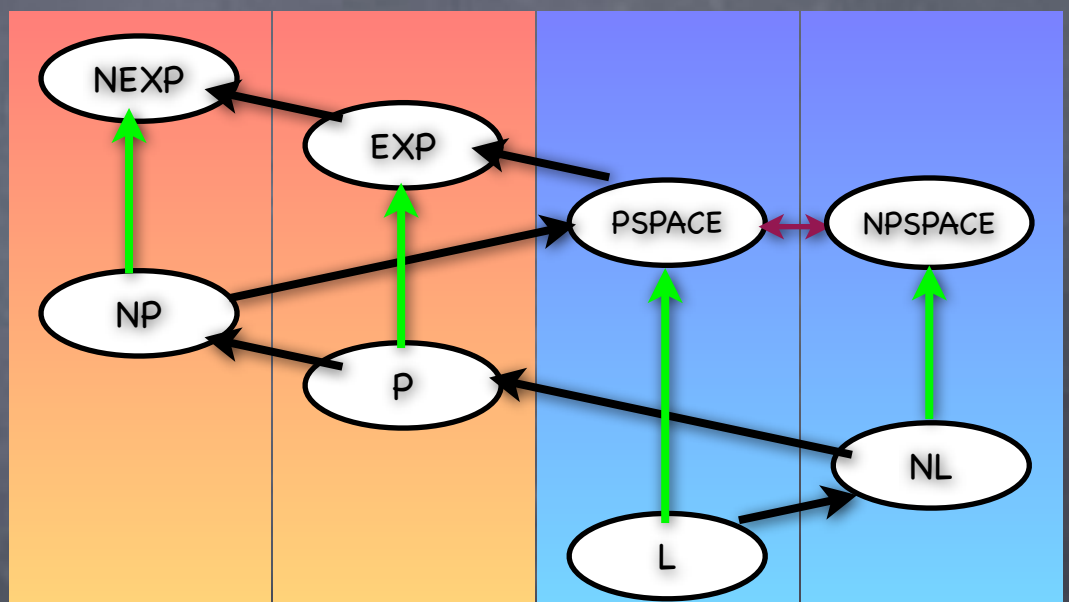
Computational Complexity

Lecture 6
NL-Completeness and NL=co-NL

Story, so far



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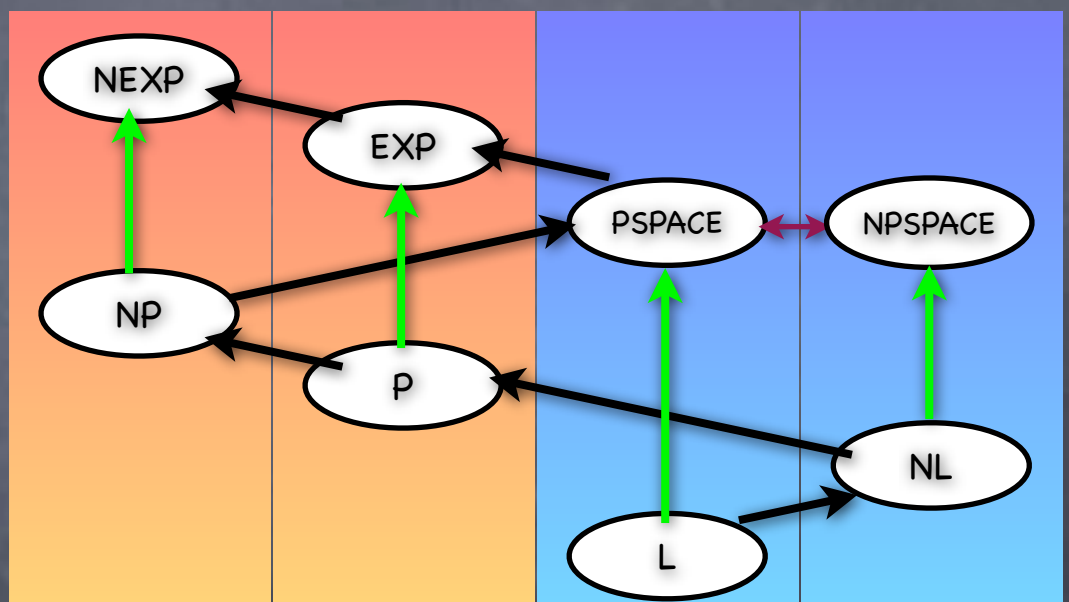

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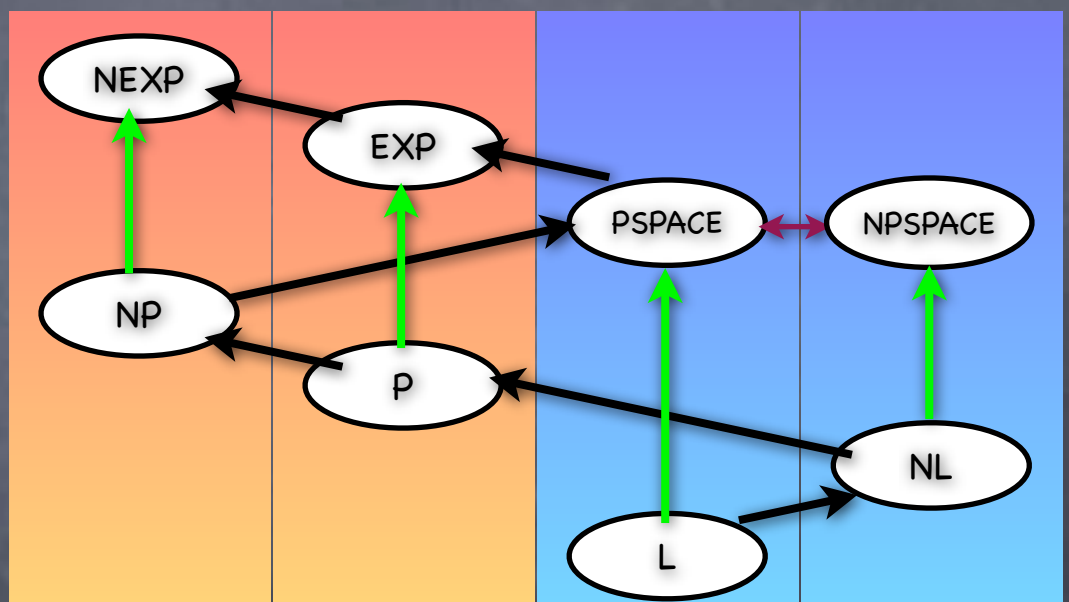

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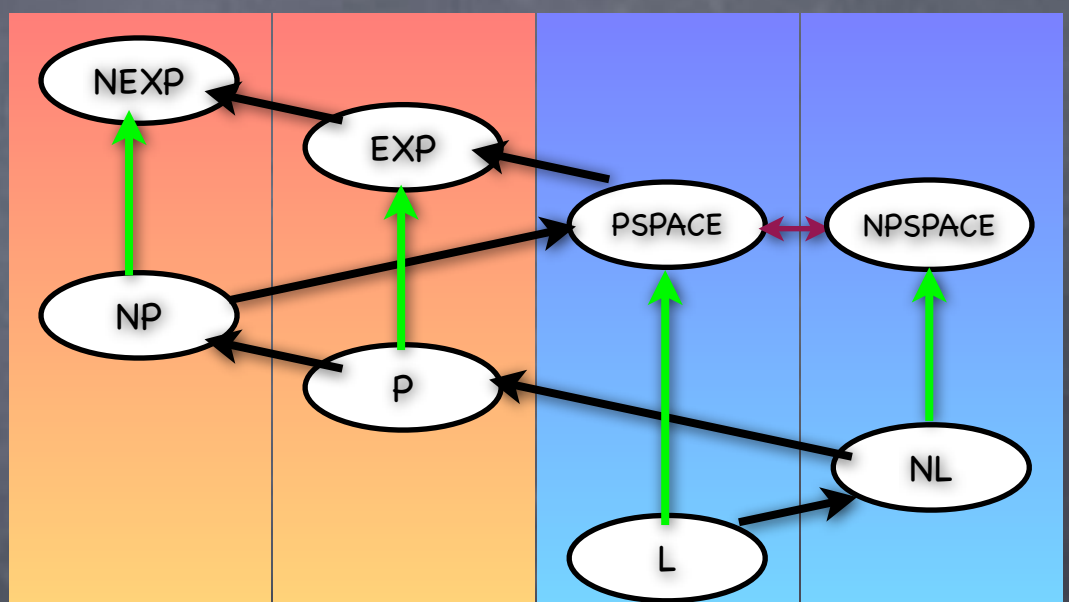

