Computational Complexity

Lecture 4
in which Diagonalization takes on itself,
and we enter Space Complexity
(But first Ladner's Theorem)

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  - Can show an NP language which is neither in P, nor NP complete (unless P = NP)

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  - Suppose f(x) = (x',pad), $|(x',pad)| \leq c|x|^c$. If |x'|>|x|/2, then $|pad| = |x'|^{H(|x'|)} > c|x|^c$ (for long enough x). So |x'| is at most |x|/2. Repeat to solve SAT

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- To define H s.t. H(n) bounded by const. iff SAT_H in P

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	✗	✗	✓	✗	✓	✗	✗	✓	✓
	✓	✓	✓	✓	✓	✓	✓	✗	✗
	✓	✓	✗	✓	✗	✓	✗	✓	✓
	✓	✓	✗	✗	✗	✗	✓	✗	✗
	✗	✗	✗	✗	✗	✗	✓	✓	✗
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	☑	☒	☒	☑	☒	☒	☒	☒	☑
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	✓	✓	✗	✓	✗	✓	✗	✓	✓
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	✓	✓	✓	✓	✓	✓	✓	✗	✗
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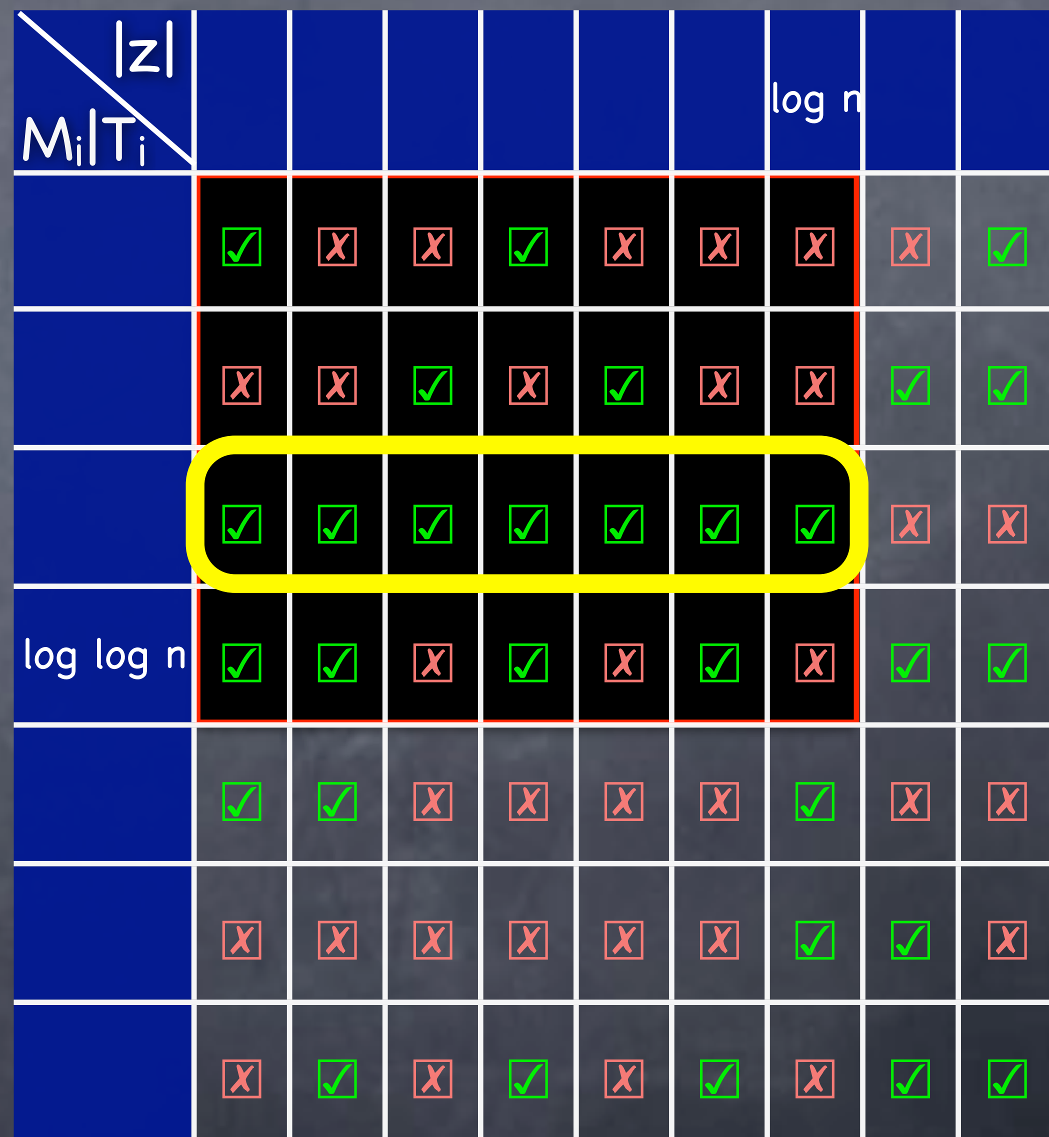

