Computational Complexity

Lecture 7

Polynomial Hierarchy

Charting (some of) the space between P and PSPACE

(where much of the action happens)

Between P and PSPACE

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- Recall NP

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Between P and PSPACE

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- How about languages $\{ x \mid \exists u_1 \forall u_2 \ldots Q u_k\ F(x,u_1,u_2,\ldots,u_k) \}$

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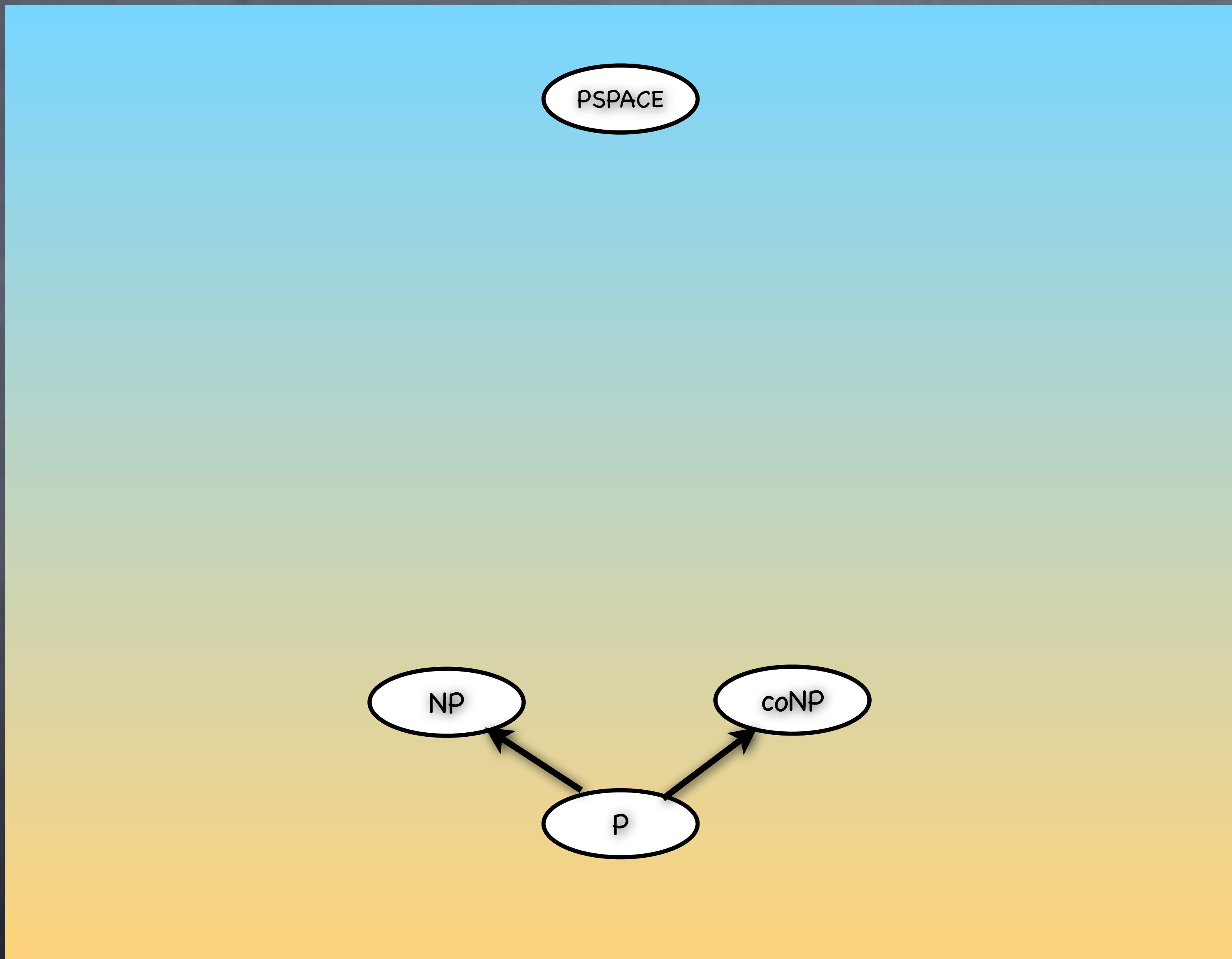
  - Such languages in PSPACE: same way TQBF is (Recall?)

Between P and PSPACE



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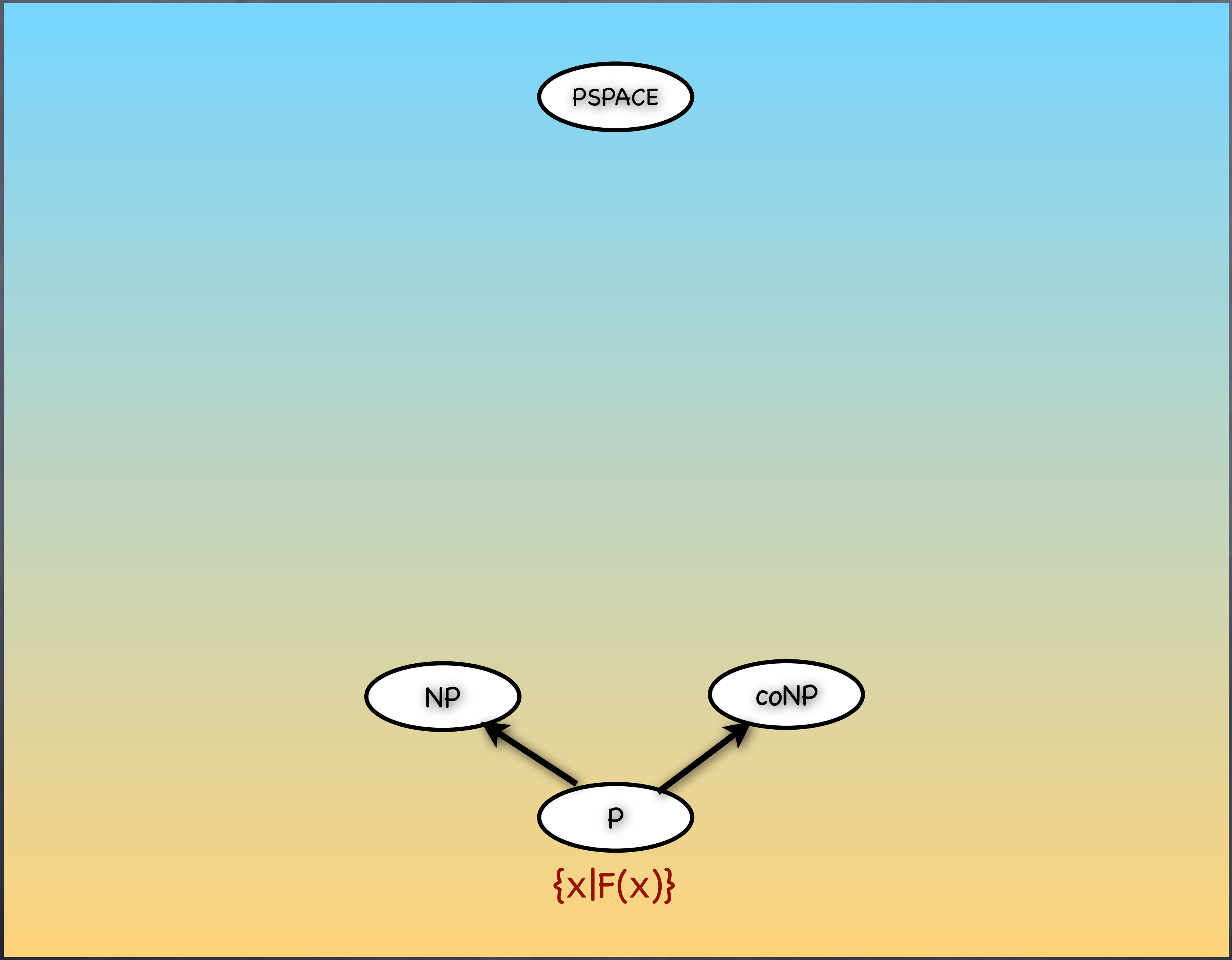