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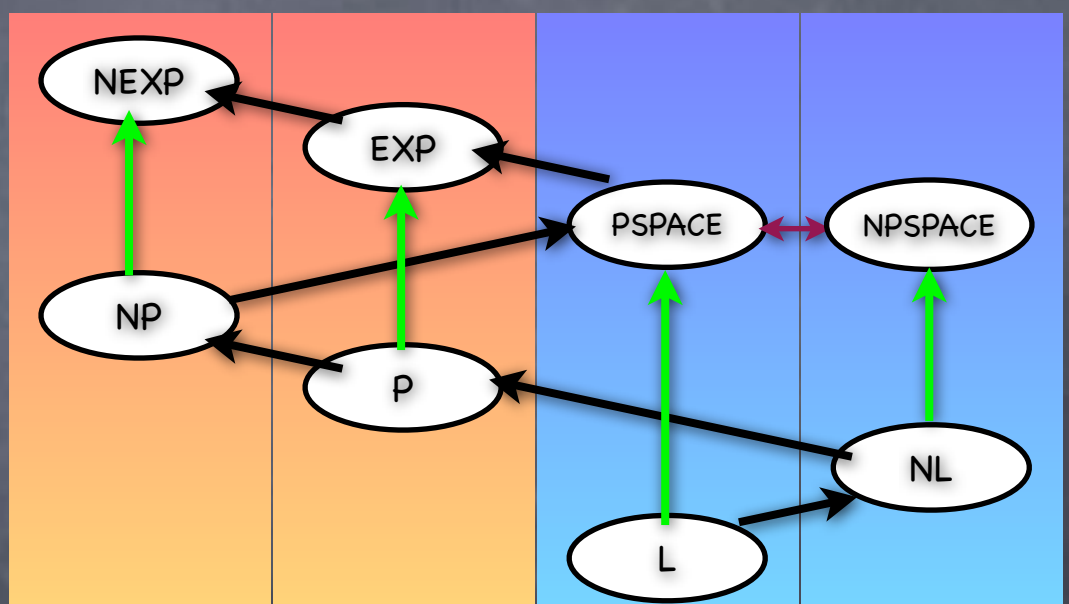

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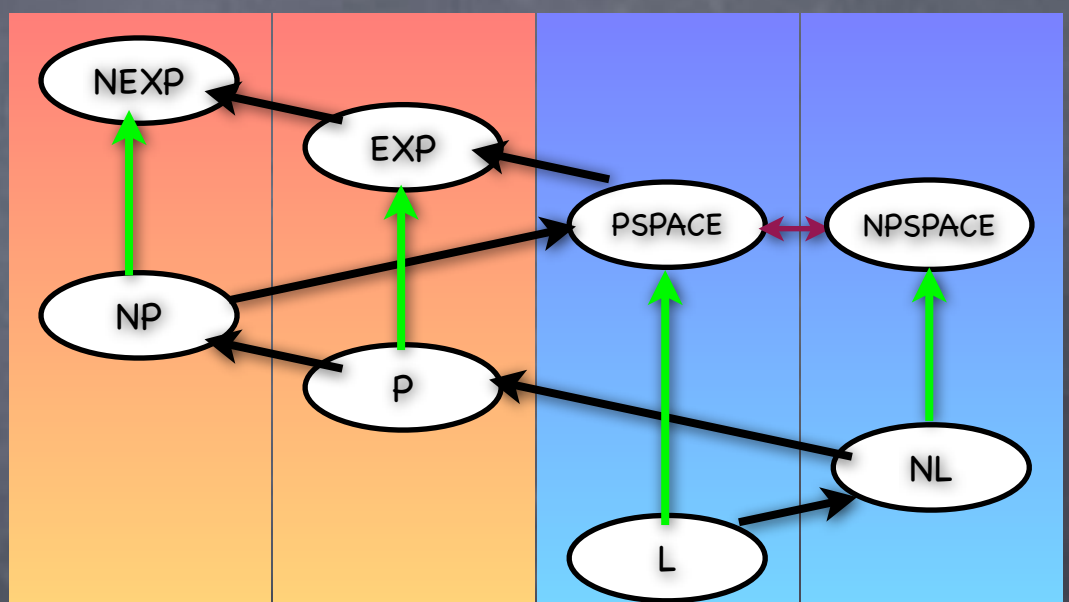

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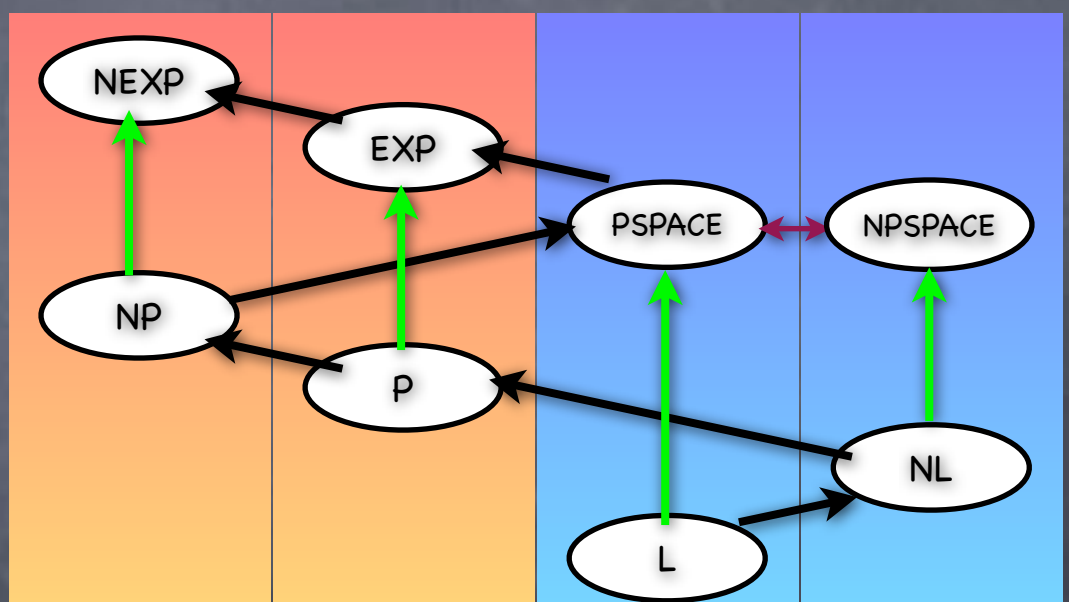

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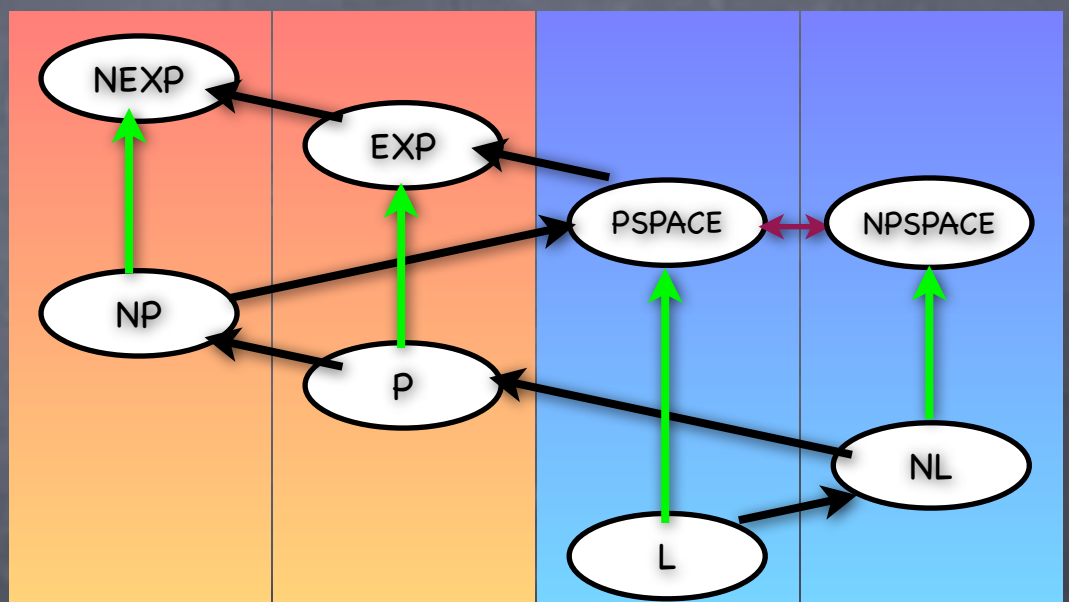
  - An NL-complete language: PATH
  - NSPACE = co-NSPACE (one less kind to worry about!)



