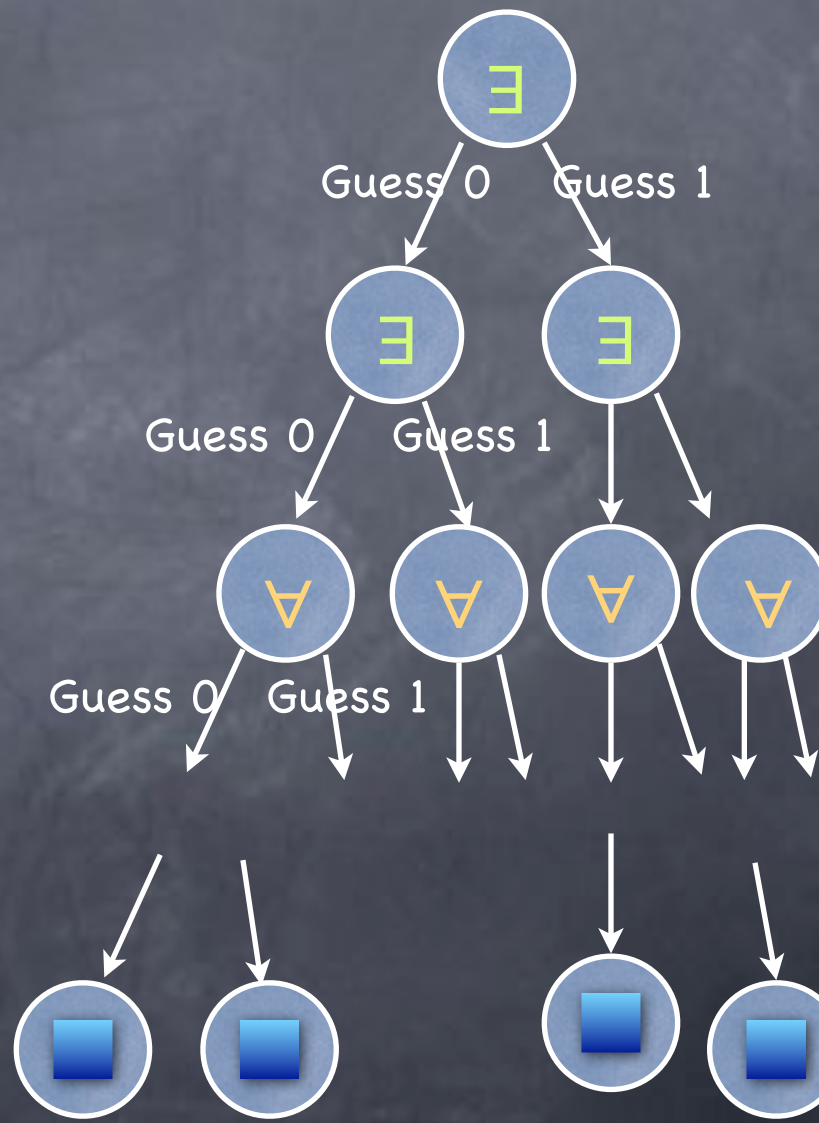
Computational Complexity

Lecture 9

Alternation

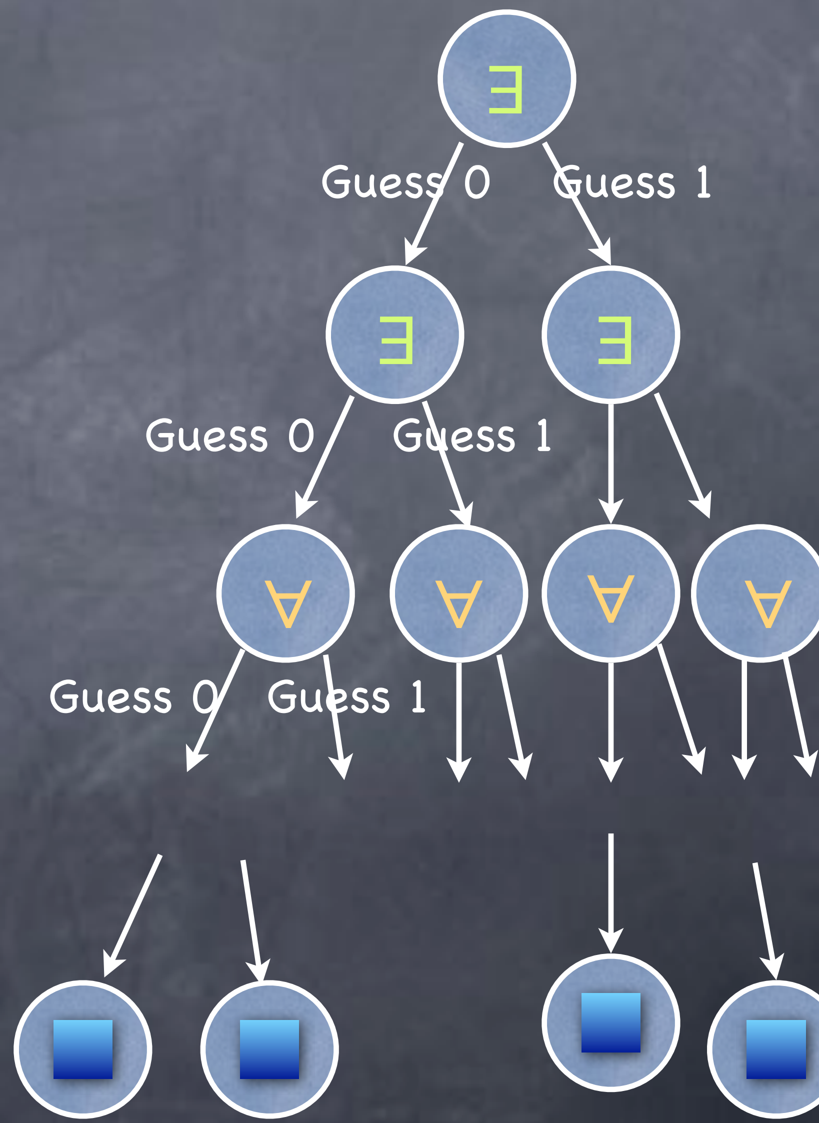
(Continued)

