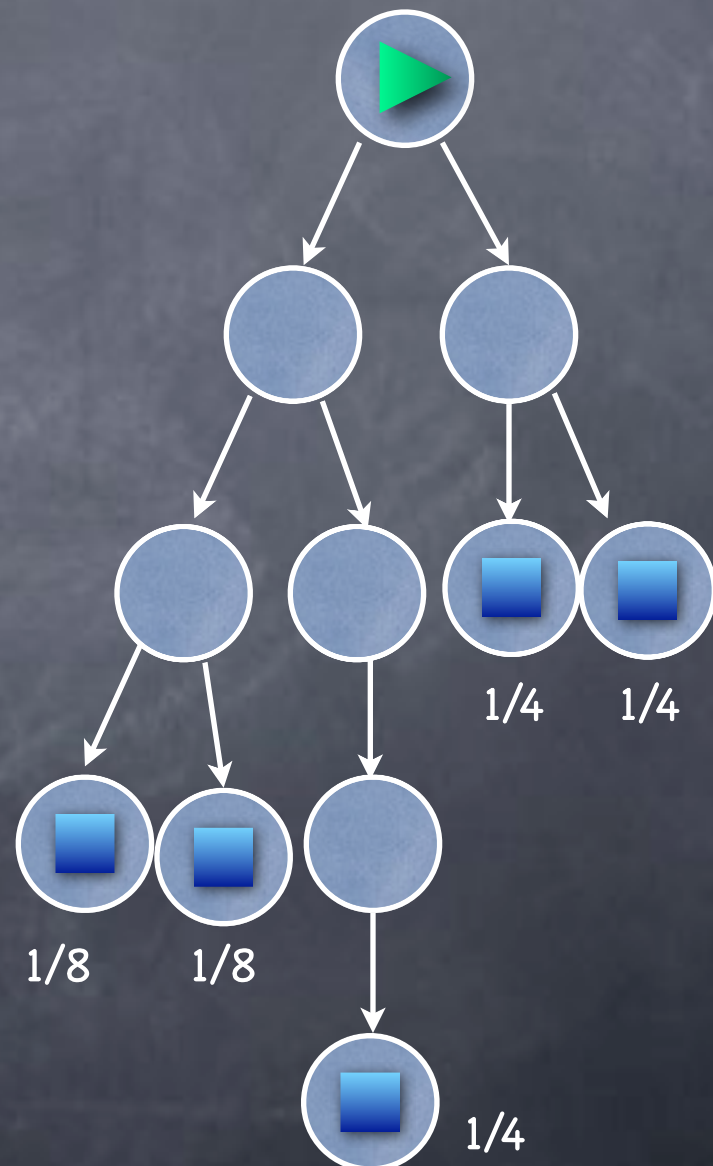


Random Tape

0 0 0	0 0 0
0 0 1	0 0 1
0 1	0 1 0
	0 1 1
1 0	1 0 0
	1 0 1
1 1	1 1 0
	1 1 1



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- (Not standard nomenclature!)

PTIME, BPTIME and RTIME

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  - Just like NTIME(T)
- BPTIME(T) = class of languages decided by BPTMs in time T
  - Similarly PTIME(T) and RTIME(T)

PP, BPP and RP

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- $\text{PP} = \cup_{c>0} \text{PTIME}(O(n^c))$

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  - co-RP

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  - co-RP: if no, may still say yes w/p at most 1/3
    - i.e., if co-RTM says yes, can be wrong

co-BPP, co-PP

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co-BPP, co-PP

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co-BPP, co-PP

- BPP = co-BPP
  - co-BPTMs are same as BPTMs
- In fact PP = co-PP
  - PTMs and co-PTMs differ on accepting inputs with Pr[yes]=1/2
  - But can modify a PTM so that Pr[M(x)=yes] $\neq$ 1/2 for all x, without changing language accepted

$PP = \text{co-}PP$

PP = co-PP

- Modifying a PTM M to an equivalent PTM M′, so that for all x $\Pr[M'(x)=\text{yes}] \neq 1/2$