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  - So if $X \subseteq P$, all languages in X are X-complete (w.r.t $\leq_p$)
- Need a tighter notion of reduction to capture "(almost) as hard as it gets" within X

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Not suitable for use as a subroutine

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- Interesting NLC language: PATH

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    - Note: w is scanned only once

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  - PATH $\in$ DSPACE($\log^2(n)$)

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- Note: in fact O(S)-space reduction from $L \in NSPACE(S)$ to PATH

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    - If Y $\subseteq$ X, then co-Y $\subseteq$ co-X. Consider X = NL, Y = co-NL.

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  - In particular NL = co-NL
- And indeed, PATH $\in$ co-NL!
  - There is a (polynomial sized) certificate that can be verified in log-space, that there is no path from s to t in a graph G

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      - List has |C| many $v \in C$, without repeating

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  - Read $|C_{N-2}|$, believing it verify $|C_{N-1}|$, forget $|C_{N-1}|$; ...

Certificate for |C|

- Let C_i := set of nodes within distance i of s. Then $C = C_N$
- Tail recursion to verify $|C_N|$:
  - Read $|C_{N-1}|$, believing it verify $|C_N|$, forget $|C_N|$;
  - Read $|C_{N-2}|$, believing it verify $|C_{N-1}|$, forget $|C_{N-1}|$; ...
  - Base case: $|C_0|=1$

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- Believing $|C_{i-1}|$ verify $|C_i|$: for each vertex v certificate that $v \in C_i$ or that $v \notin C_i$ (these certificates are poly(N) long)
- Certificate that $v \notin C_i$ given (i.e., believing) $|C_{i-1}|$: list of all vertices in C_{i-1} in order, with certificates. As before verify C_{i-1} believing $|C_{i-1}|$ (scan and ensure list is correct/complete), but also check that no node in the list has v as a neighbor

Certificate for $t \notin C_N$

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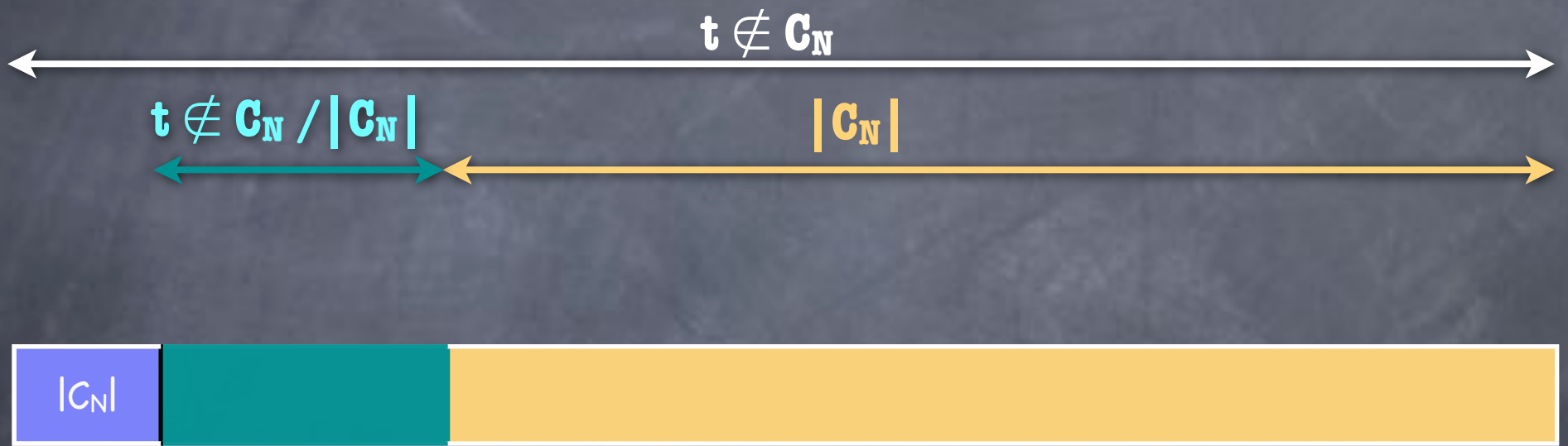
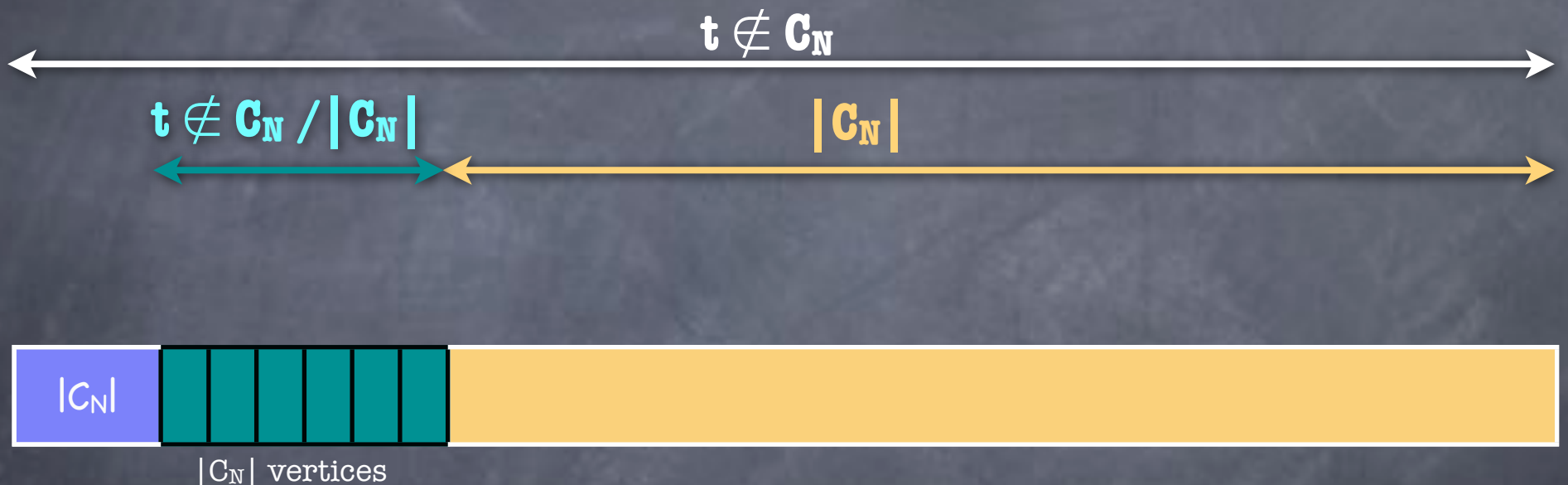
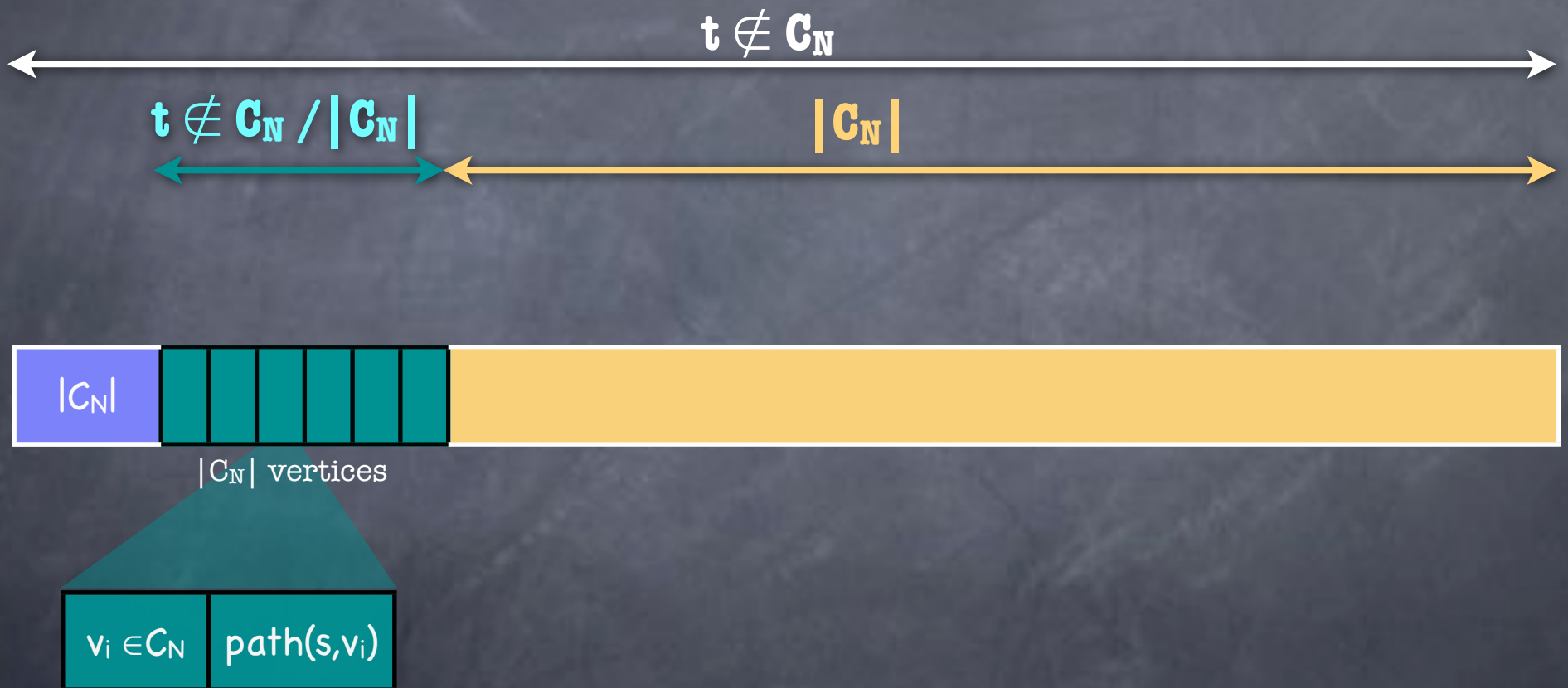
Certificate for $t \notin C_N$