ATM



ATM

- Alternating Turing Machine



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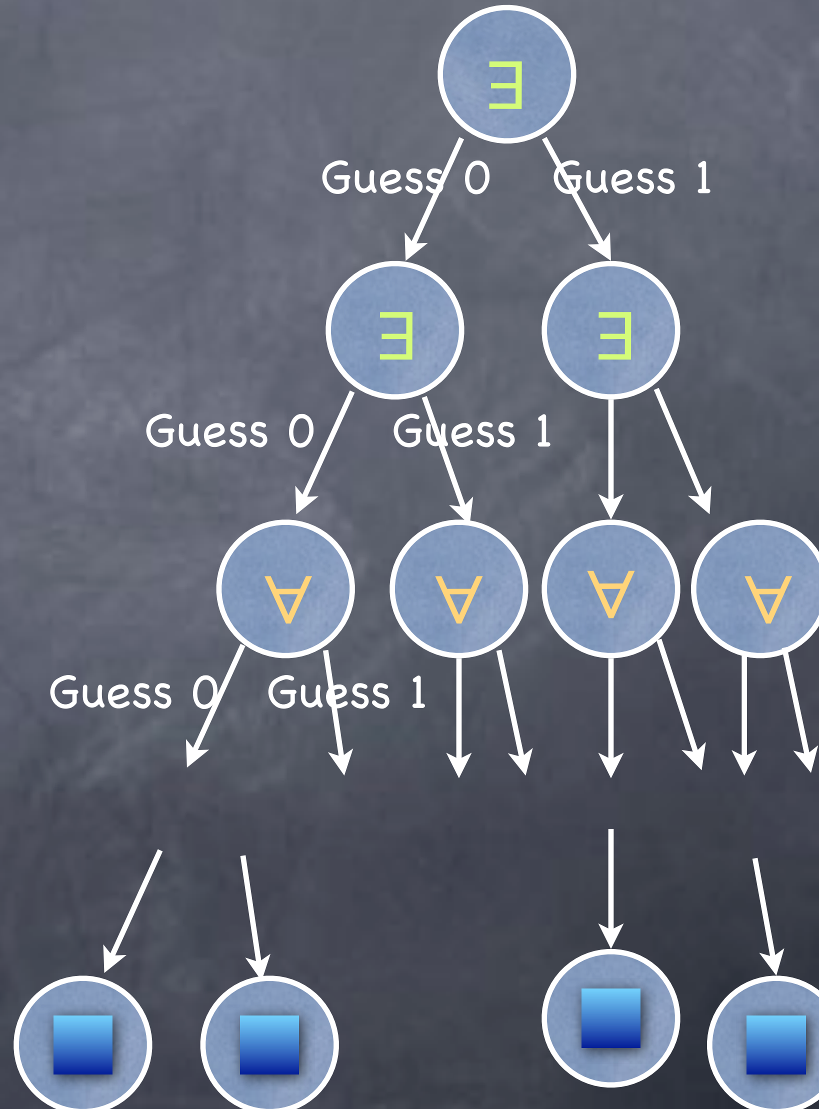
  - At each step, execution can fork into two (like NTM or co-NTM)



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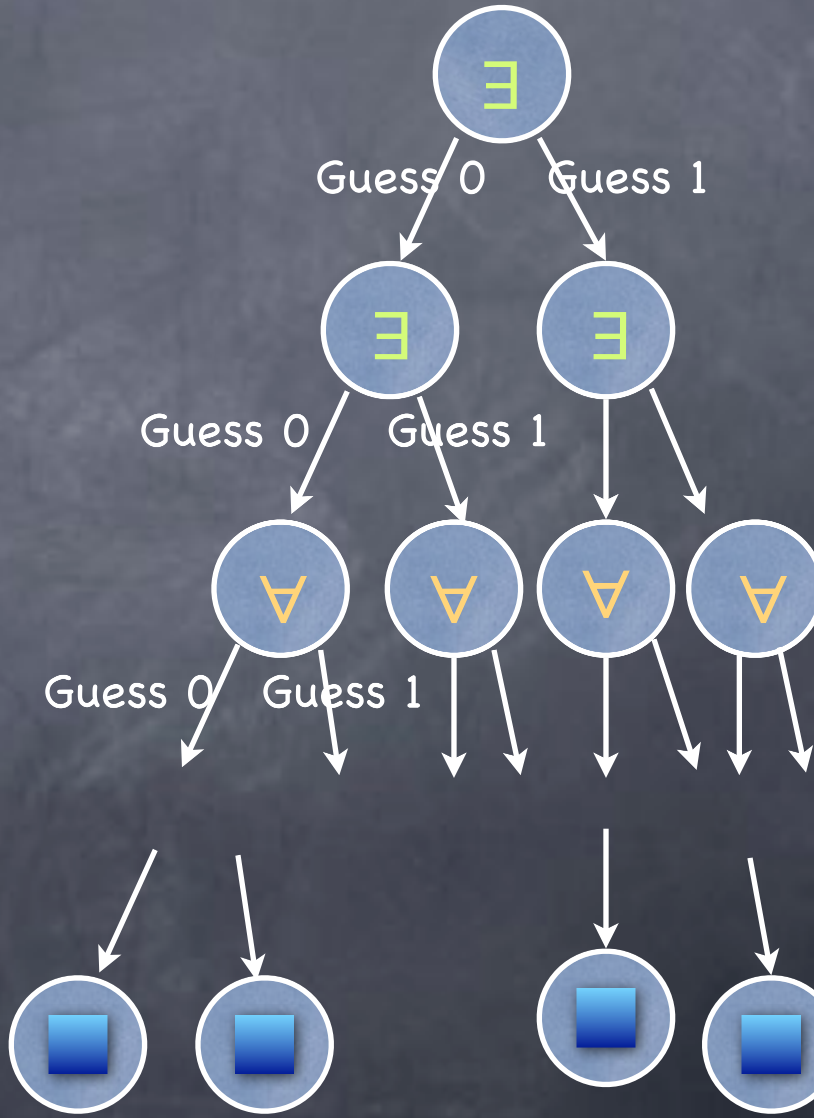