PP = co-PP

- Modifying a PTM M to an equivalent PTM M′, so that for all x Pr[M′(x)=yes] ≠ 1/2
    - Consider M′(x): w.p. 1/2 run M(x); w.p. 1/2, ignore input and say yes w.p. 1/2 - ε, and say no w.p. 1/2 + ε

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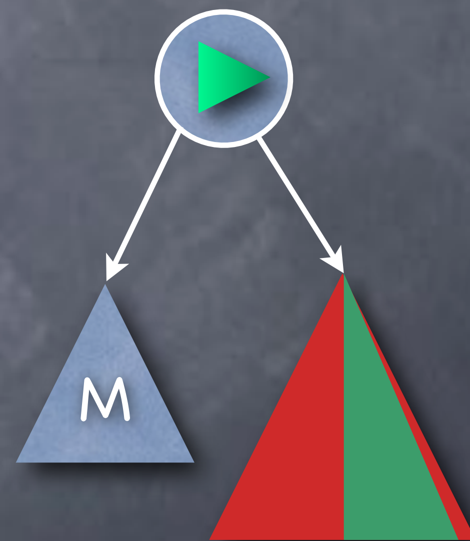

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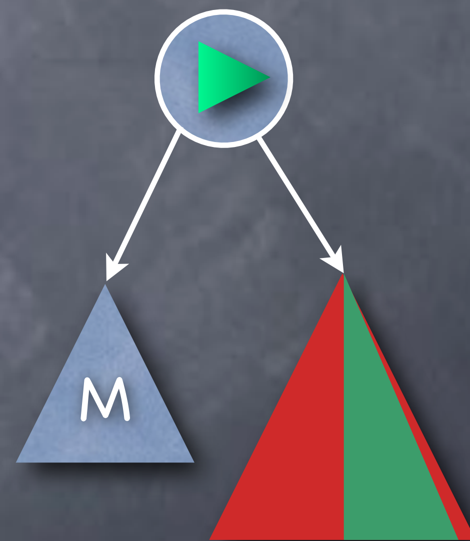
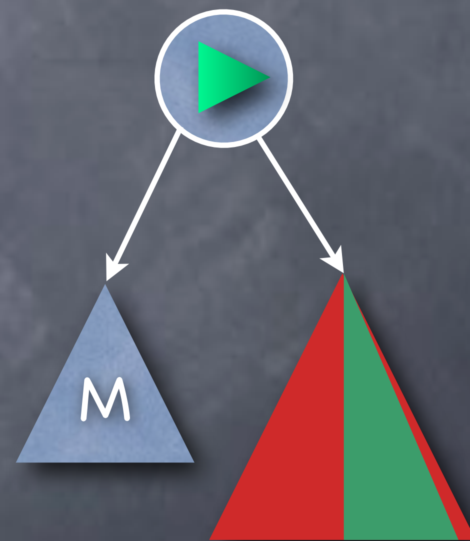
    - $\text{pr}[M'(x)=\text{yes}] = \text{pr}[M(x)=\text{yes}]/2 + (1/2 - \varepsilon)/2$



PP = co-PP

- Modifying a PTM M to an equivalent PTM M′, so that for all x Pr[M′(x)=yes] $\neq$ 1/2
  - Consider M′(x): w.p. 1/2 run M(x); w.p. 1/2, ignore input and say yes w.p. 1/2 - ε, and say no w.p. 1/2 + ε
    - pr[M′(x)=yes] = pr[M(x)=yes]/2 + (1/2 - ε)/2
  - If pr[M(x)=yes] > 1/2 $\Rightarrow$ pr[M(x)=yes] > 1/2 + ε