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	☒	☑	☒	☑	☒	☑	☒	☑	☑

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Equivalence carries over!

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  - No! Different in different worlds!
    - There exist languages A, B such that $P^A = NP^A$, but $P^B \neq NP^B$!

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  - EXPTM = { $(M,x,1^n)$ | TM represented by M accepts x within time 2^n }

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Building B and L, s.t. L in $NP^B \backslash P^B$

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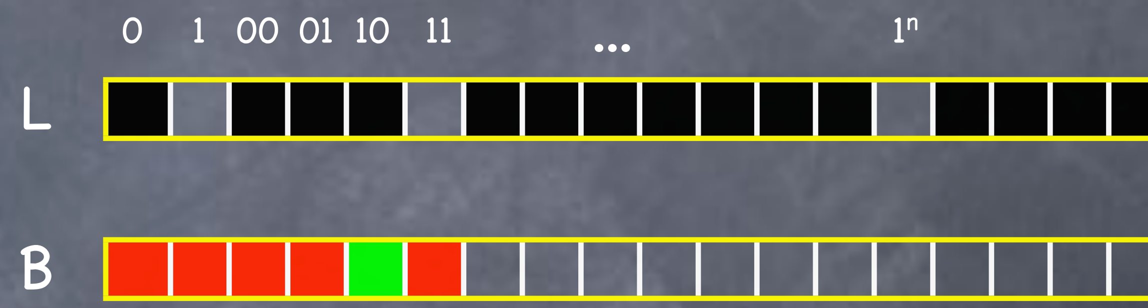

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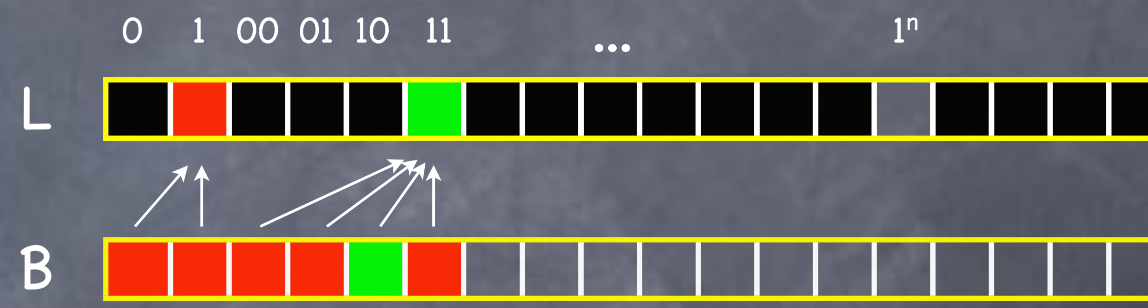