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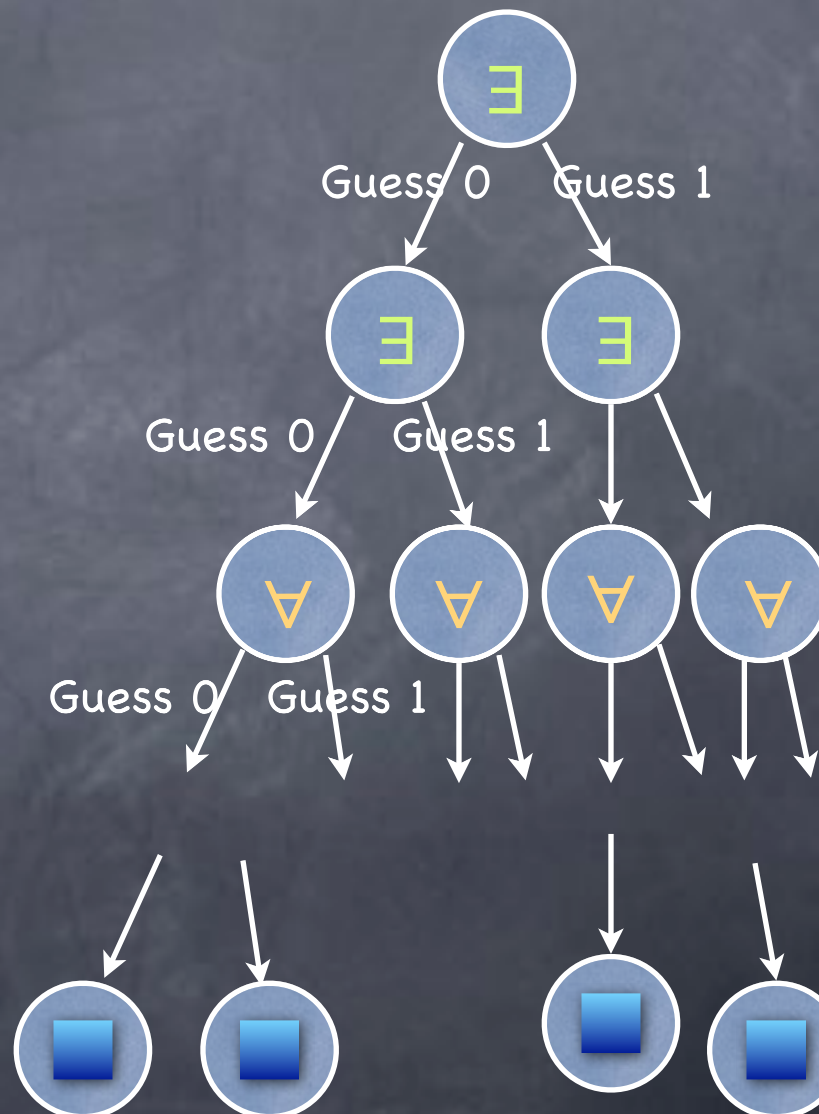

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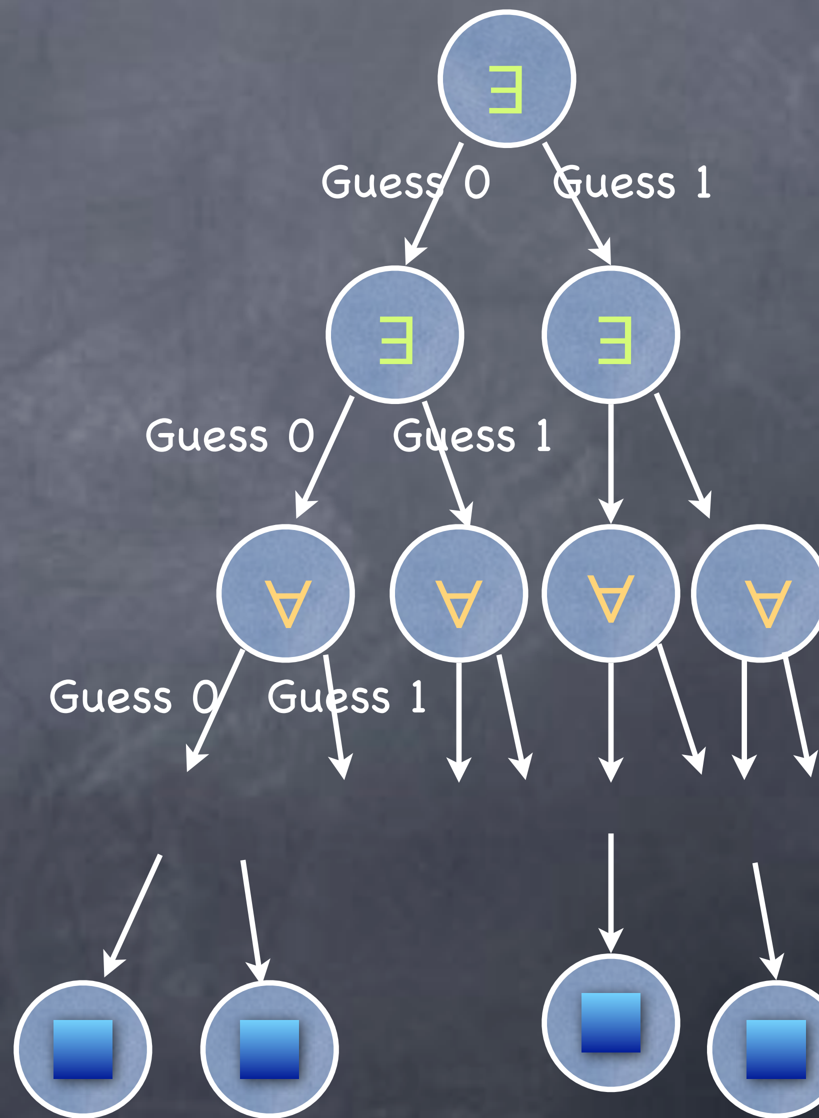