    then M and M′ equivalent



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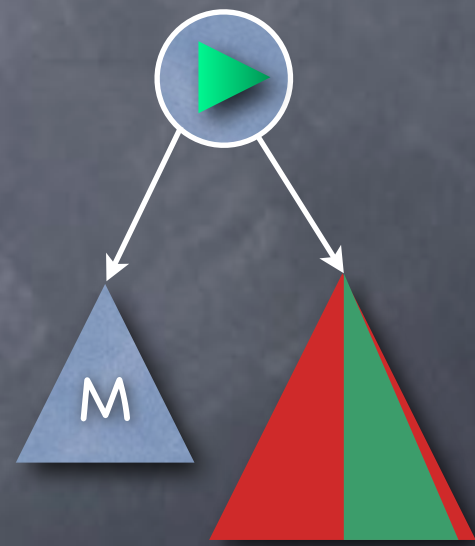

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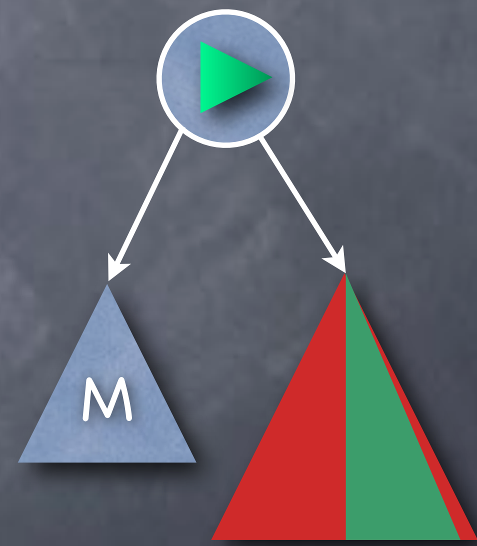

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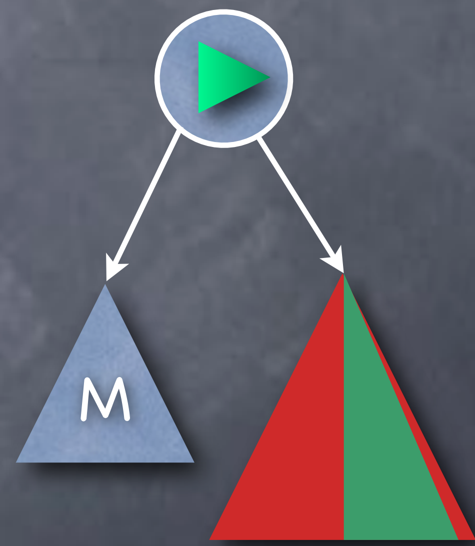
      - M′ tosses at most m+2 coins



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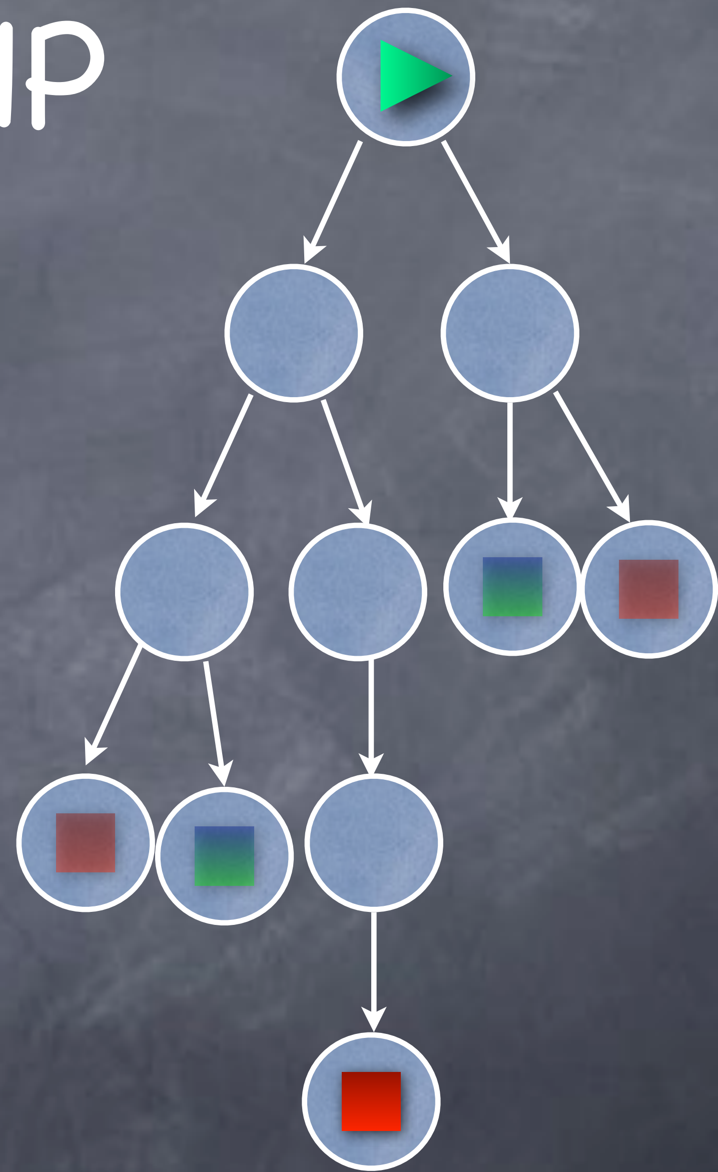