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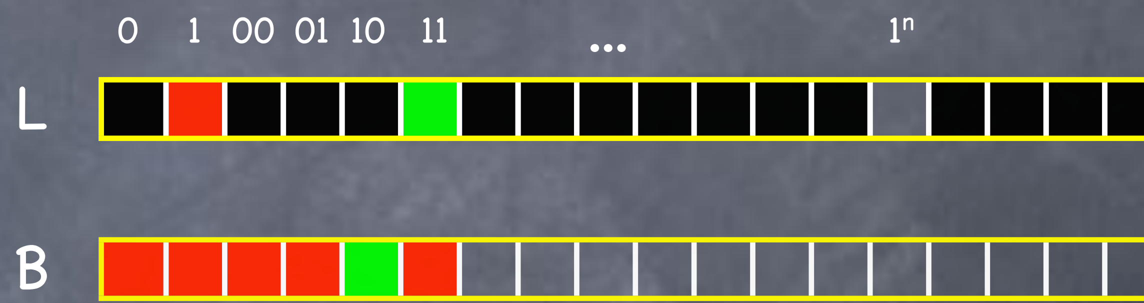

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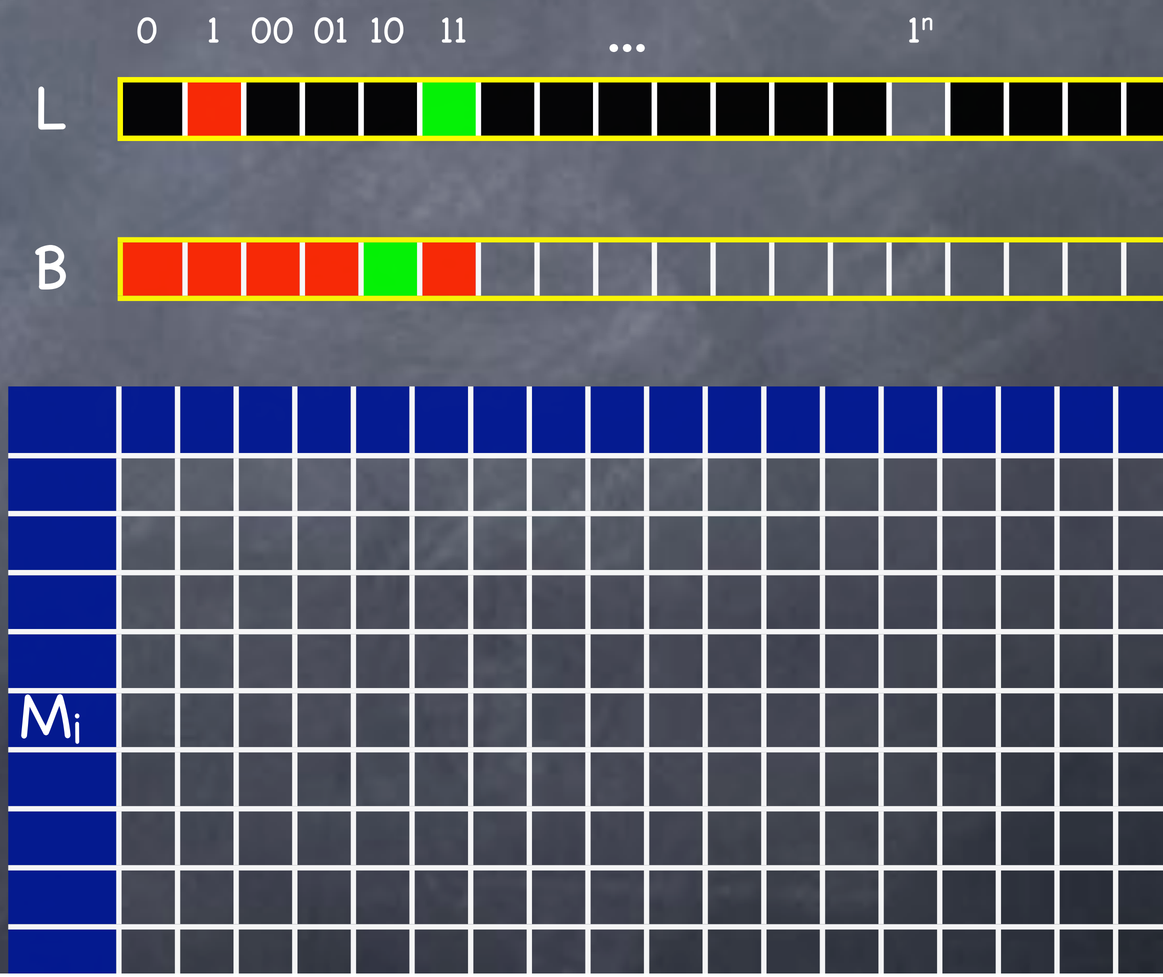