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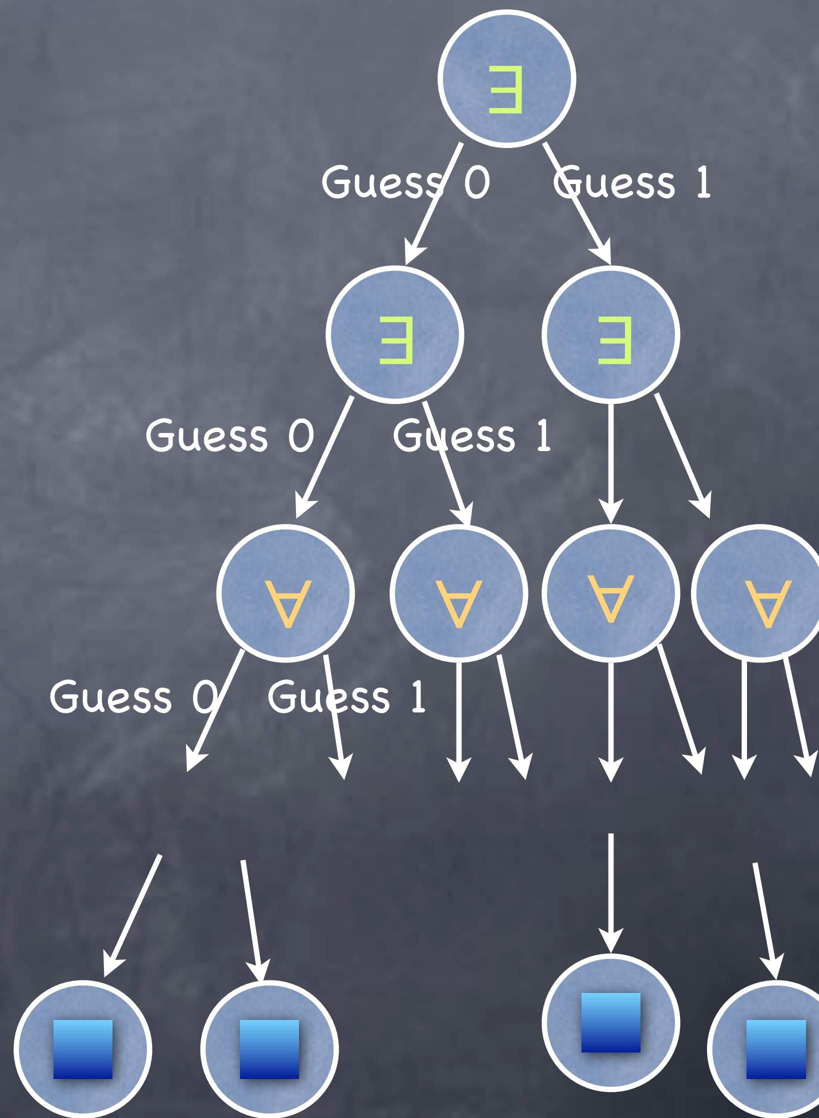
  - ATM accepts if start config accepts according to this rule



ATIME, ASPACE

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    - Naive recursion: Extra O(S) space to store i,j at each level for $2^{O(S)}$ levels!

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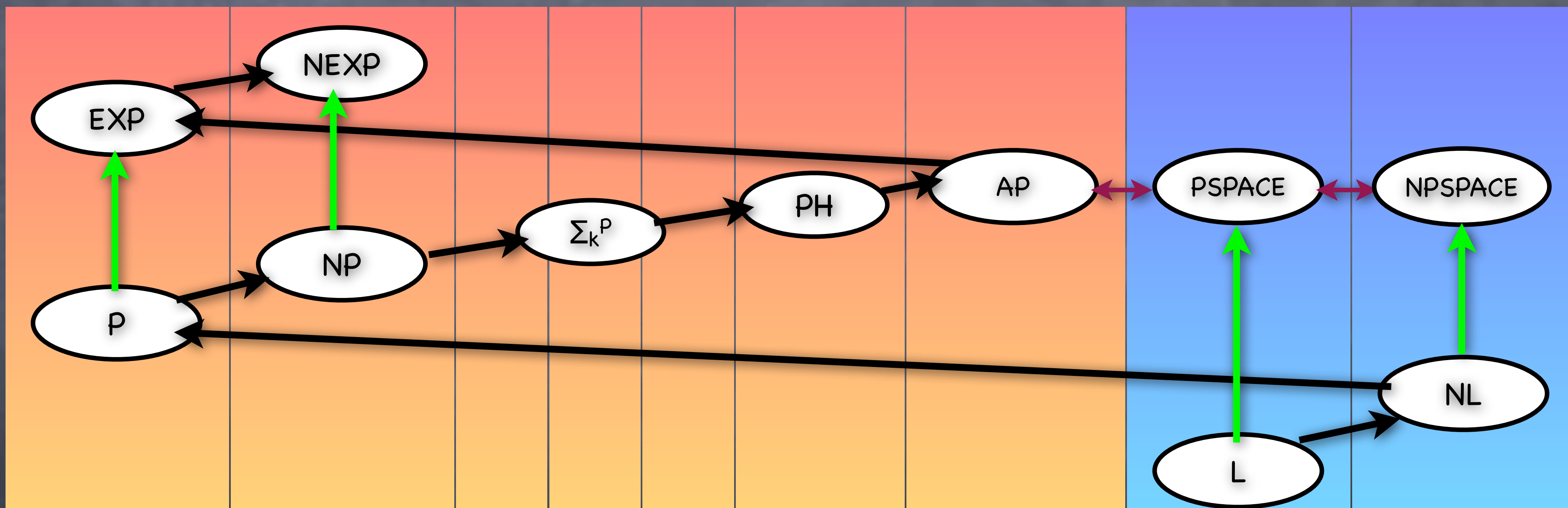
Gets the AND check for free. No need to use a stack.

Zoo



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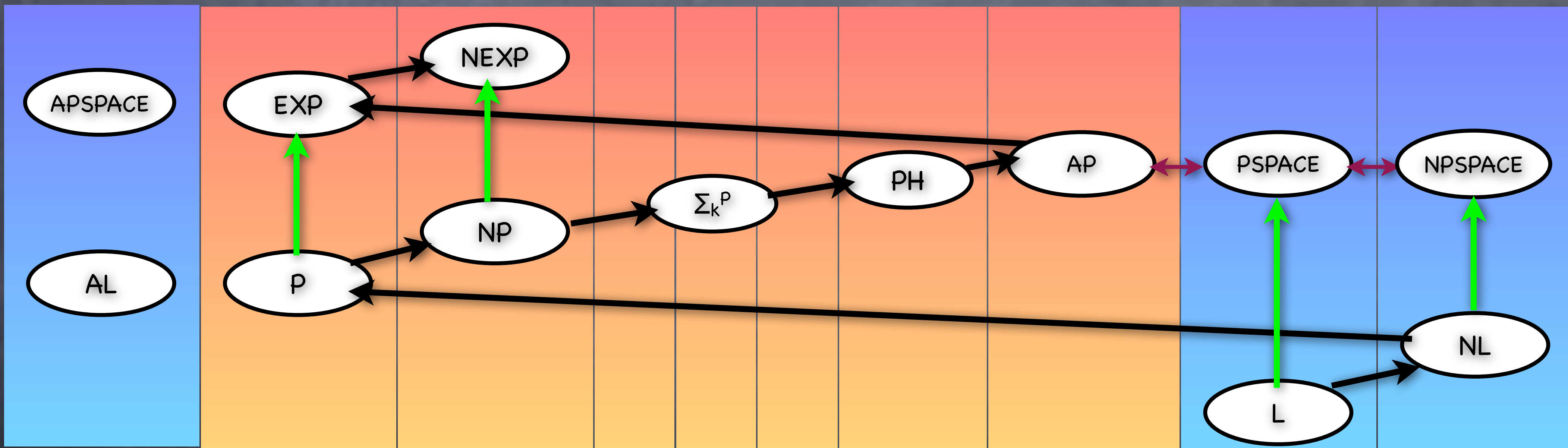