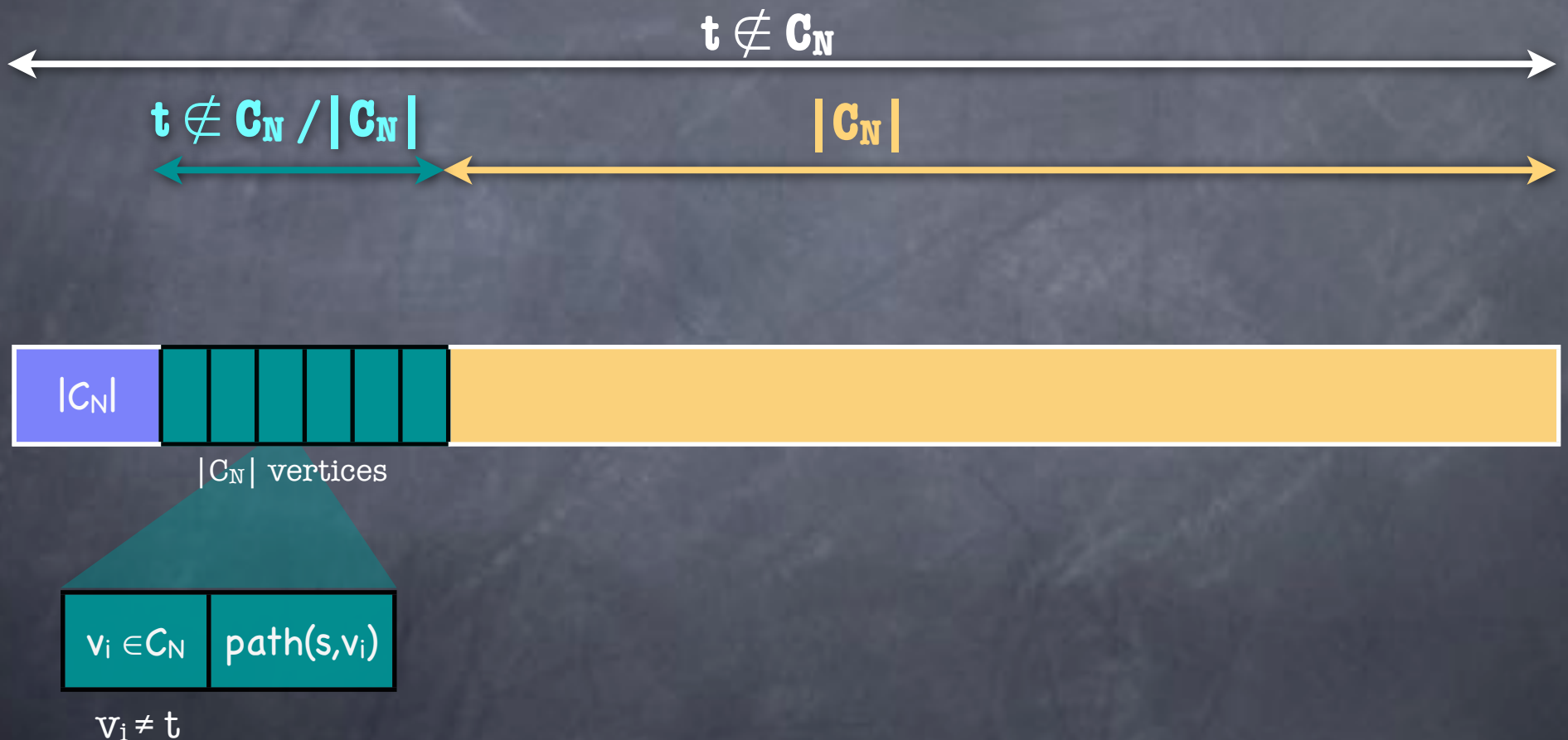
Certificate for $t \notin C_N$



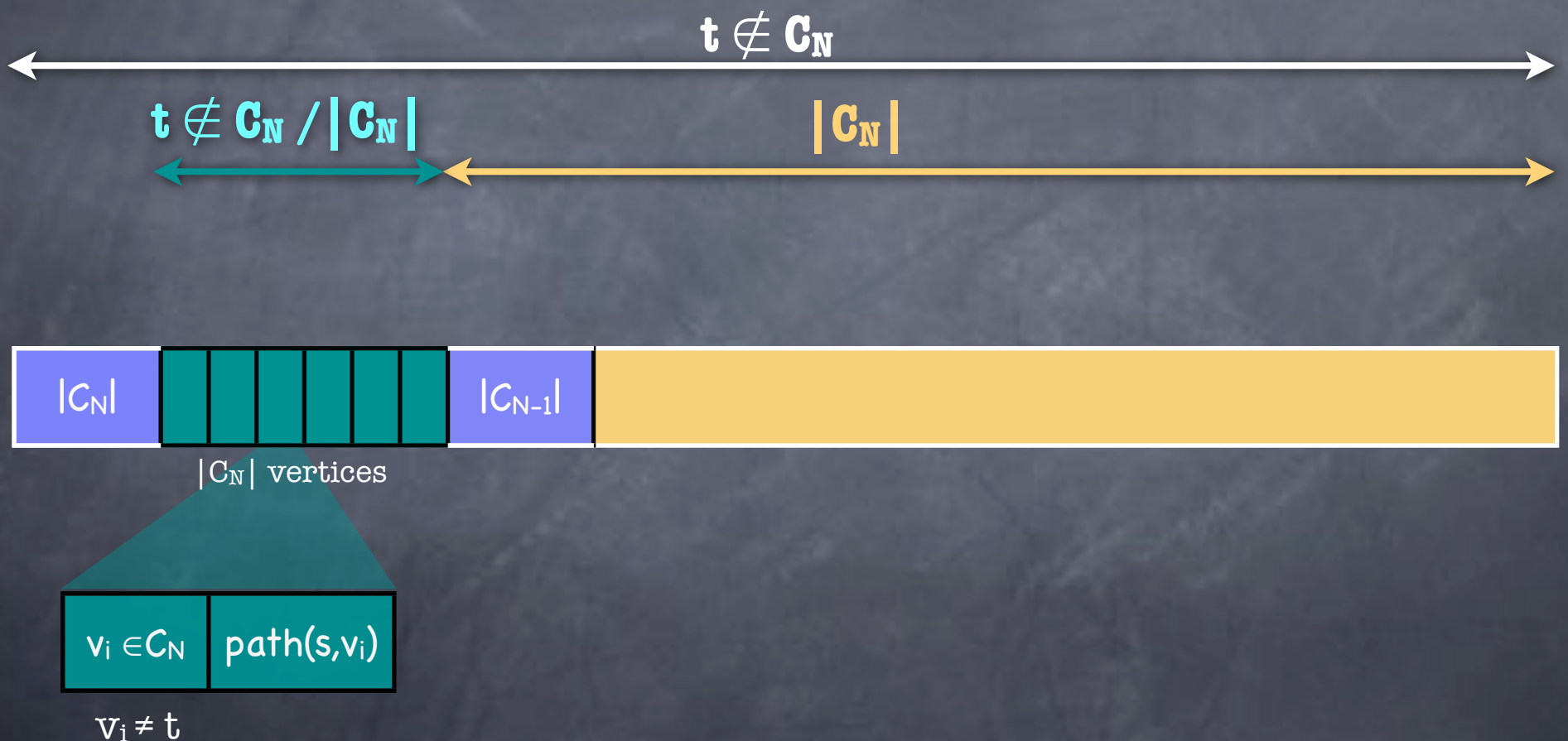
Certificate for $t \notin C_N$



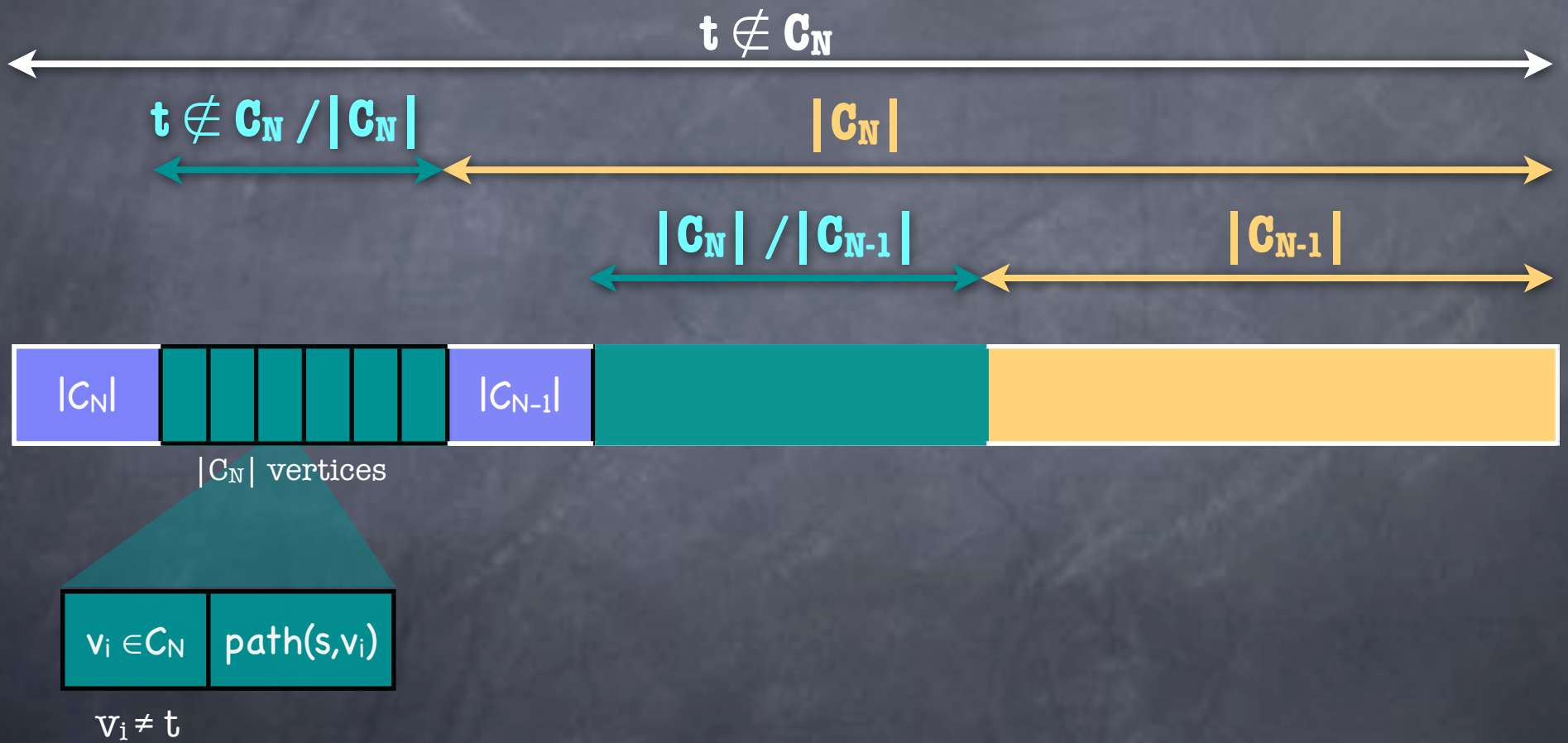