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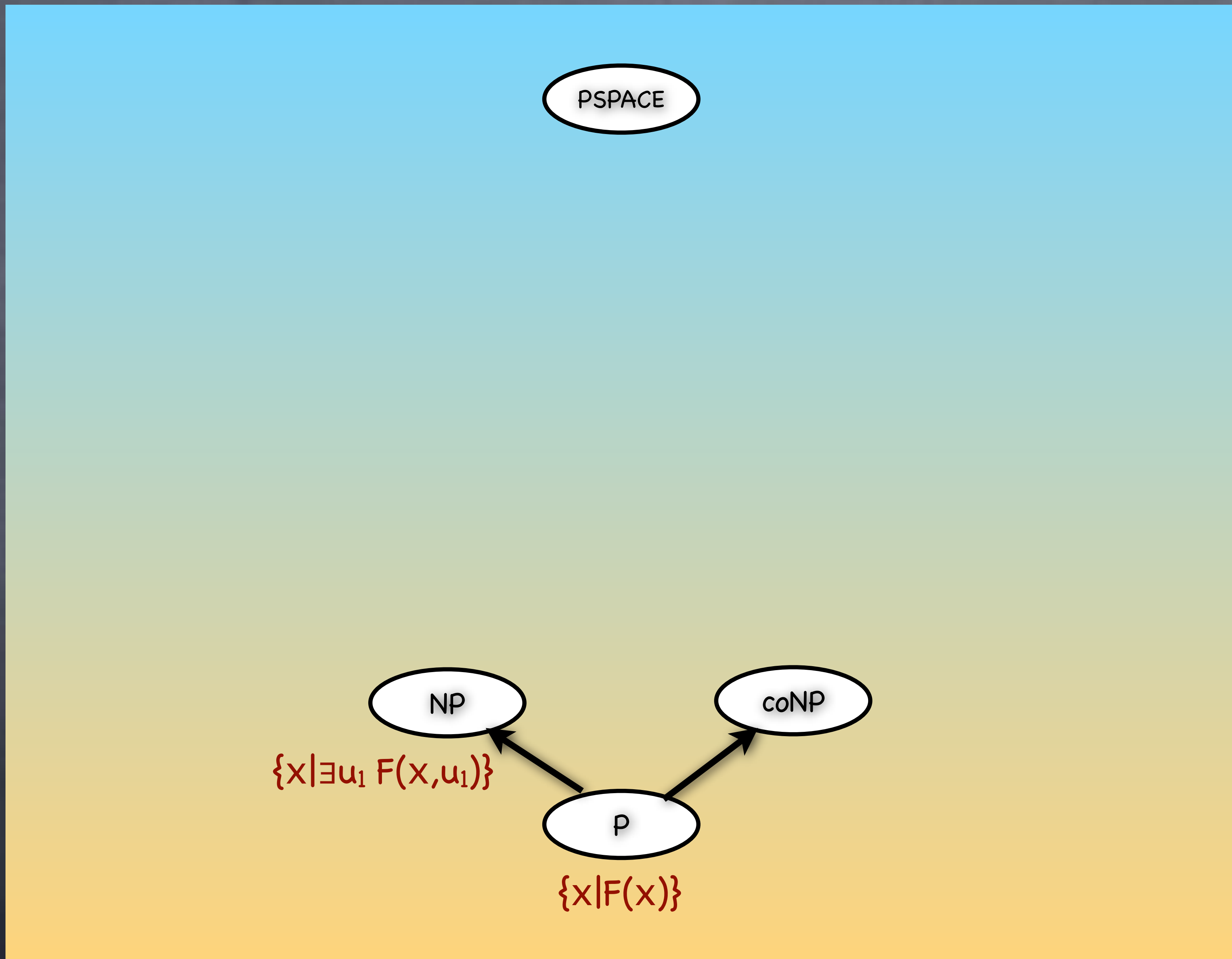

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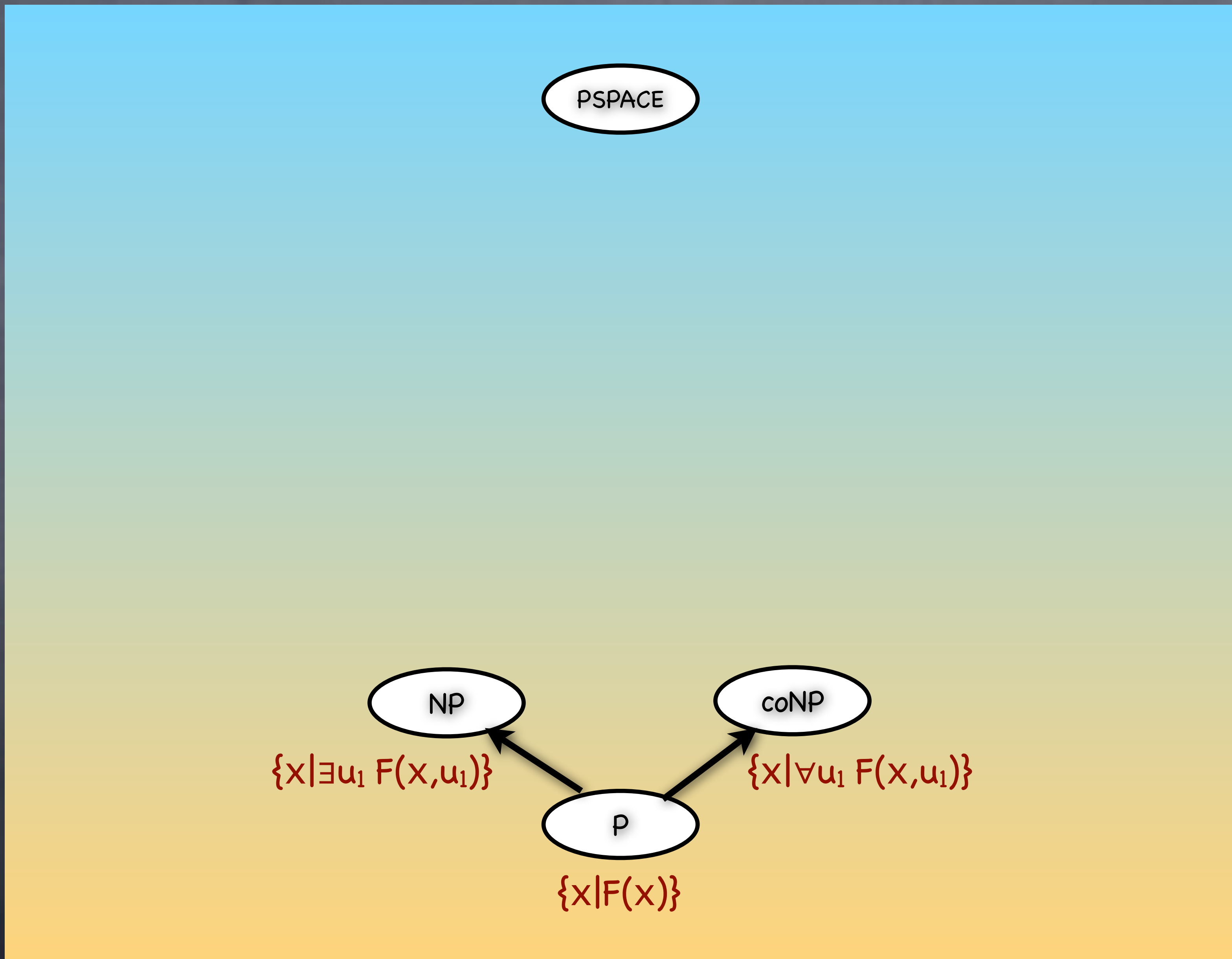

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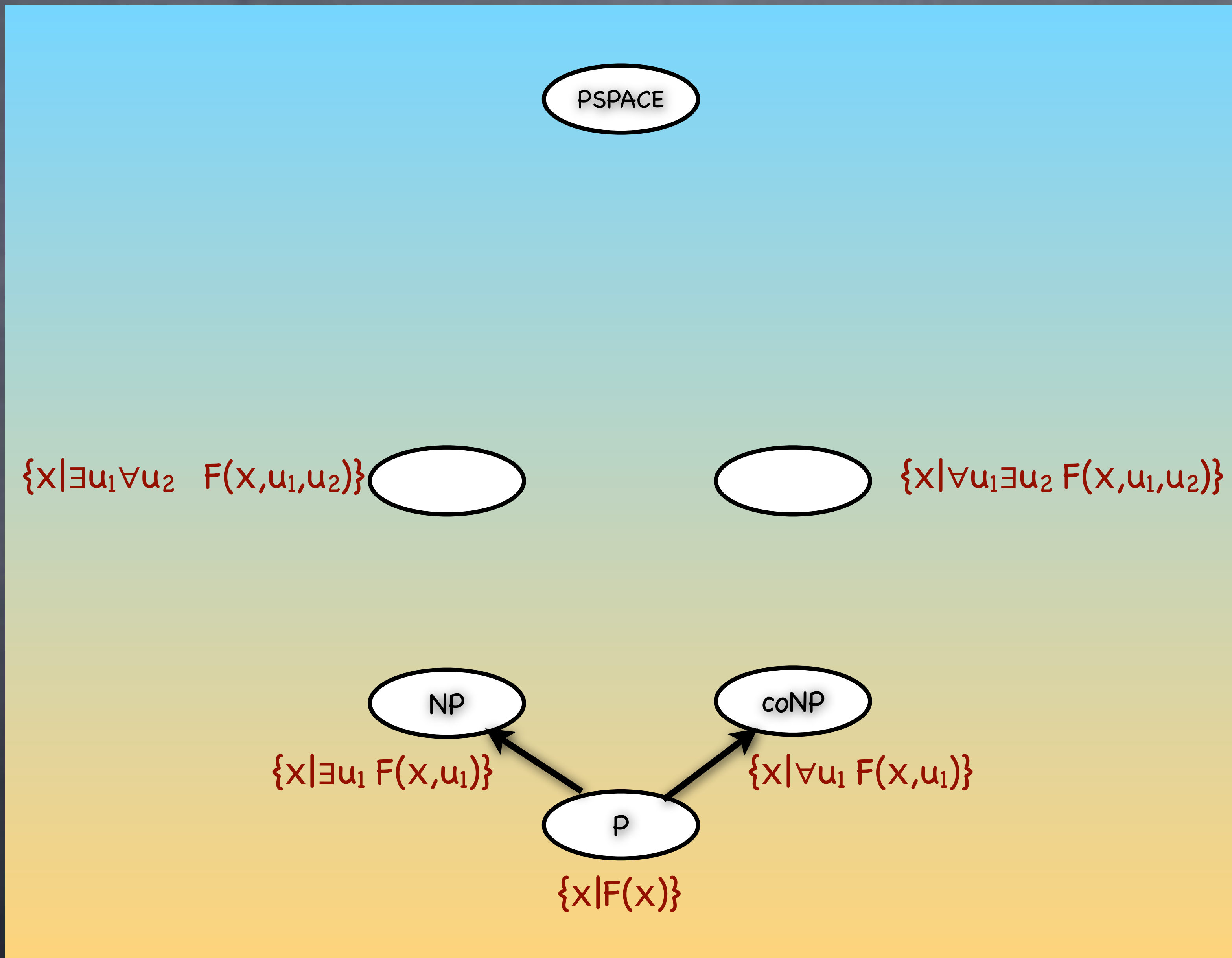