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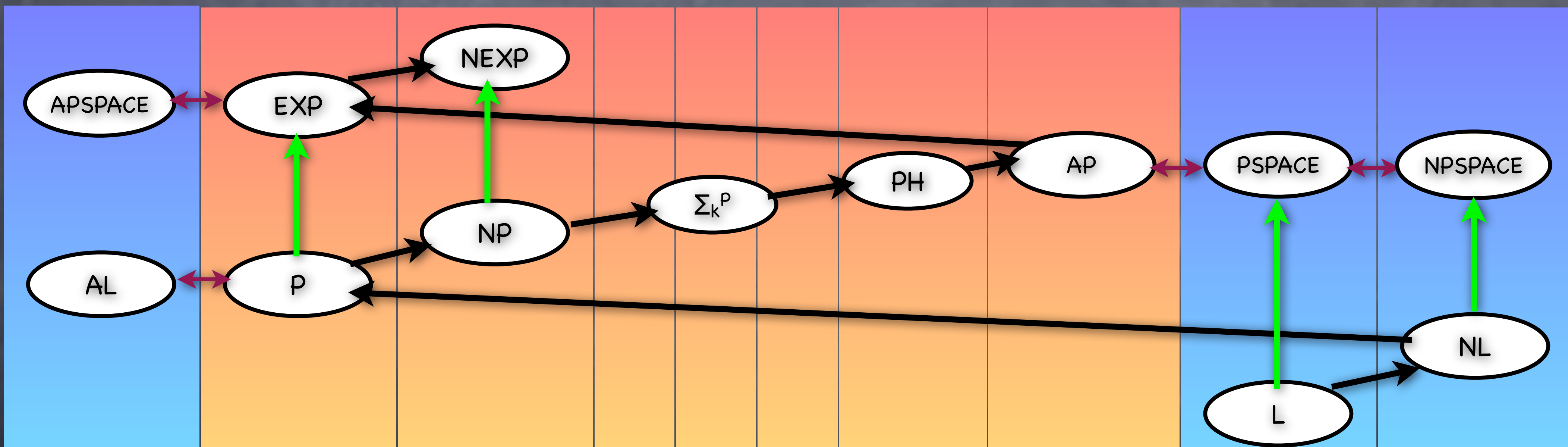

Non-Uniform Computation

Lecture 10

Non-Uniform Computational Models: Circuits

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- Non-uniform: A different "program" for each input size
  - Then complexity of building the program and executing the program
  - Sometimes will focus on the latter alone
  - Not entirely realistic if the program family is uncomputable or very complex to compute

Non-uniform advice

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- Program: TM M and advice strings $\{A_n\}$

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  - M given $A_{|x|}$ along with x

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    - e.g. advice to decide undecidable unary languages

P/poly and P/log

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NP vs. P/log, P/poly

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- P/log contains P
  - Does P/log or P/poly contain NP?

$NP \subseteq P/\log \Rightarrow NP = P$

NP $\subseteq$ P/log $\Rightarrow$ NP=P

- Recall finding witness for an NP language is Turing reducible to deciding the language

Recall

Search using Decision

- Suppose given "oracles" for deciding all NP languages, can we easily find certificates?
  - Yes! So, if decision easy (decision-oracles realizable), then search is easy too!
- Say, given x, need to find w s.t. $(x,w) \in L'$ (if such w exists)
  - consider L_1 in NP: $(x,y) \in L_1$ iff $\exists z$ s.t. $(x,yz) \in L'$. (i.e., can y be a prefix of a certificate for x).
  - Query L_1-oracle with (x,0) and (x,1). If $\exists$w, one of the two must be positive: say $(x,0) \in L_1$; then first bit of w be 0.
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Use L_2 so that (x,z,pad) in L_2 iff (x,z) in L_1. Can query L_2 with same size instances

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- If no advice worked (one of them was correct), then input not in language

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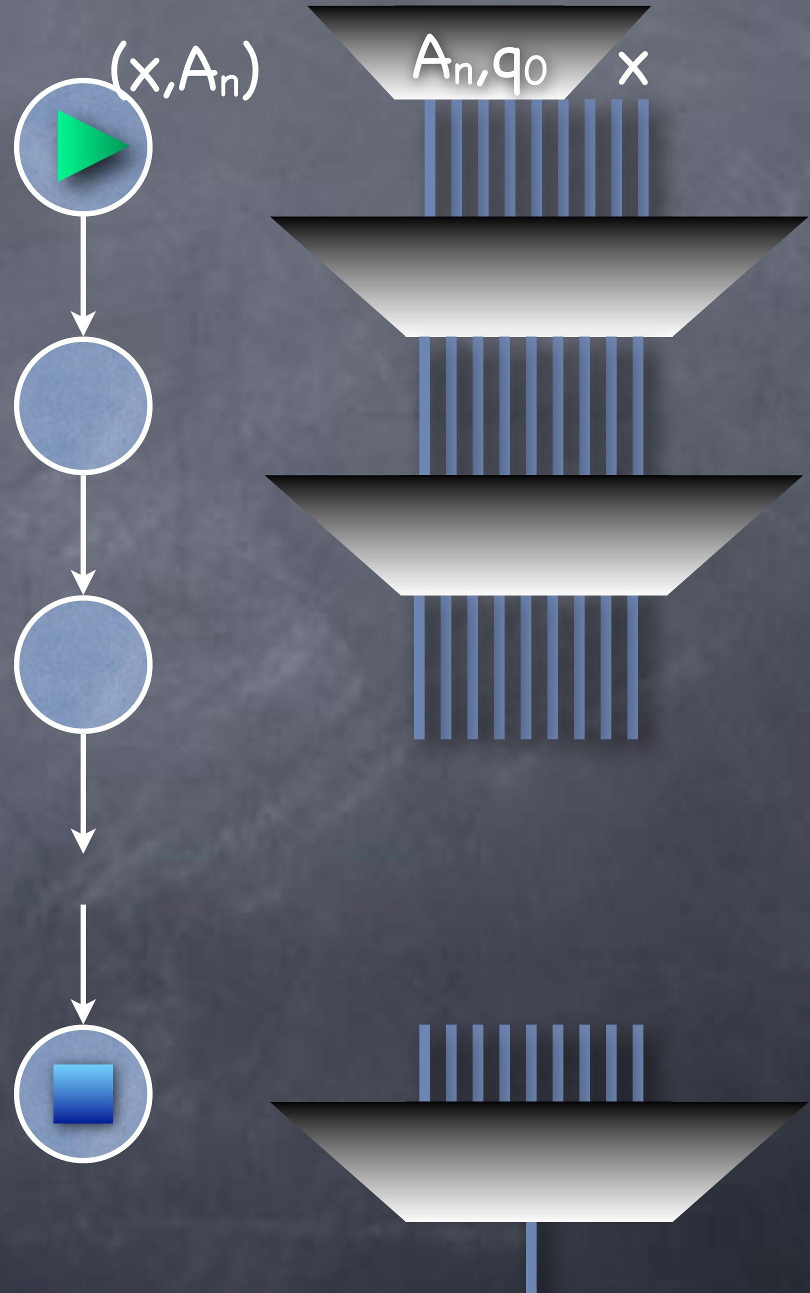