Certificate for $t \notin C_N$



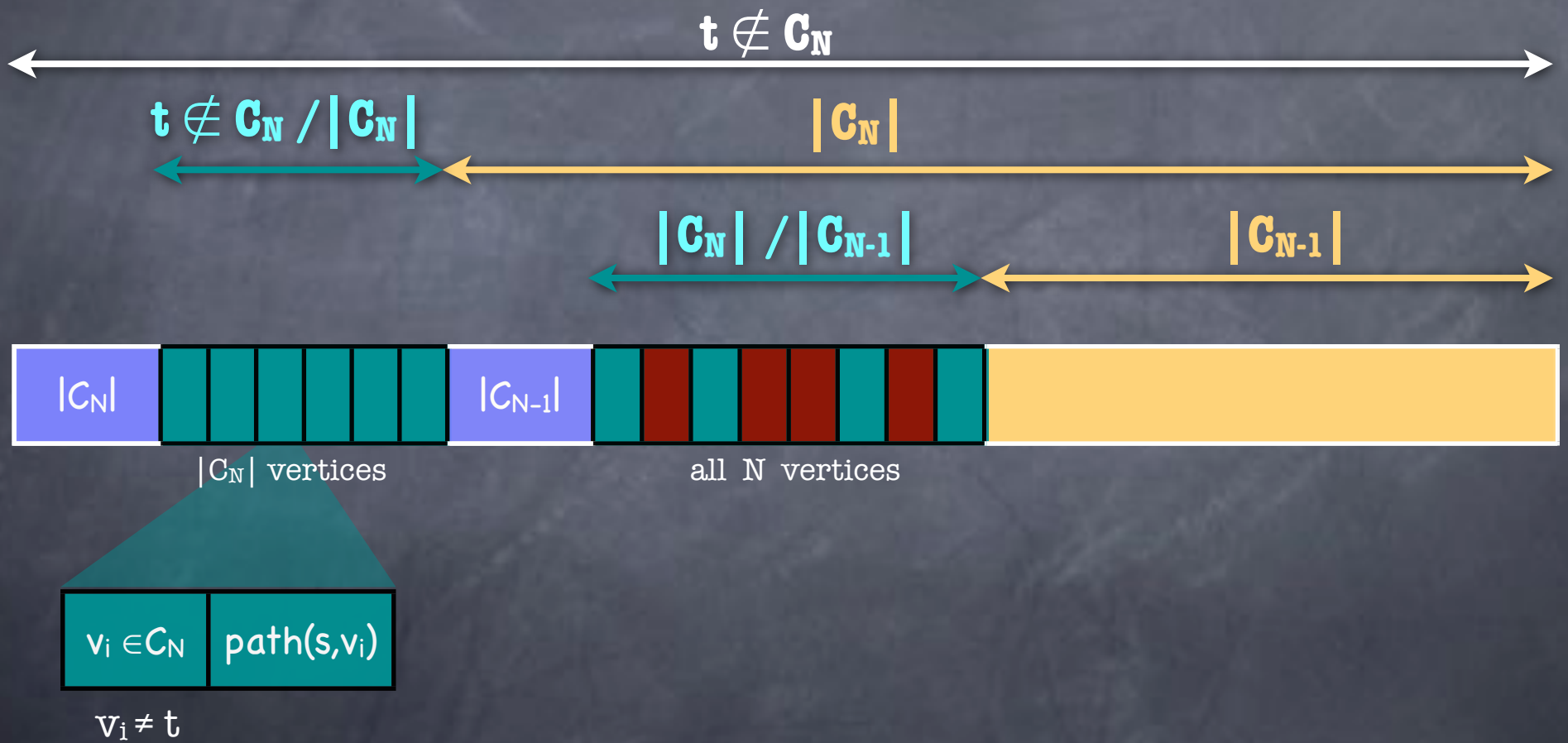
Certificate for $t \notin C_N$



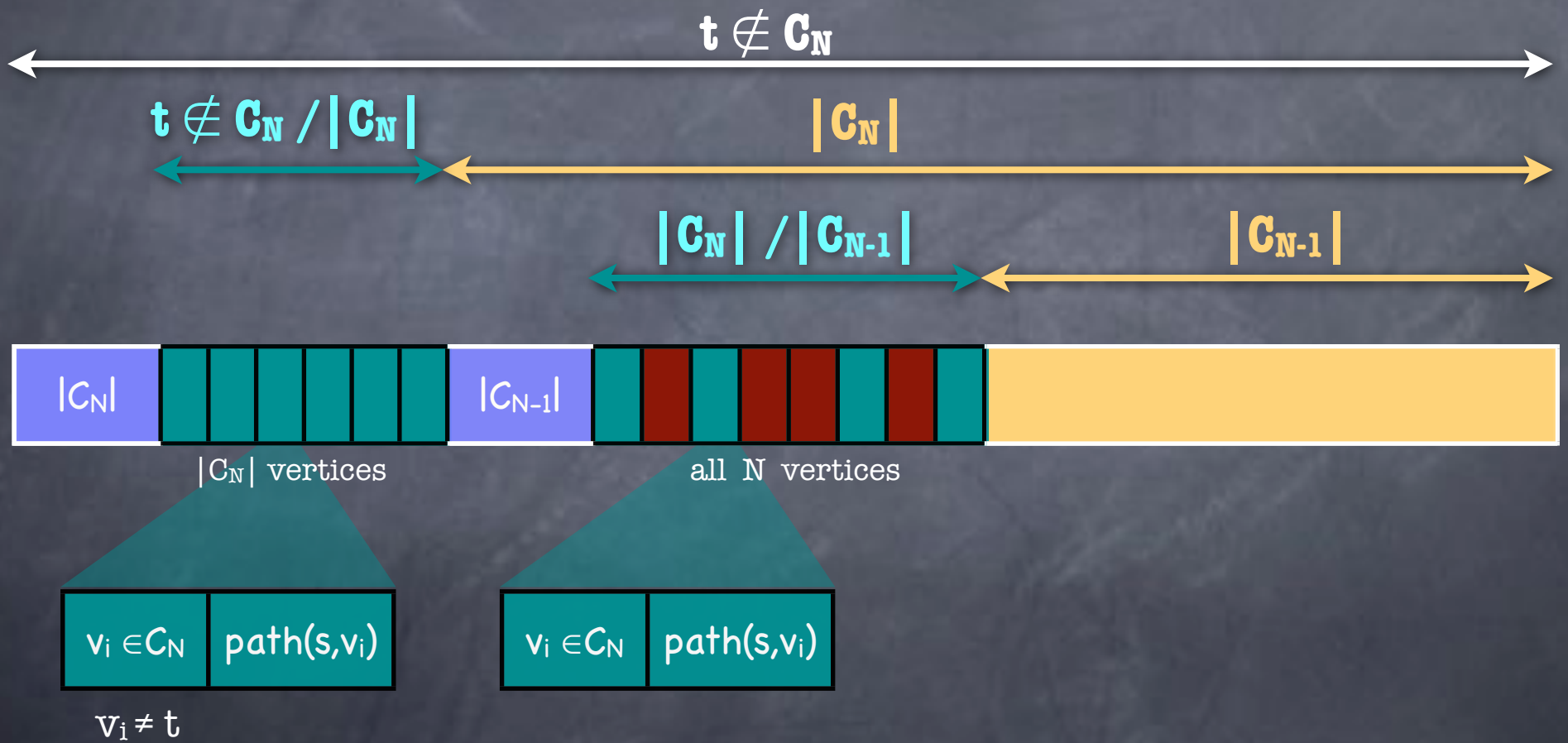
Certificate for $t \notin C_N$