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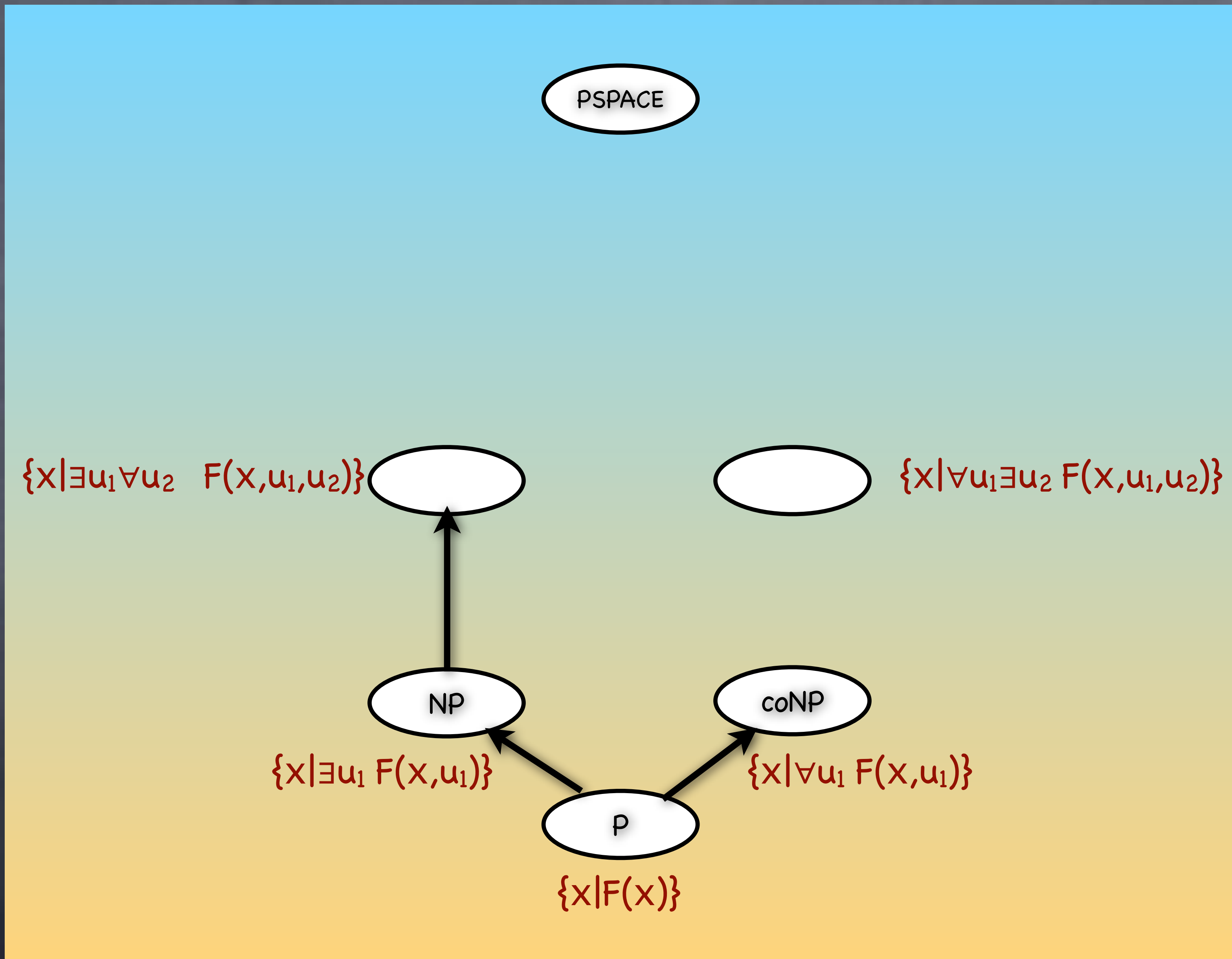

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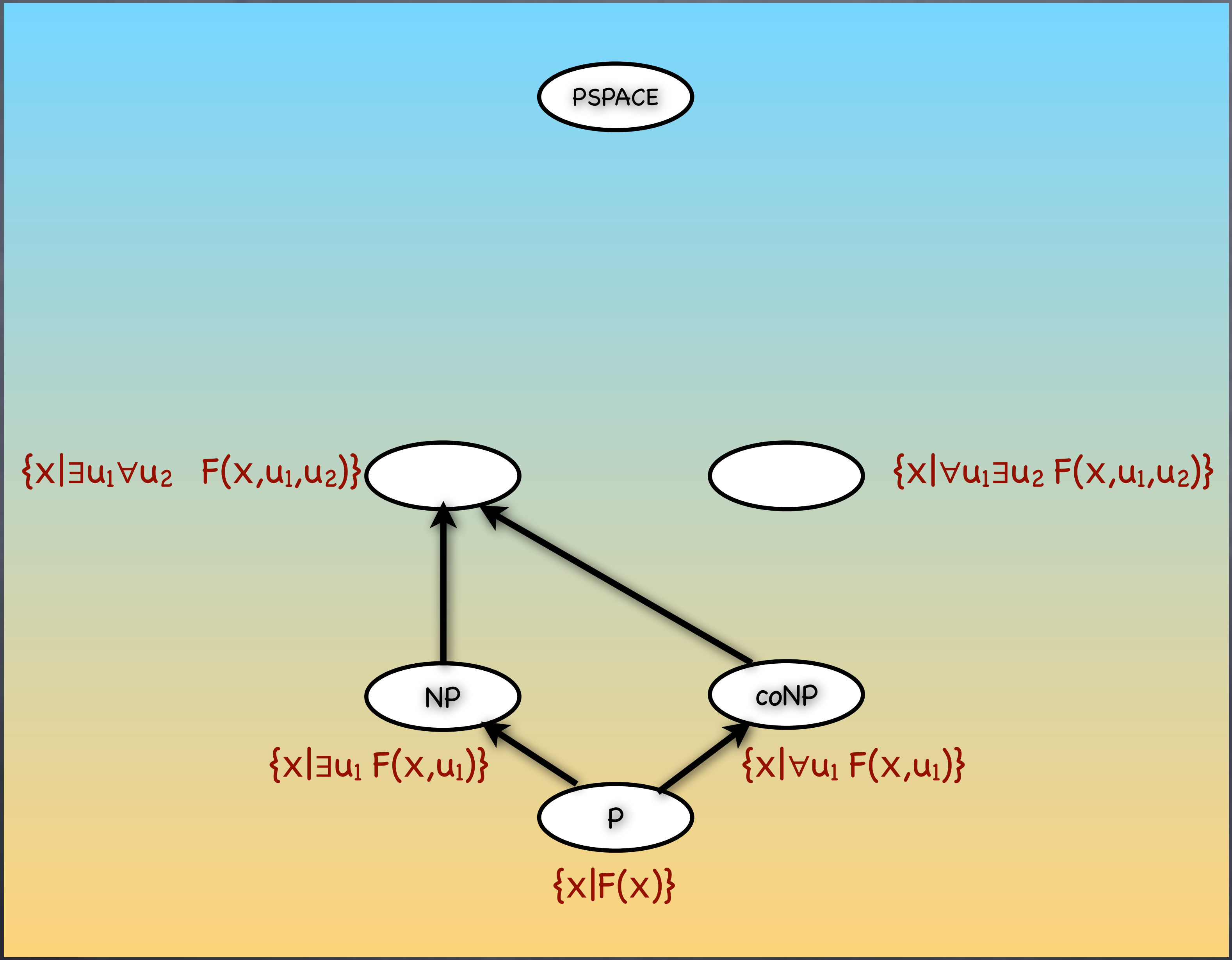