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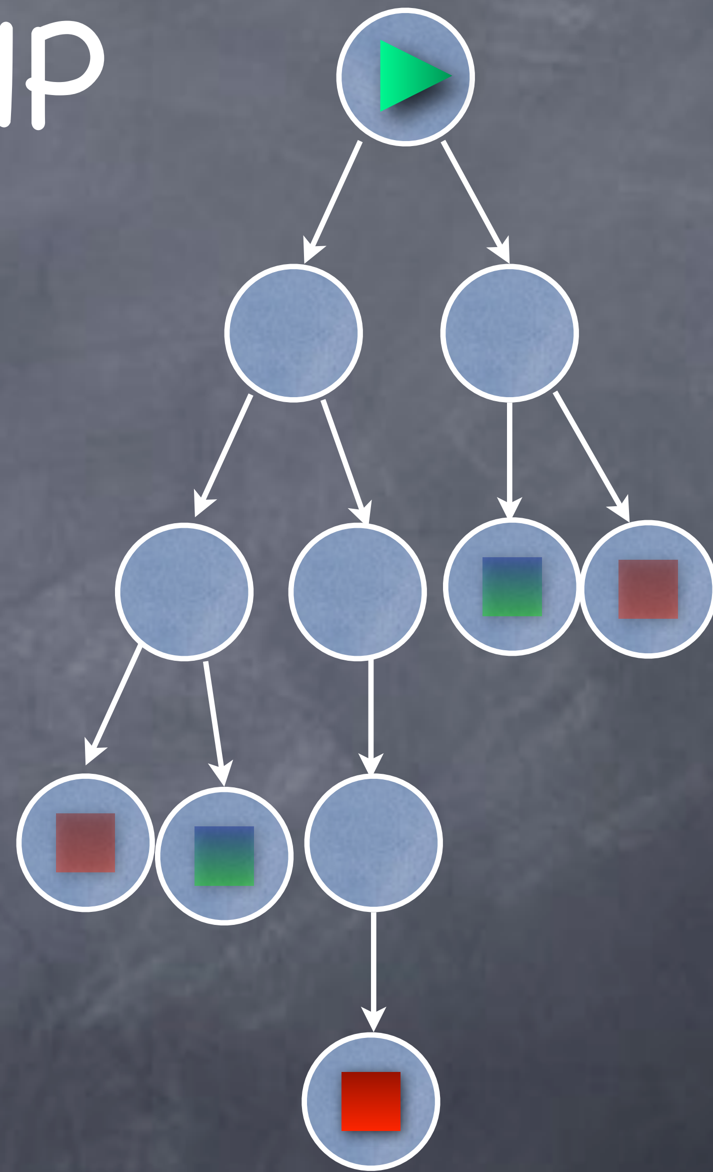