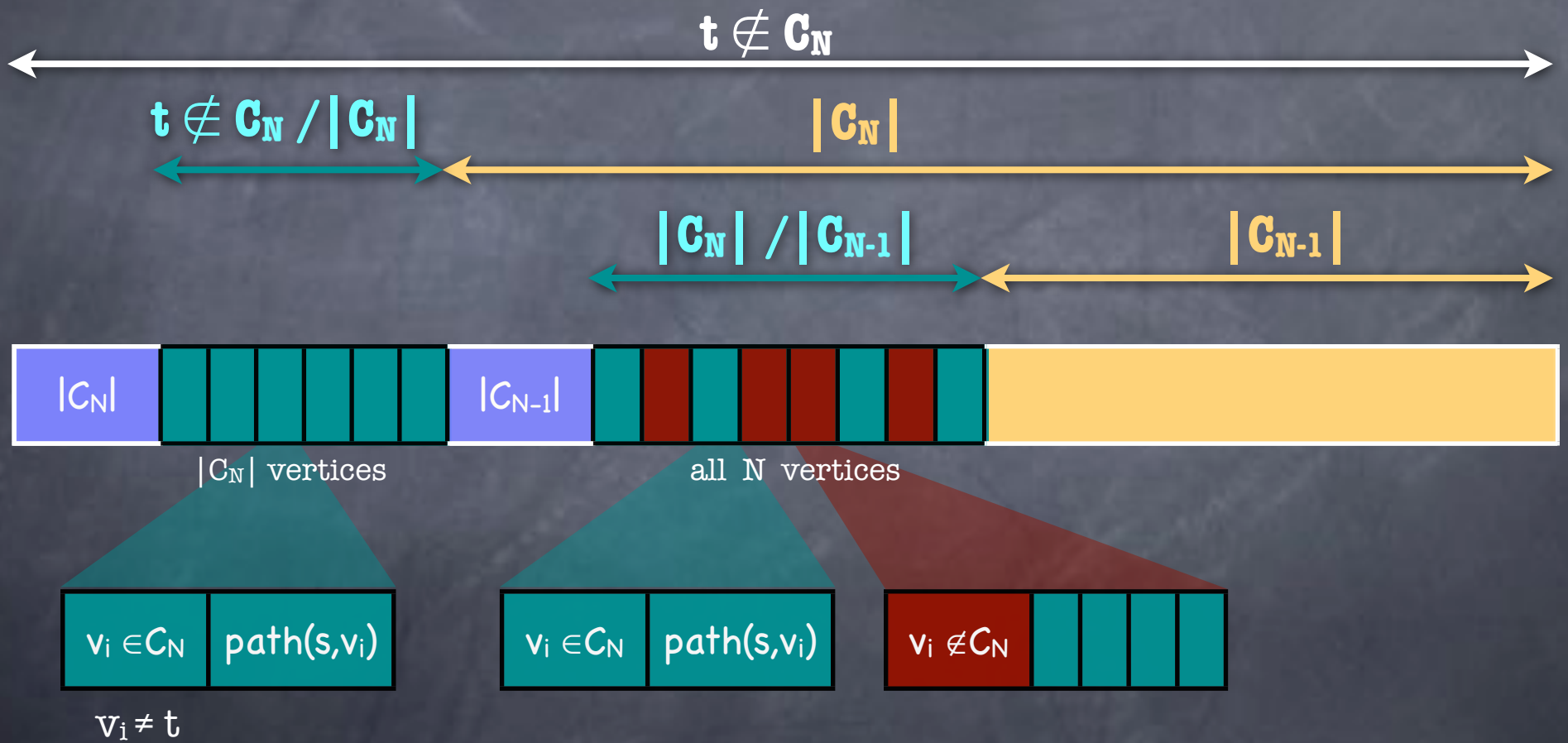
Certificate for $t \notin C_N$



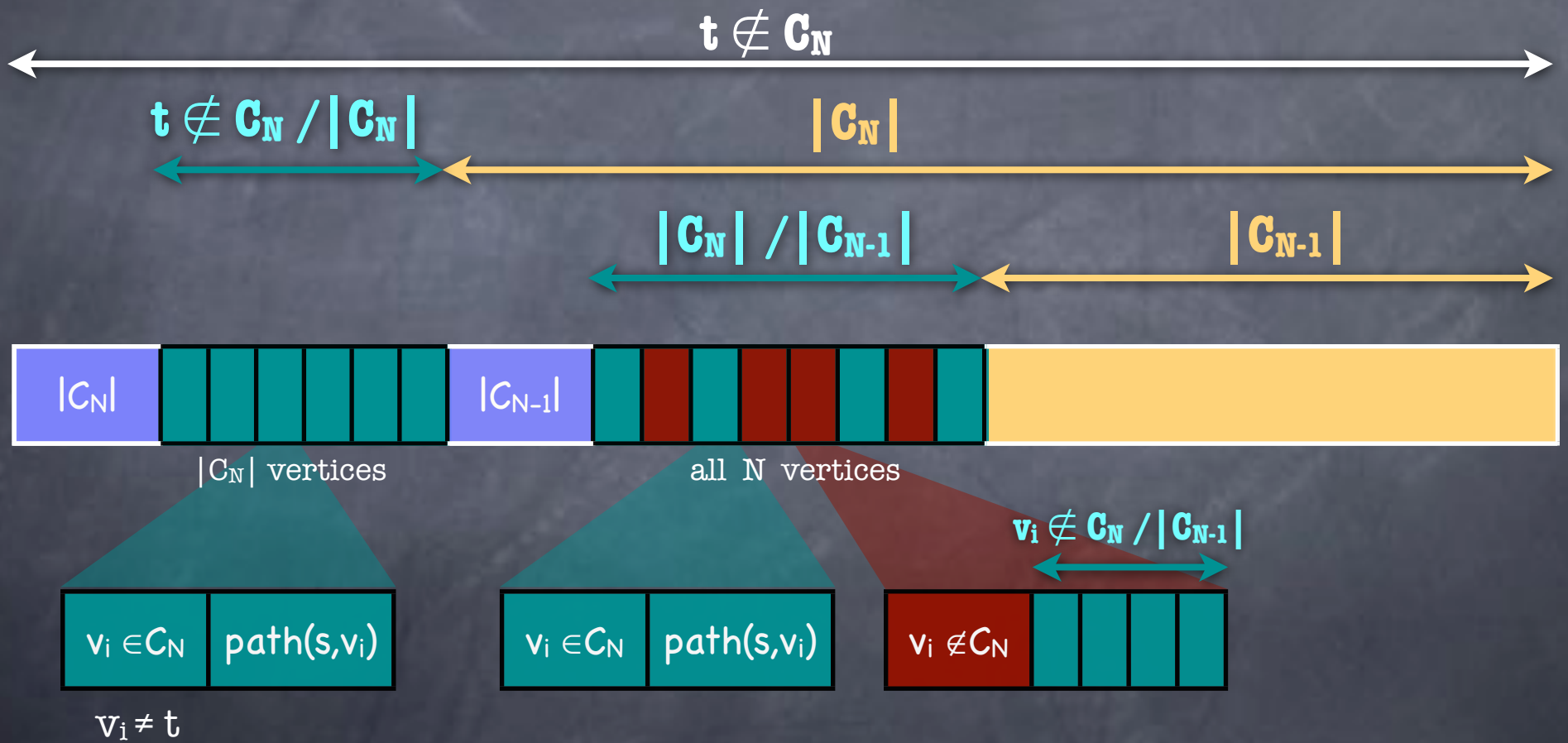
Certificate for $t \notin C_N$



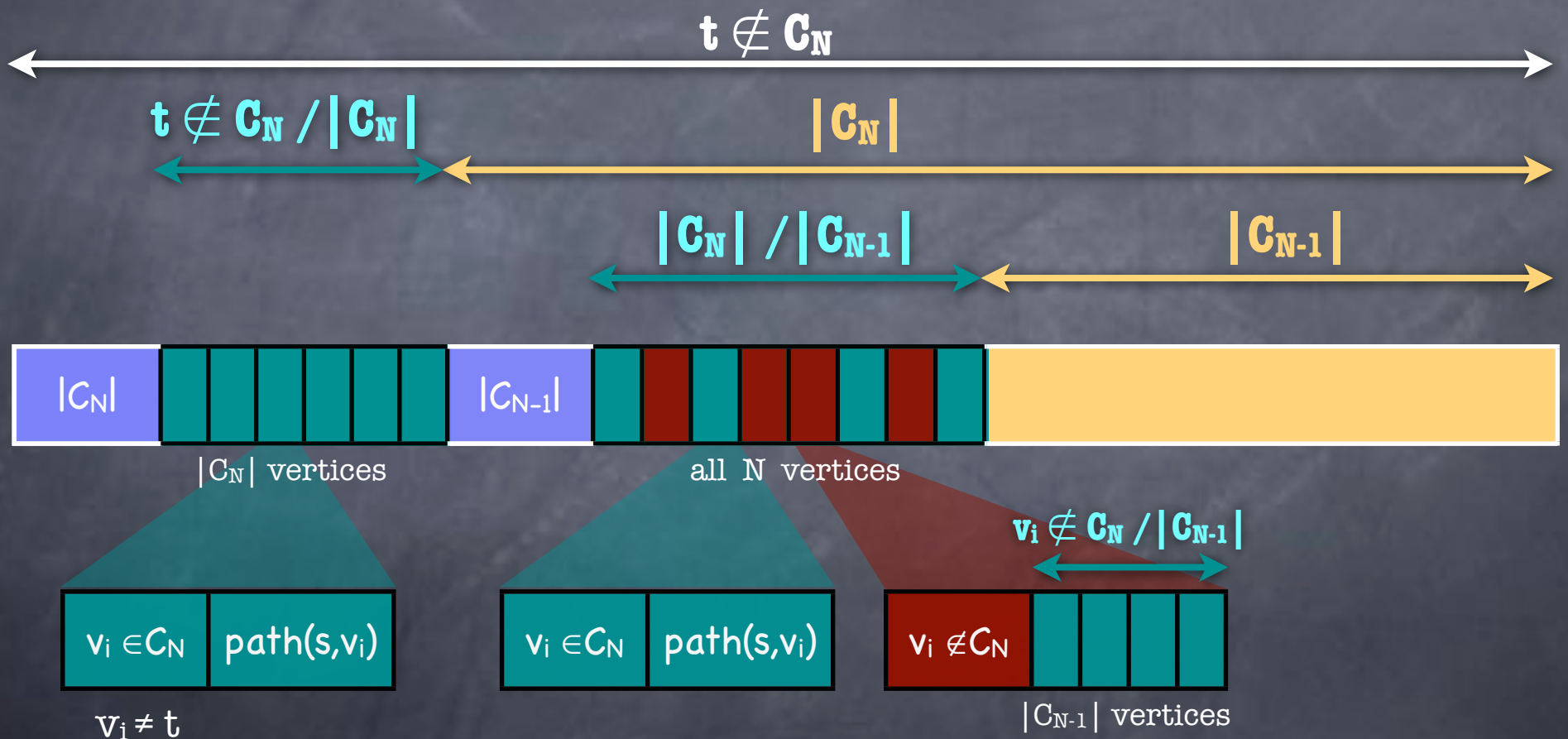