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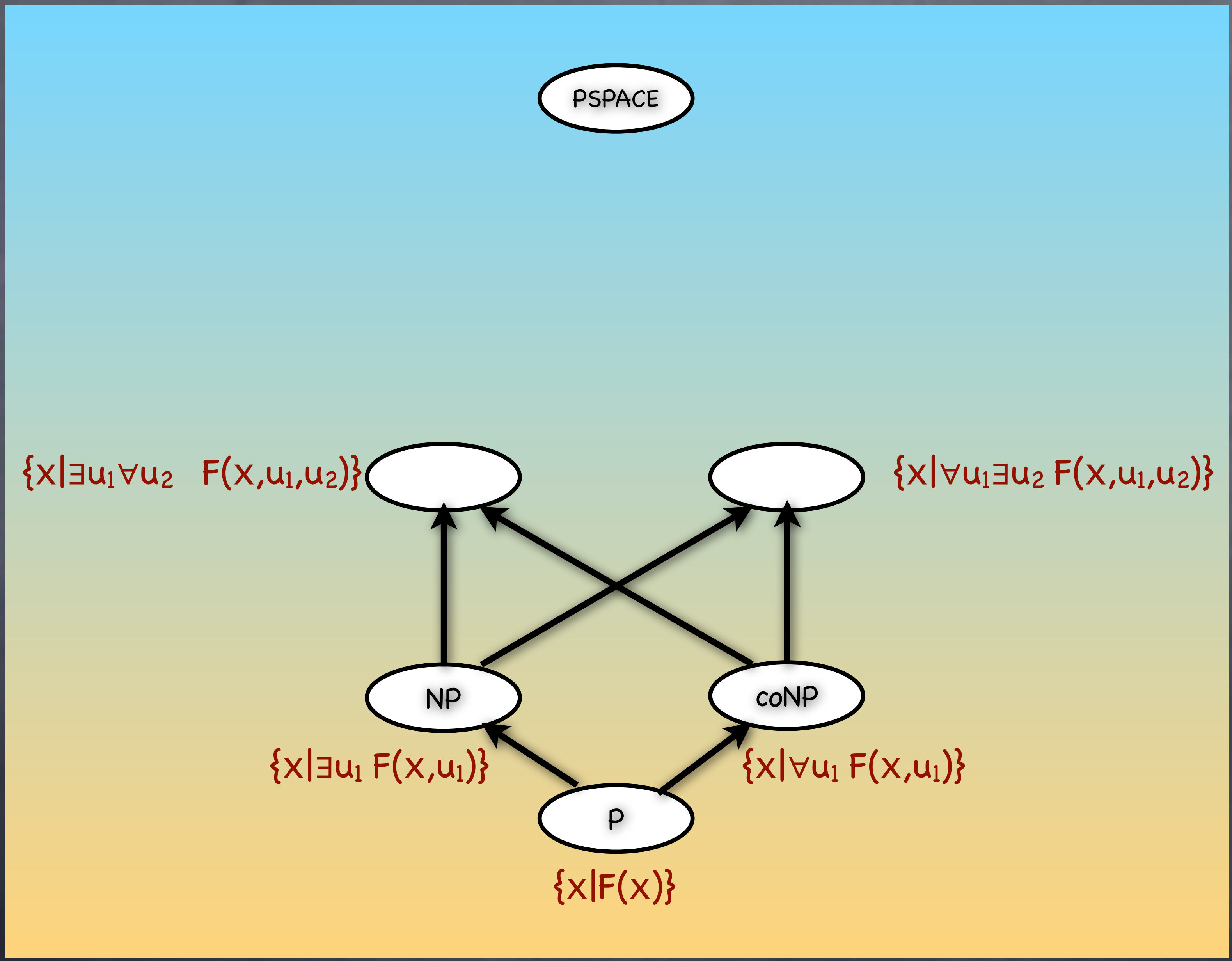

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  - Does Alice have a move such that for all moves of the adversary, Alice wins
  - Is "$\exists u_1\ \forall u_2\ \varphi(u_1,u_2)$" true?
  - Seems inherently more complex than deciding $\exists u_1\ \varphi(u_1)$ or $\forall u_1\ \varphi(u_1)$

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- Π_2^P = co-Σ_2^P
  - { x | ¬ ($\exists u_1 \forall u_2$ F′(x,u_1,u_2)) }
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- MIN-CKT = { C| $\forall C' \exists x$ C′(x)≠C(x) or C′=C or |C′|>|C| }

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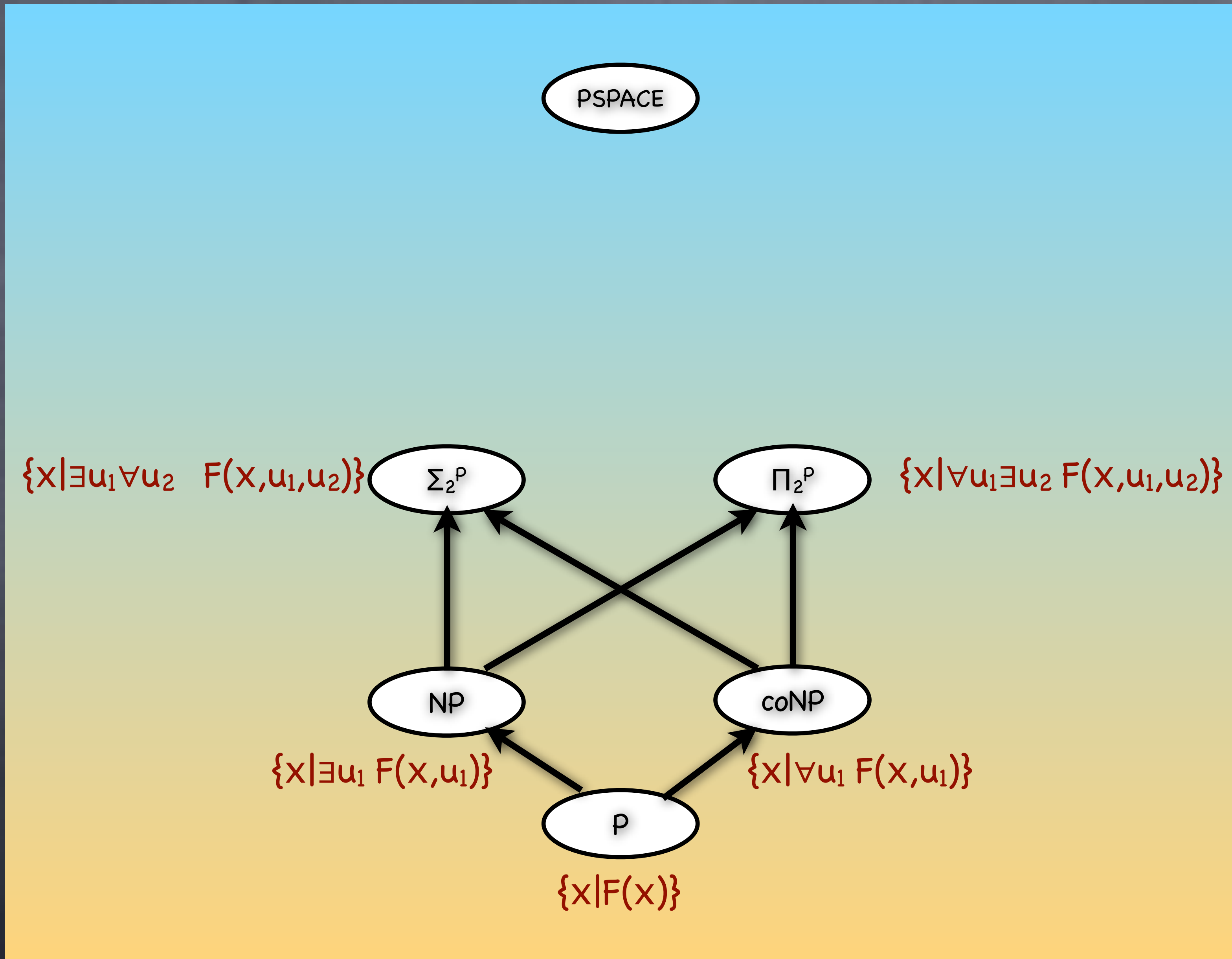