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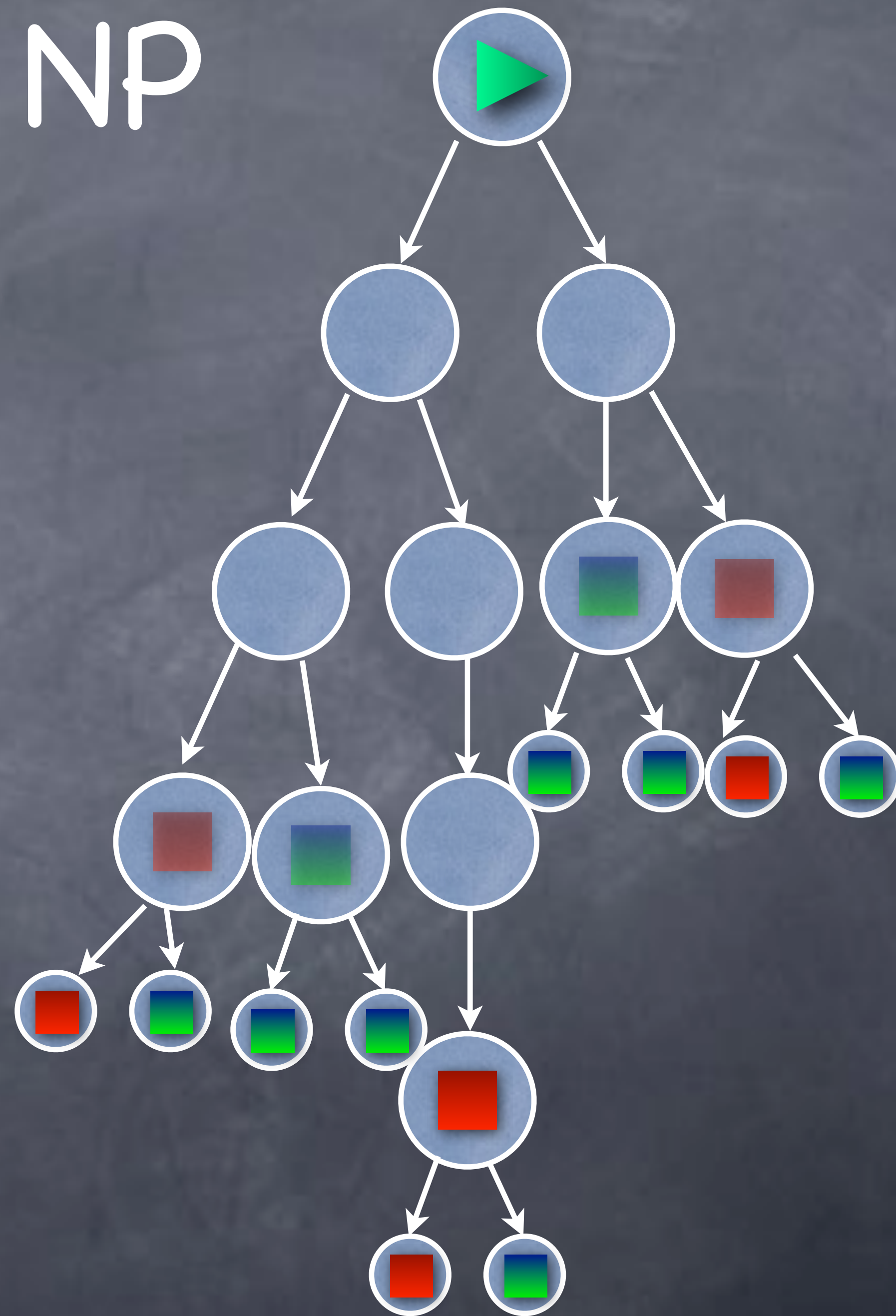