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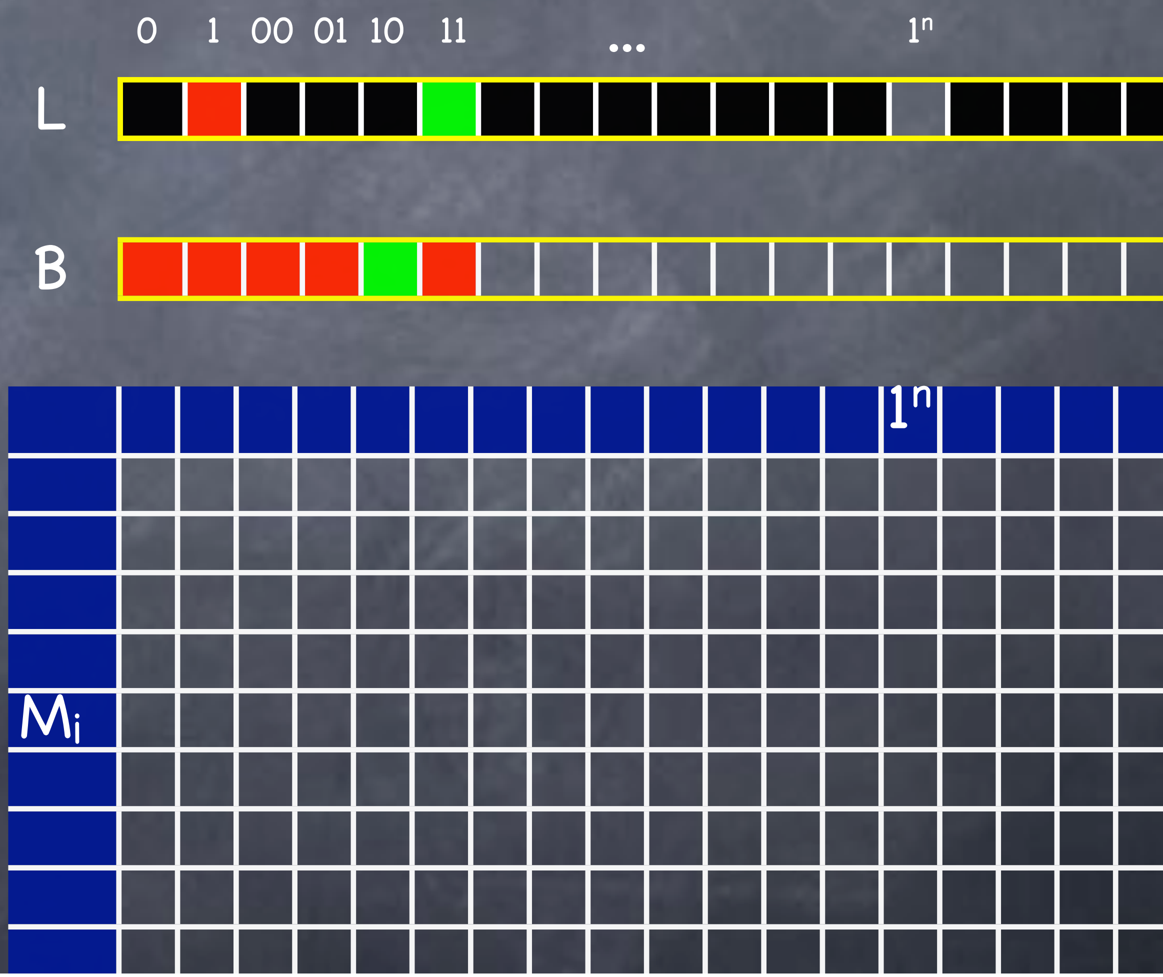