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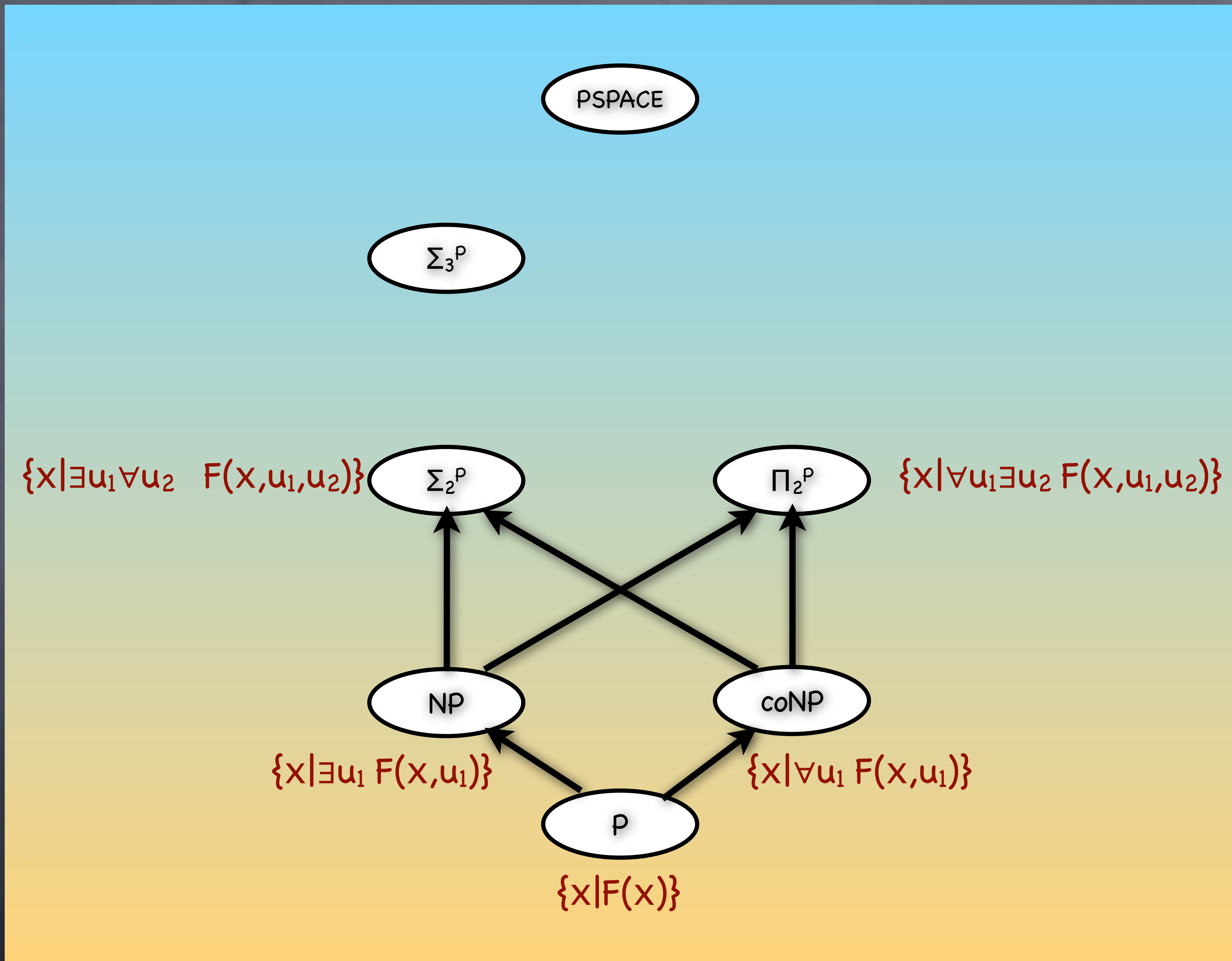

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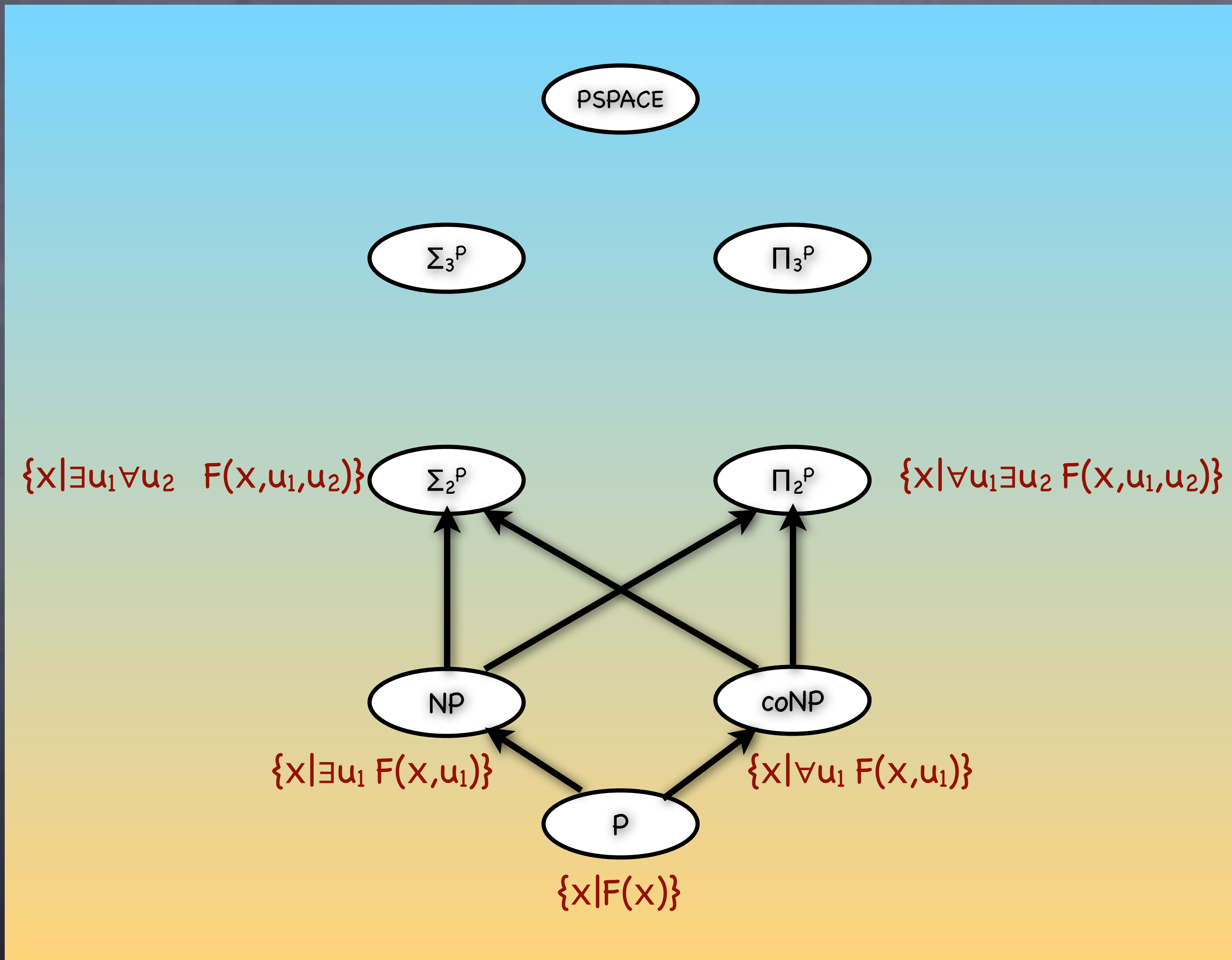

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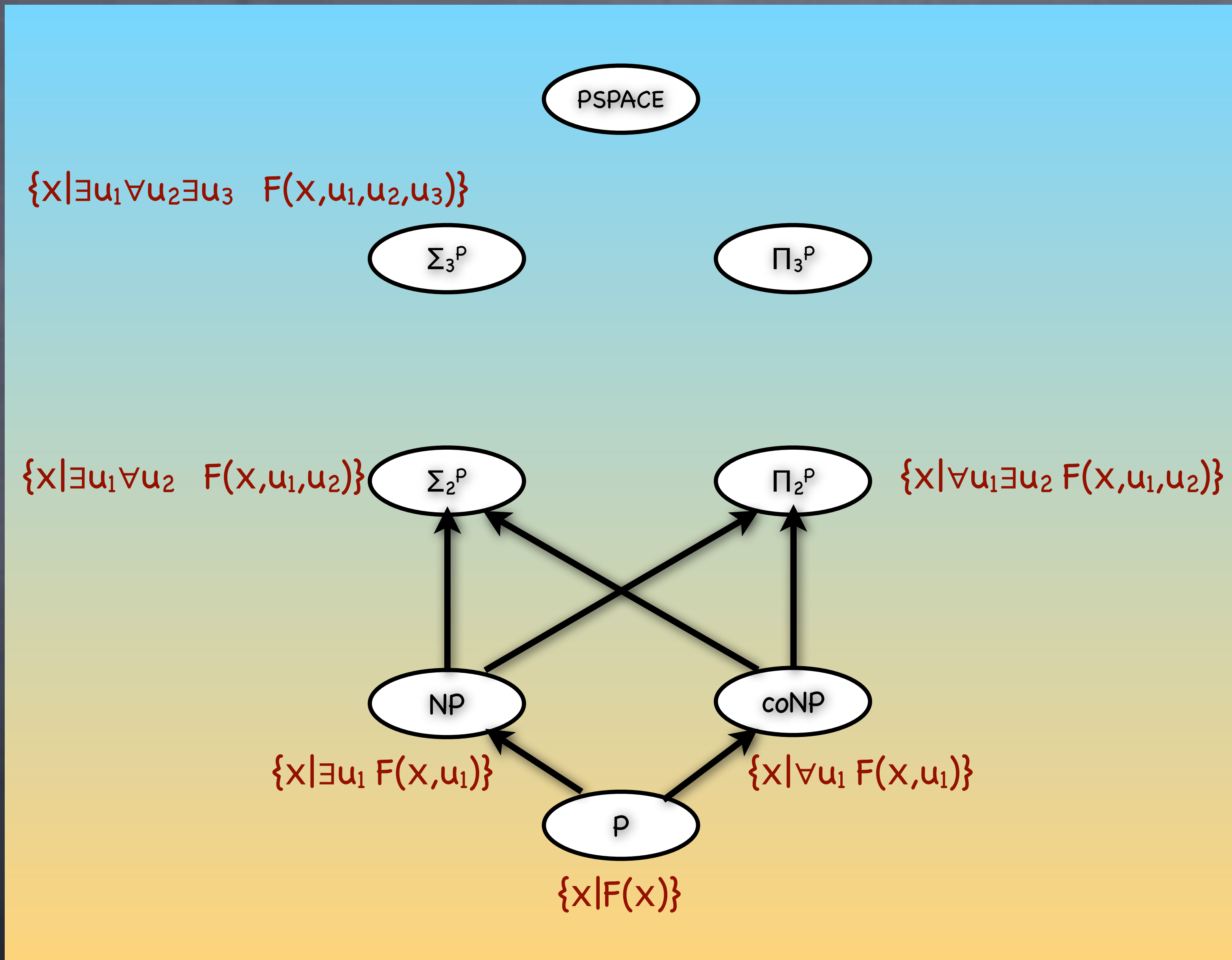