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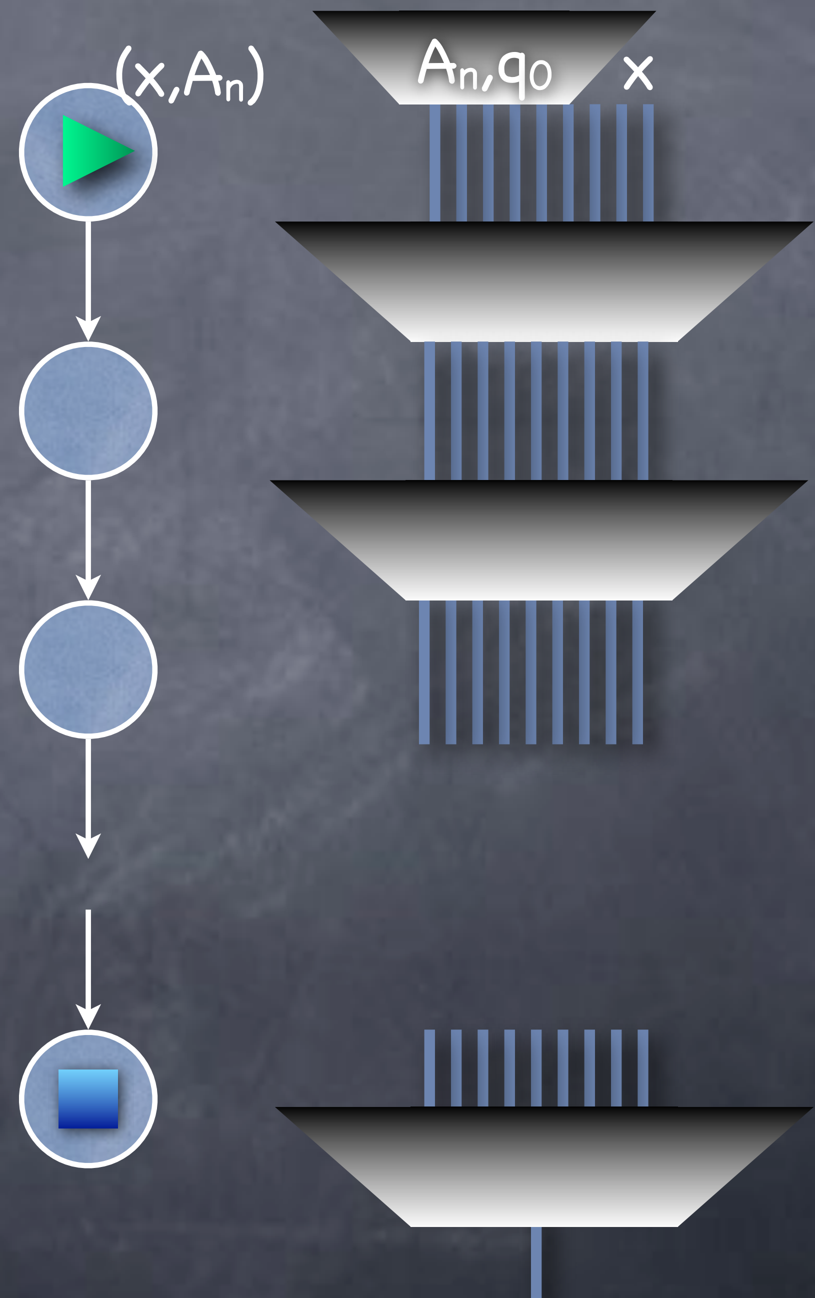