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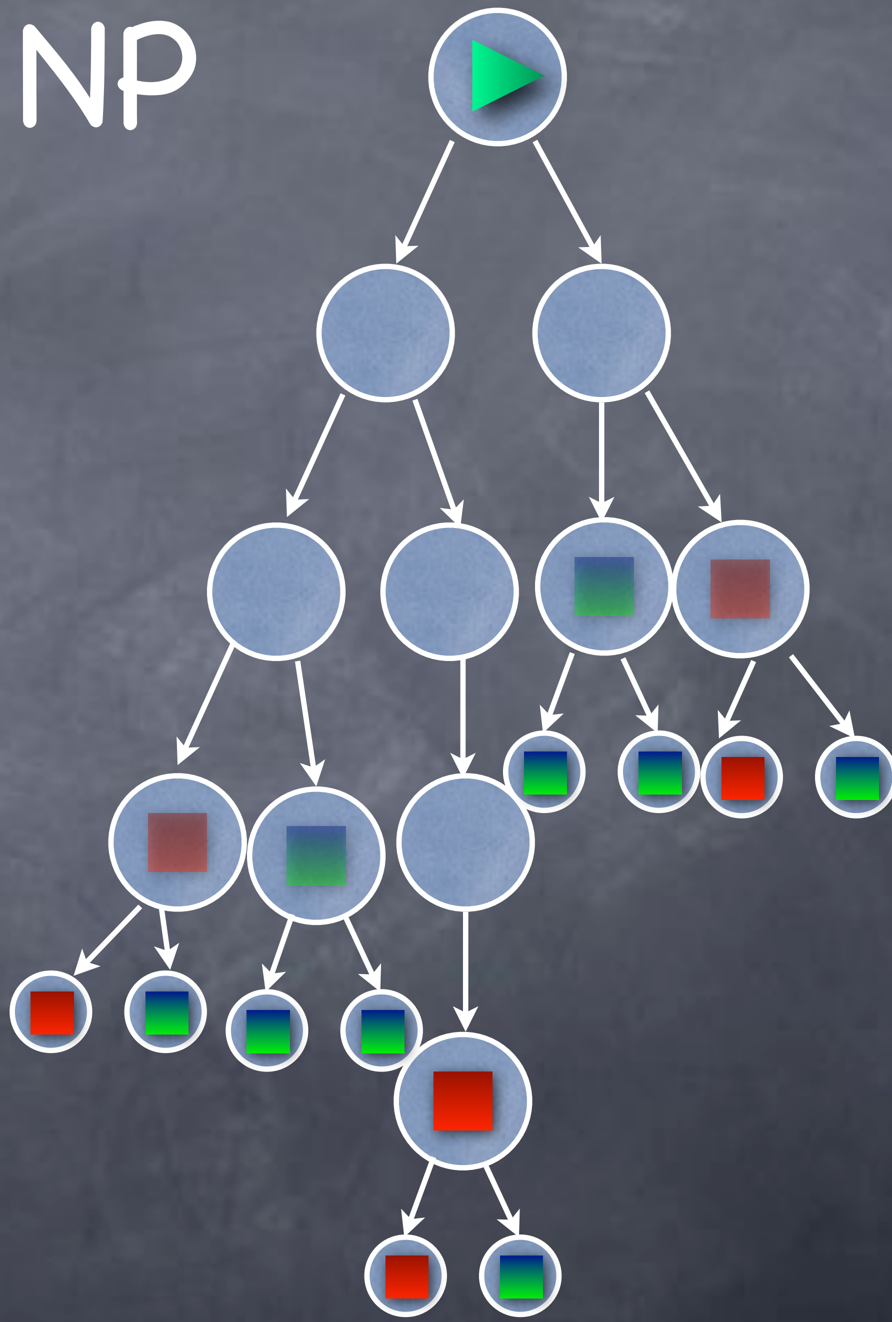