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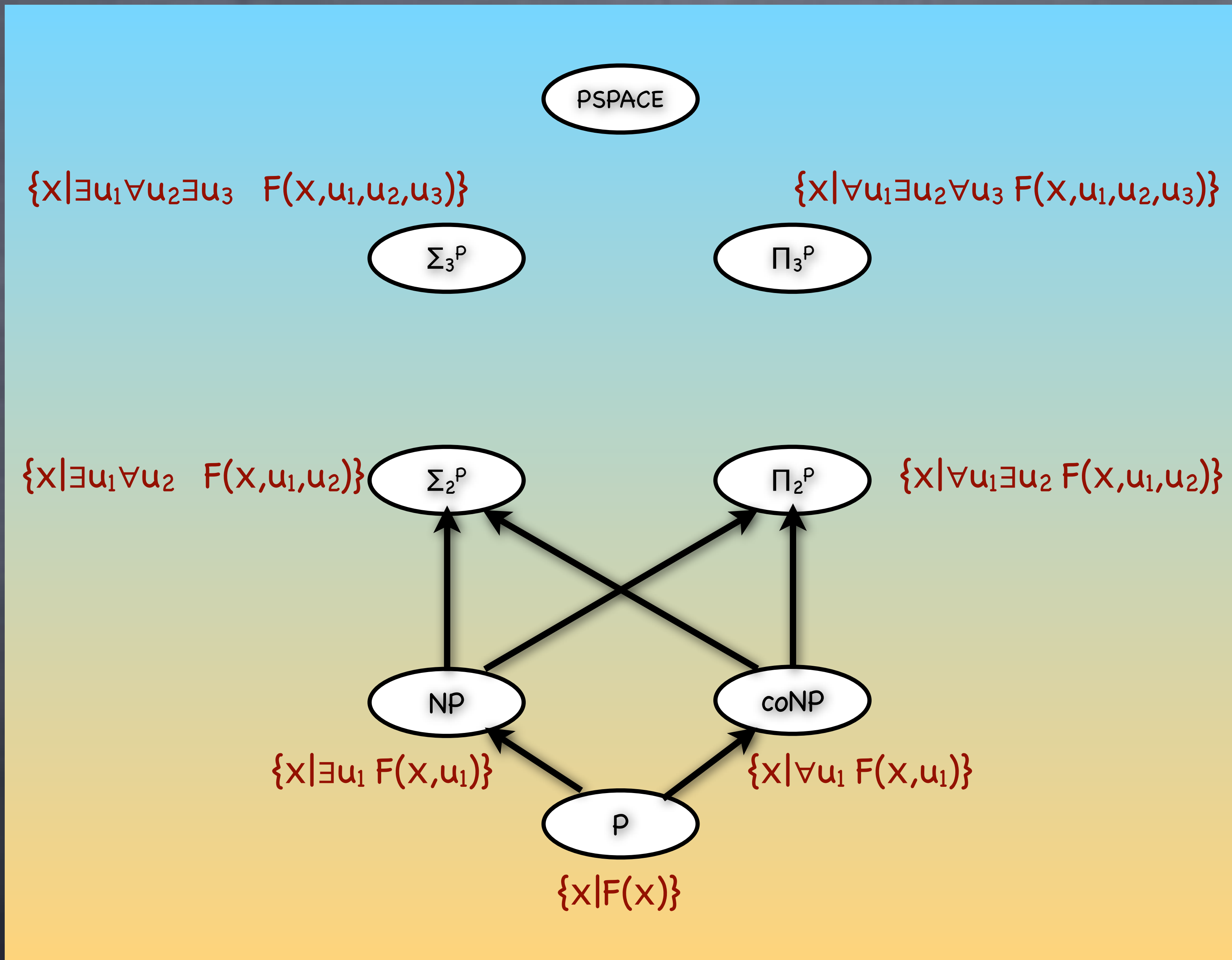

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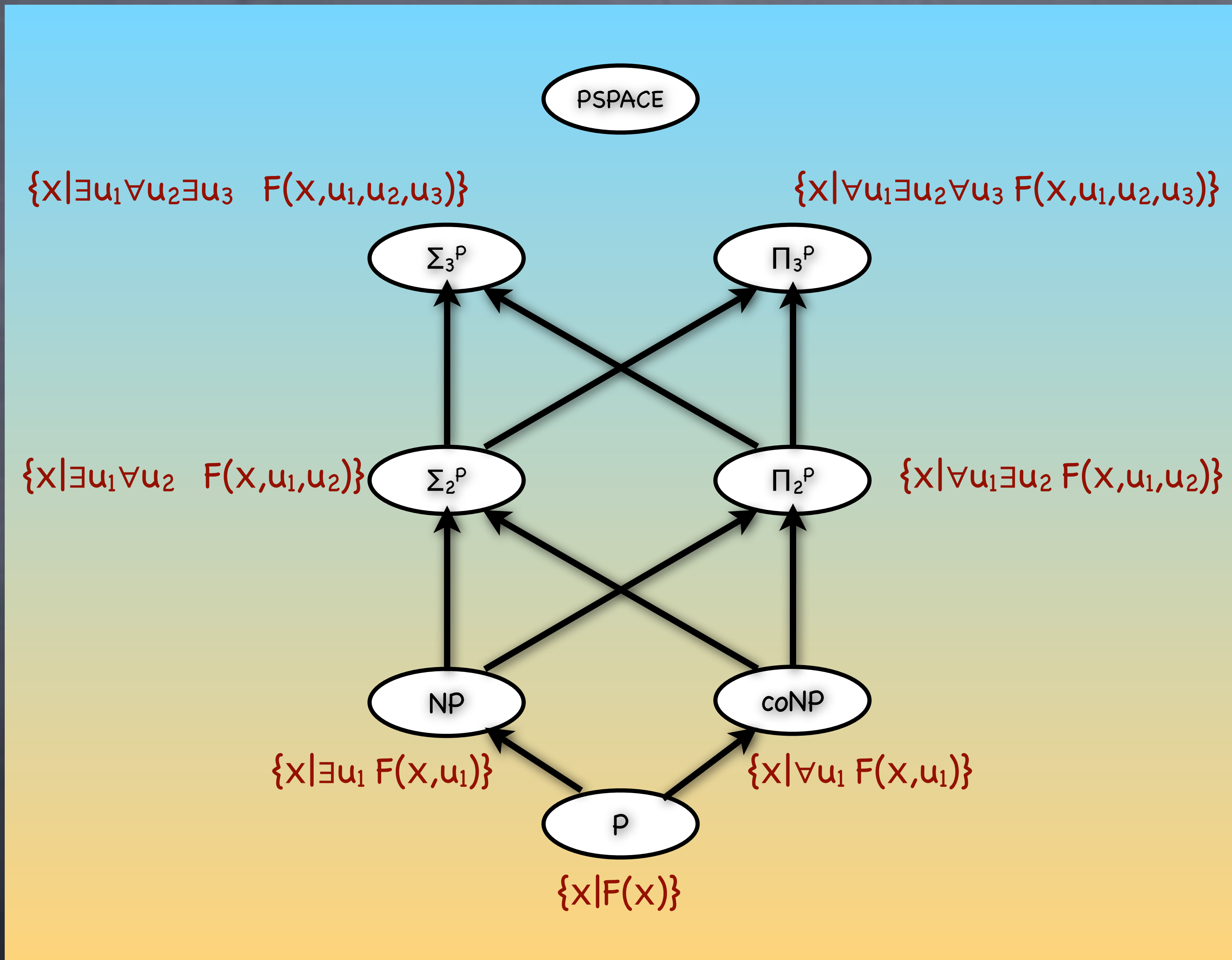

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- P = Σ_0^P = Π_0^P , NP = Σ_1^P and co-NP = Π_1^P

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- PH ⊆ PSPACE
  - Because for each k, $\Sigma_k^P \subseteq$ PSPACE
  - We don't know if PH ⊊ PSPACE
- We don't know if P ⊊ PH (or P ⊊ PSPACE)
  - Believed that $\Sigma_k^P \subsetneq \Sigma_{k+1}^P$ and $\Pi_k^P \subsetneq \Pi_{k+1}^P$ for all k