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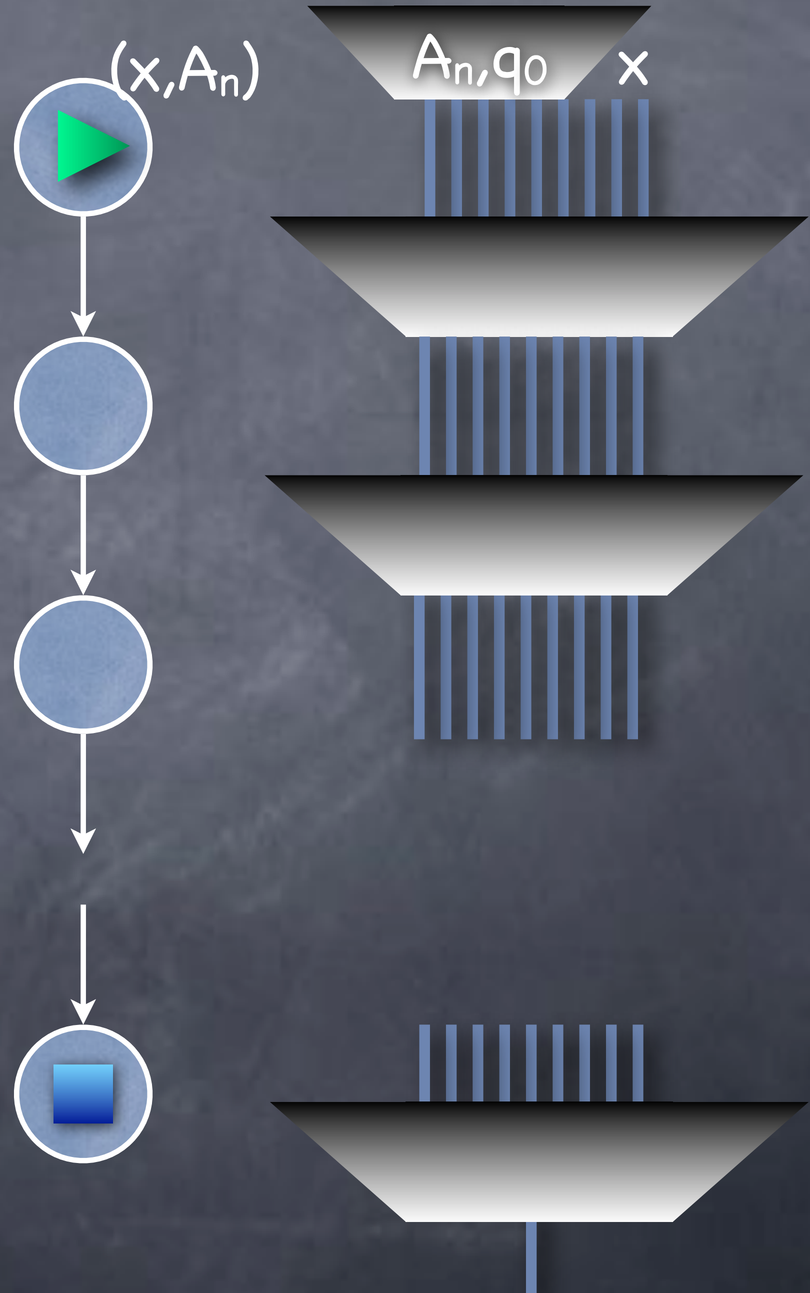

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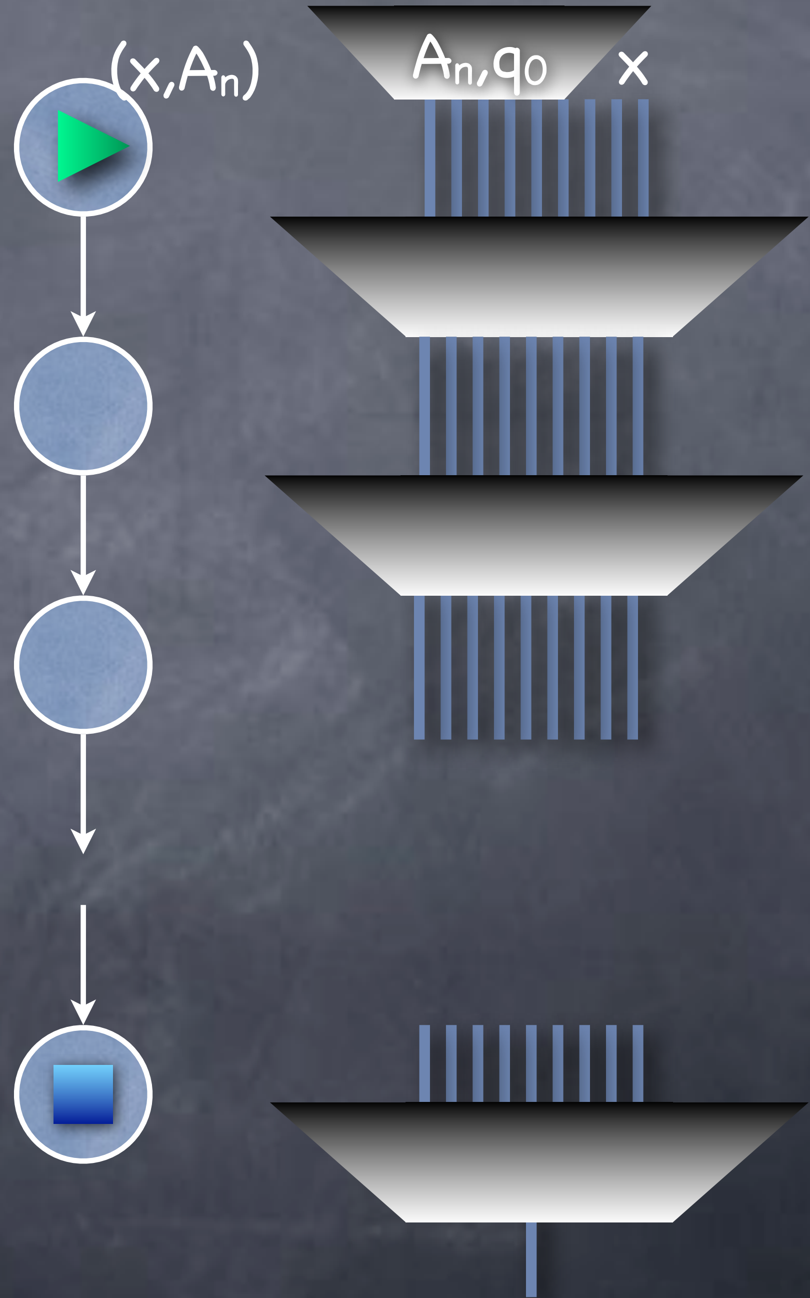