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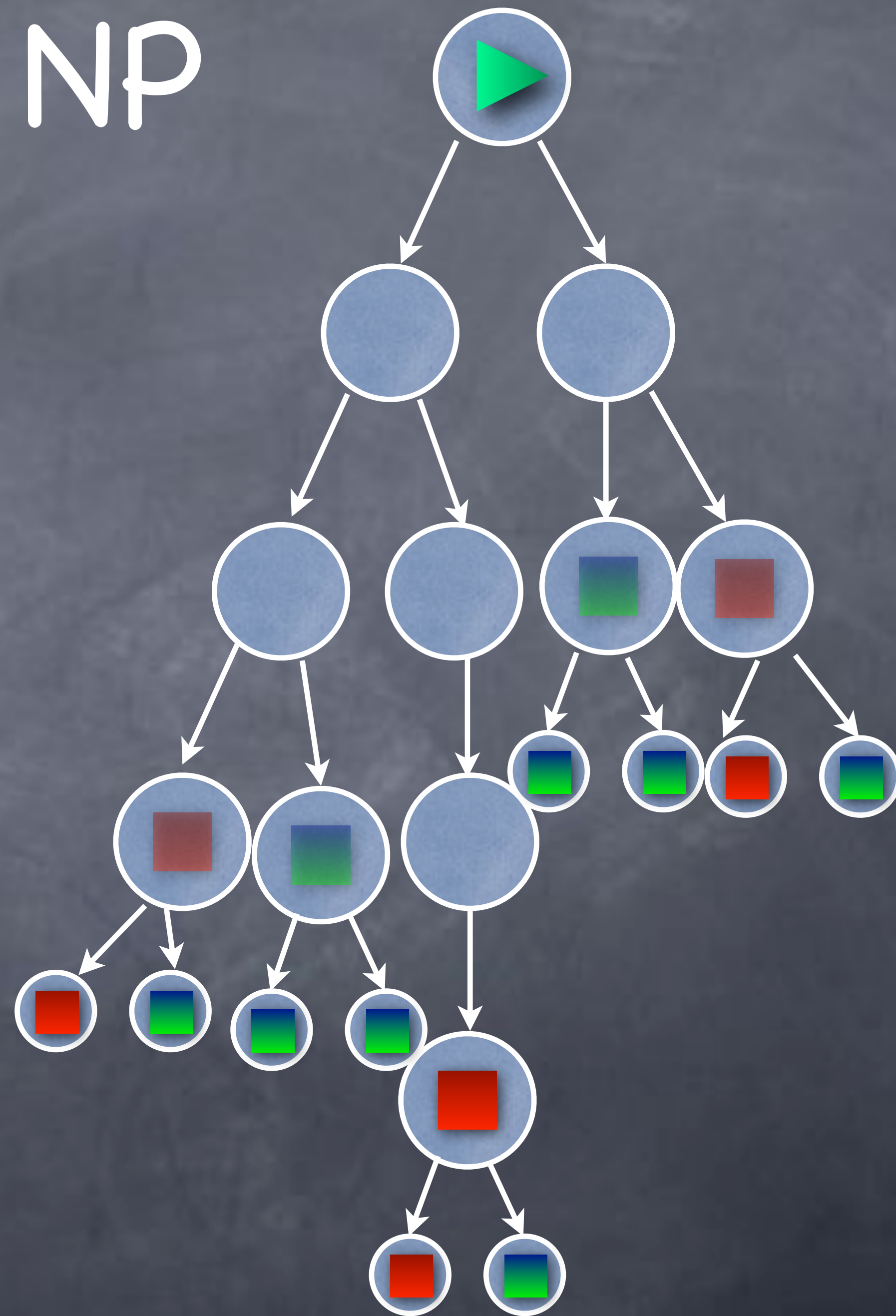
- Accepting gap can be exponentially small



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- M′(x): Repeat M(x) t times and if any yes, say yes

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  - With $t = n^d/\delta$, $\Pr[M'(x)=no] < e^{-(n\hat{\ }d)}$ $(1-\delta < e^{-\delta})$
    - For $\delta = n^{-c}$, $t=n^{d+c}$ is polynomial

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      - $t = O(n^d/\delta^2)$ enough for Pr[error] ≤ 2^{-n^d}

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- Next: more on BPP and relatives