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- For each level of PH (w.r.t Karp reductions)

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  - Why? Consider odd k Σ_k^P and even k Π_k^P (ends with ∃)

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    - Complete for Π_k^P
  - Why? Consider odd k Σ_k^P and even k Π_k^P (ends with ∃)
    - Recall: $F(X)=1$ iff $CKT_F(X)=1$ iff $\exists w\ \varphi_F(X;w)=1$

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  - Π_kSAT: True QBFs with k alternations, starting with $\forall$
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  - Why? Consider odd k Σ_k^P and even k Π_k^P (ends with $\exists$)
    - Recall: $F(X)=1$ iff $CKT_F(X)=1$ iff $\exists w\ \varphi_F(X;w)=1$

Needed a $\exists$ in going from ckt to CNF formula

Complete problems

- For each level of PH (w.r.t Karp reductions)
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For the other classes consider co-classes

Needed a ∃ in going from ckt to CNF formula

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- Corollary: If PH = PSPACE, then PH = PSPACE = Σ_k^P for some k
  - Because if PH = PSPACE, TQBF is PH-complete

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- Then entire PH collapses! (to that level)

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  - By induction PH = $\Sigma_k^P = \Pi_k^P$
  - Enough to show $\Sigma_k^P = \Pi_k^P$ (for k>0) $\Rightarrow \Sigma_{k+1}^P \subseteq \Sigma_k^P$

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- $\Rightarrow \Sigma_{k+1}^P \subseteq \Sigma_k^P$

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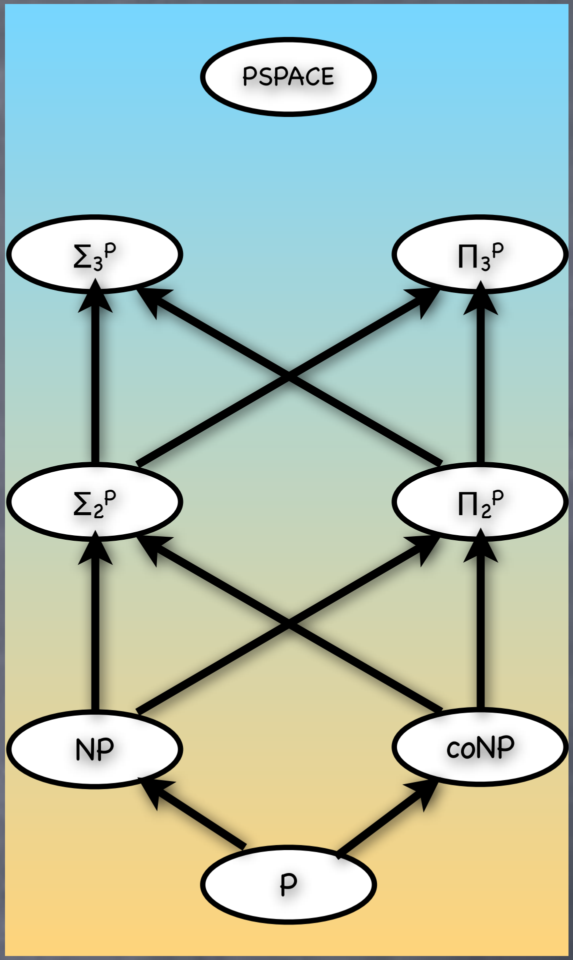
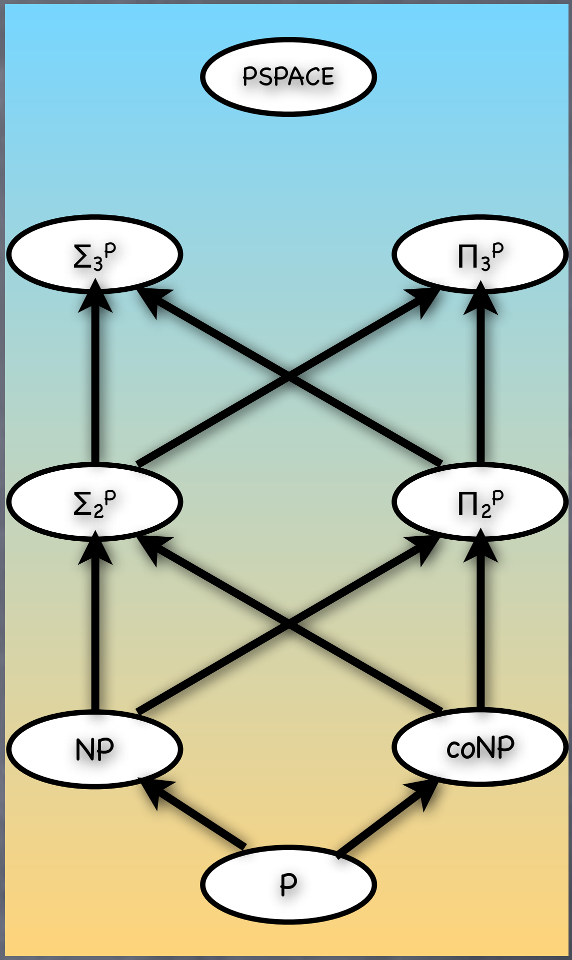
- Holds for k=0 too: i.e., NP = P $\Rightarrow$ PH = P
  - NP = P $\Rightarrow$ NP = co-NP $\Rightarrow$ PH = NP (= P)

Today



Today

- Polynomial Hierarchy



Today

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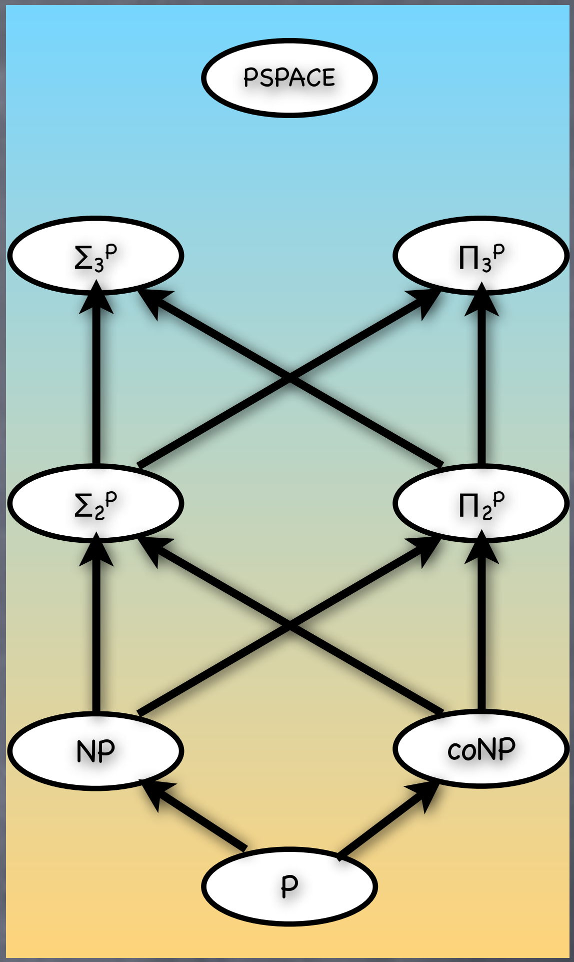

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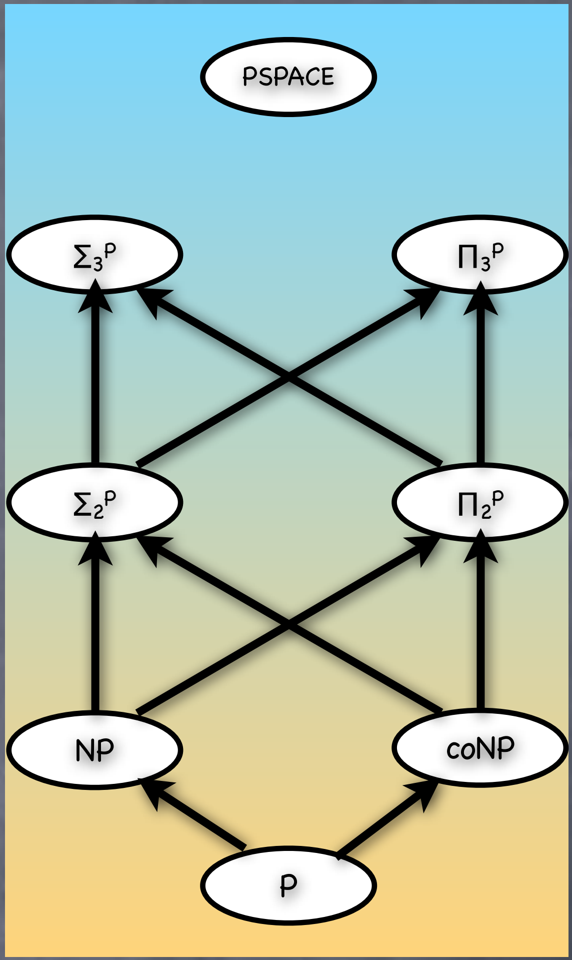
- Collapse of Polynomial Hierarchy