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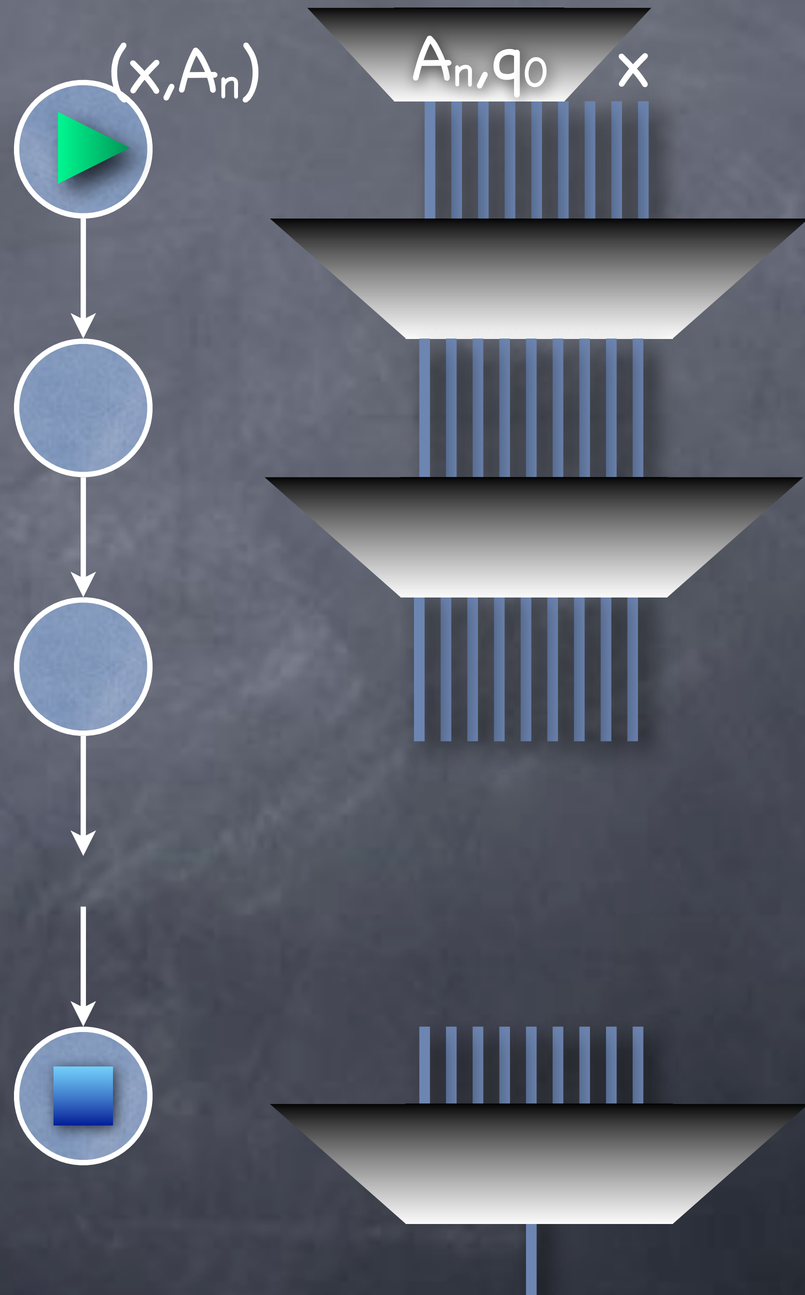

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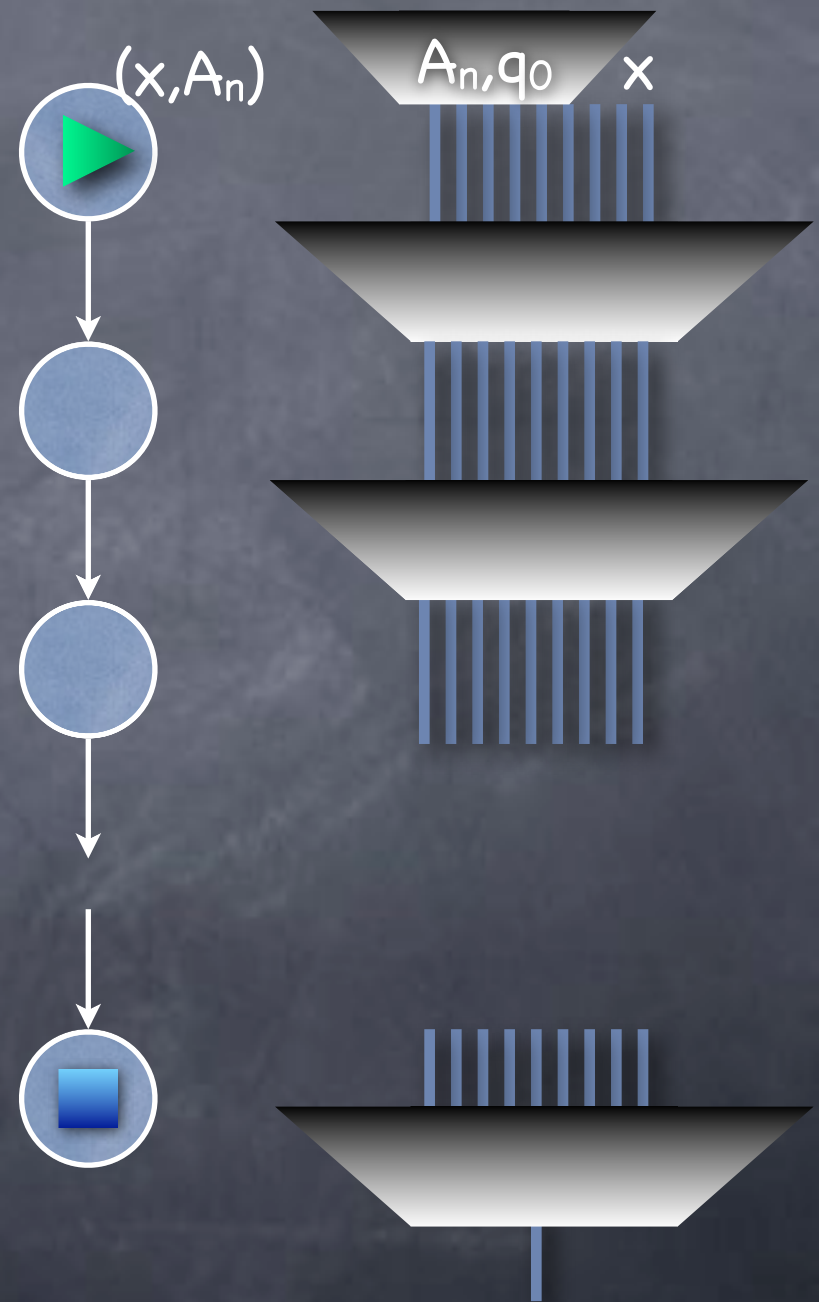

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  - P/poly $\subseteq$ SIZE(poly): Transformation from Cook's theorem, with advice string hardwired into circuit

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- Logspace-uniform:
  - An O(log n) space TM can compute the circuit

NC and AC

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- So NC = AC

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- Open problem: Is NC = P?

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  - Total “work” is size of the circuit