Certificate for $t \notin C_N$



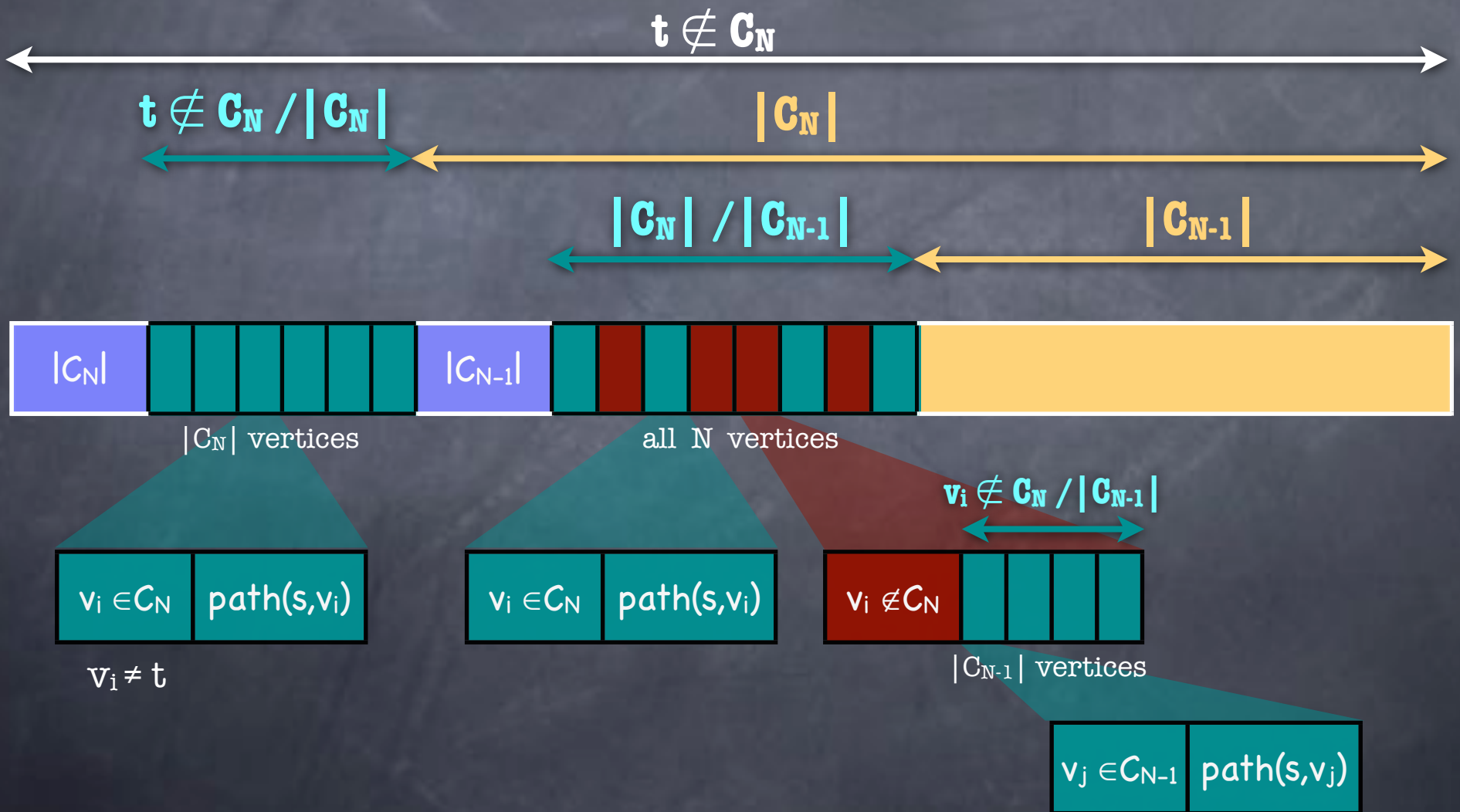
Certificate for $t \notin C_N$



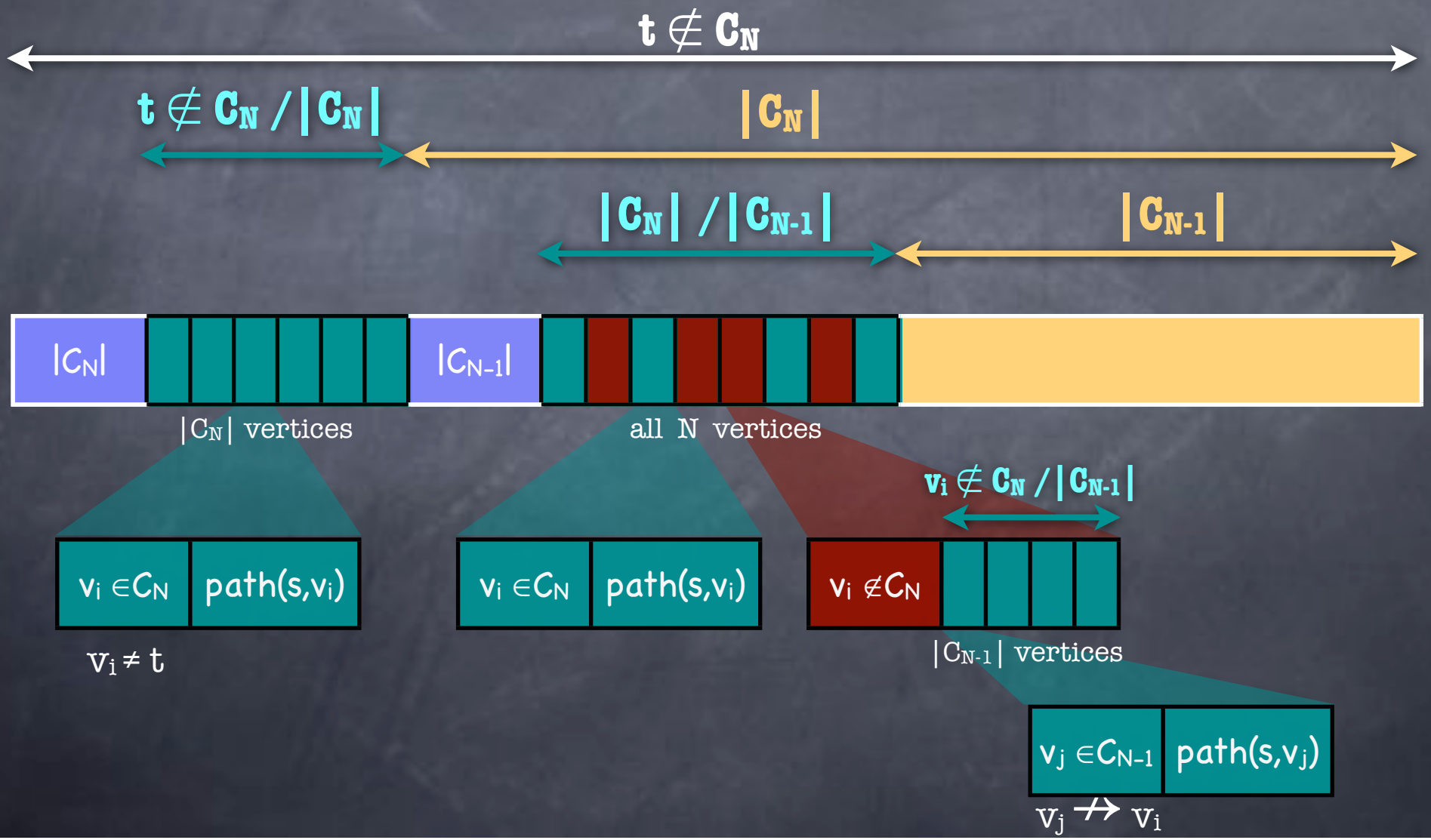
Certificate for $t \notin C_N$