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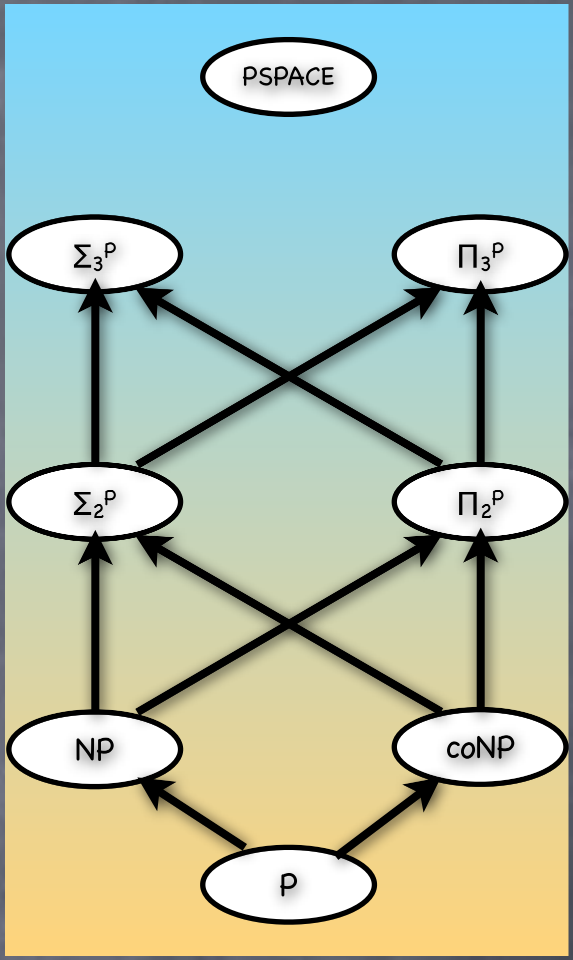

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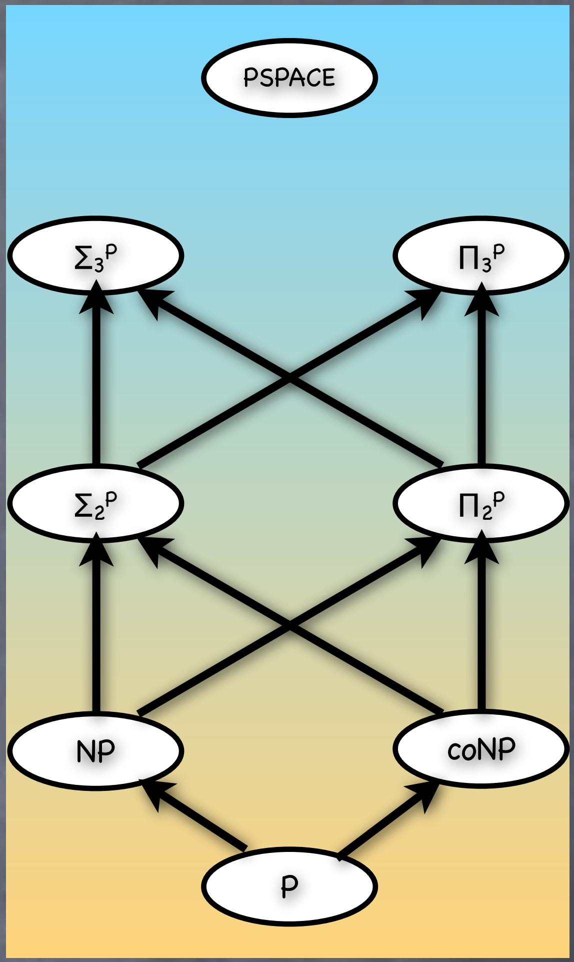

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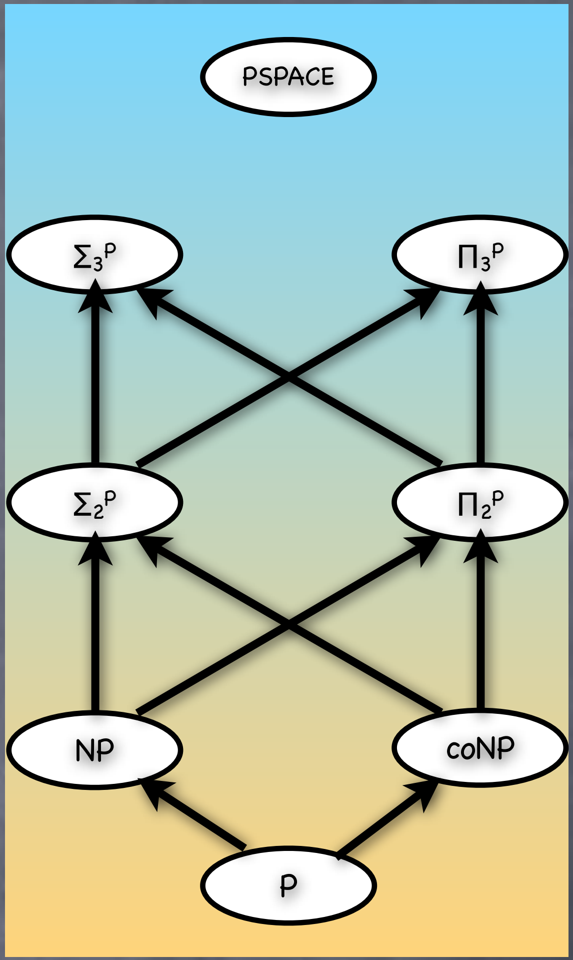

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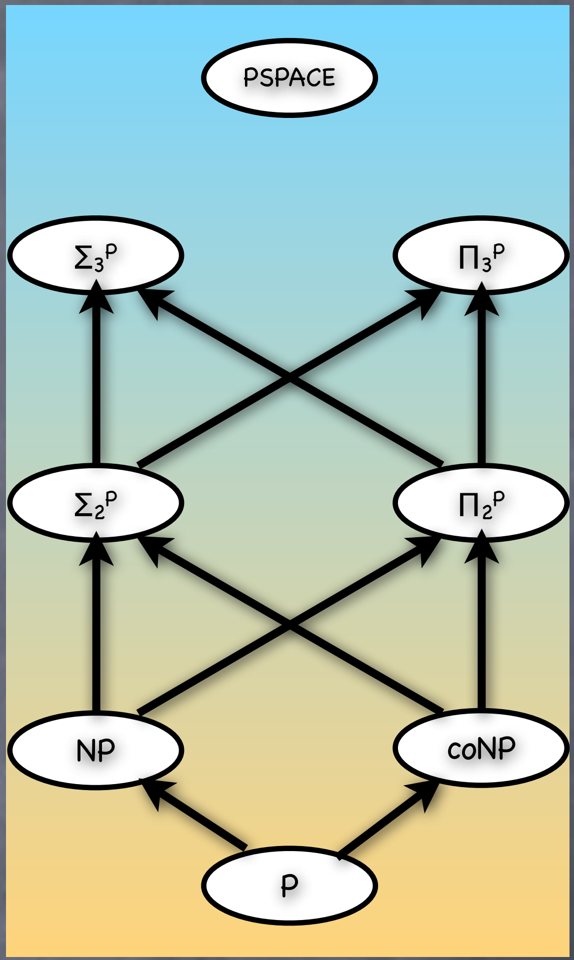

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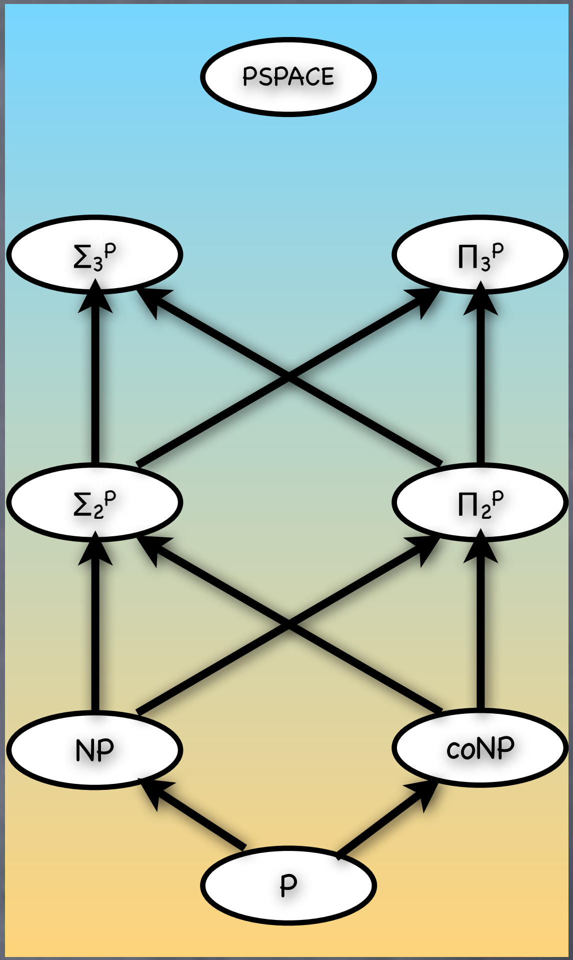

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- Coming up: More ways to look at the polynomial hierarchy