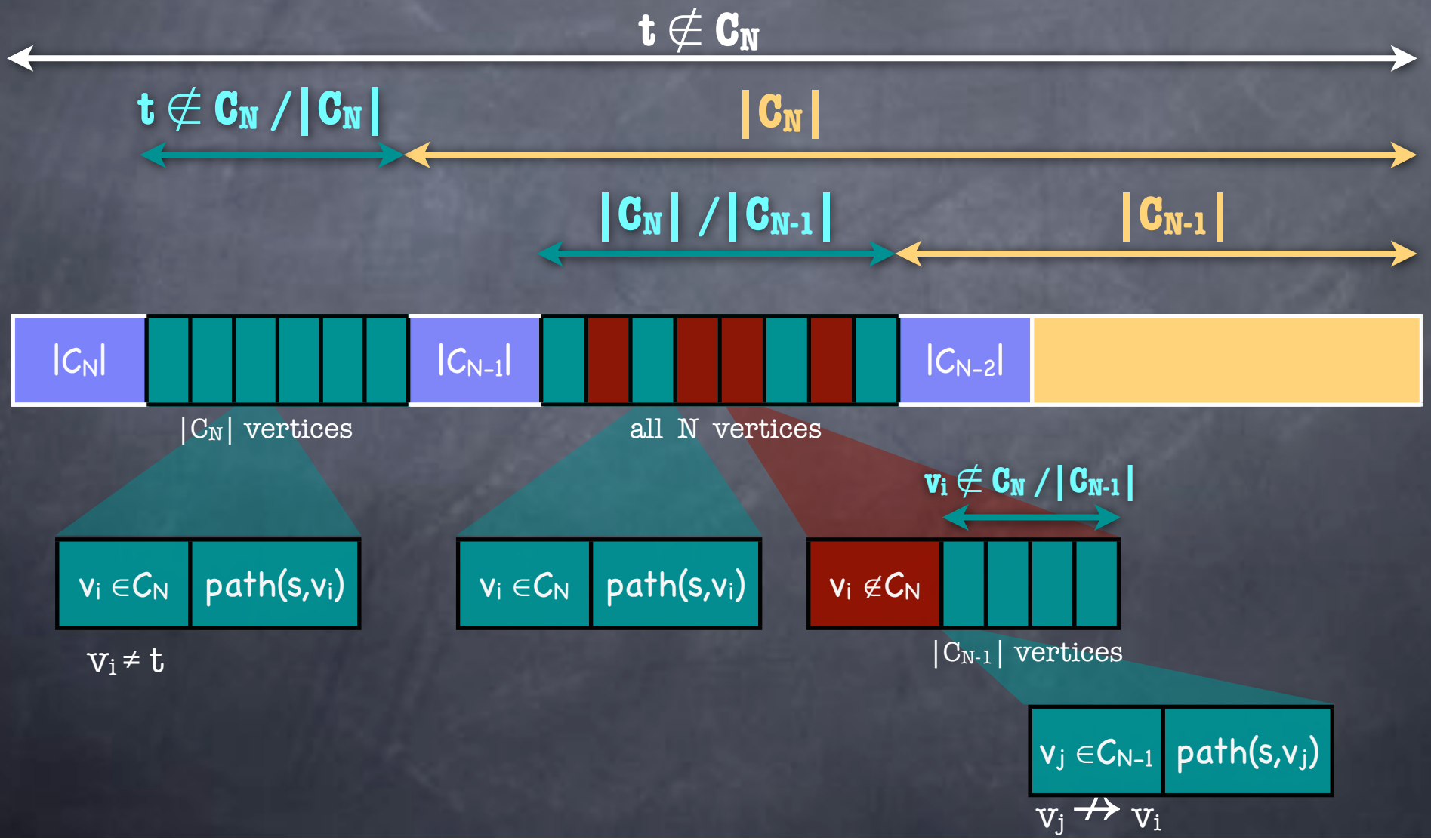
Certificate for $t \notin C_N$



Certificate for $t \notin C_N$



Certificate for $t \notin C_N$



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