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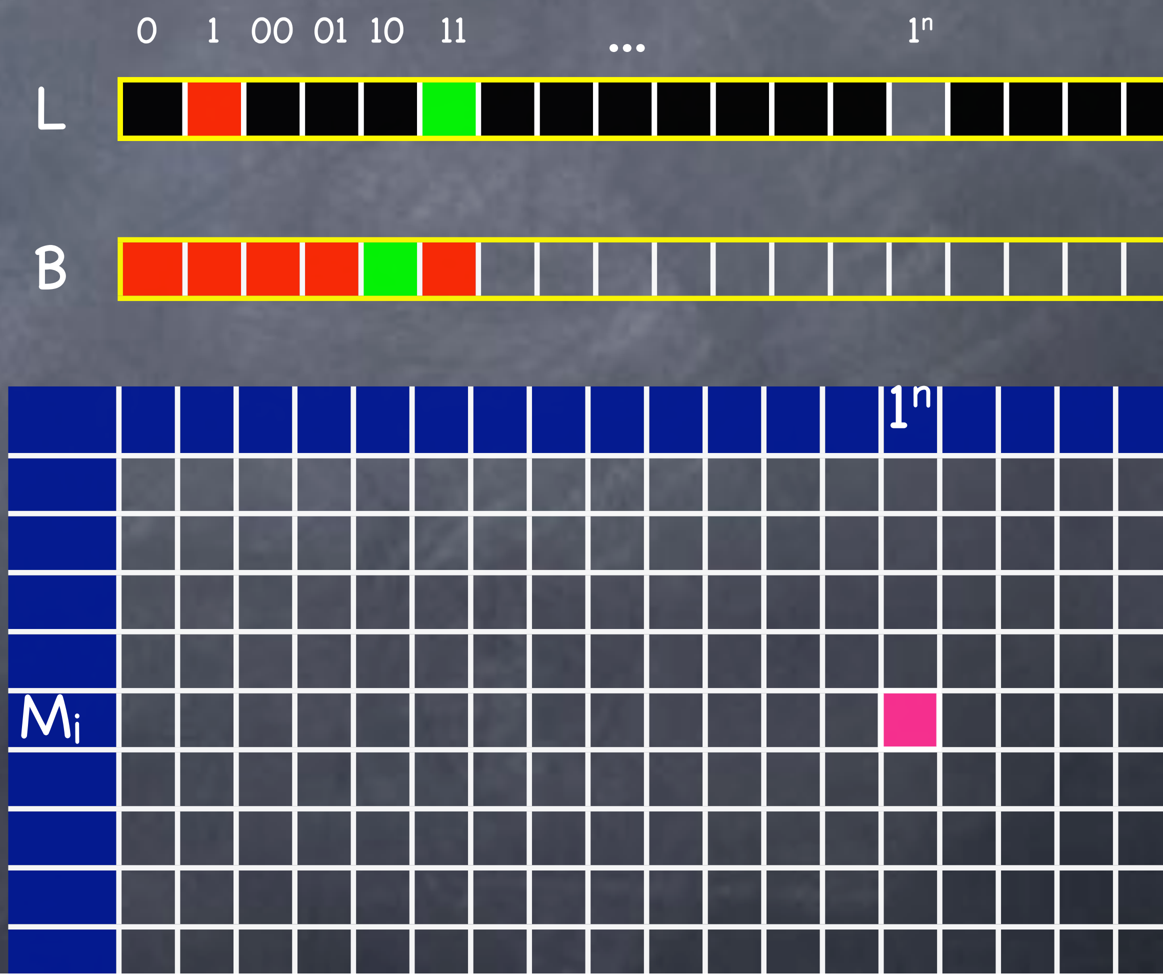

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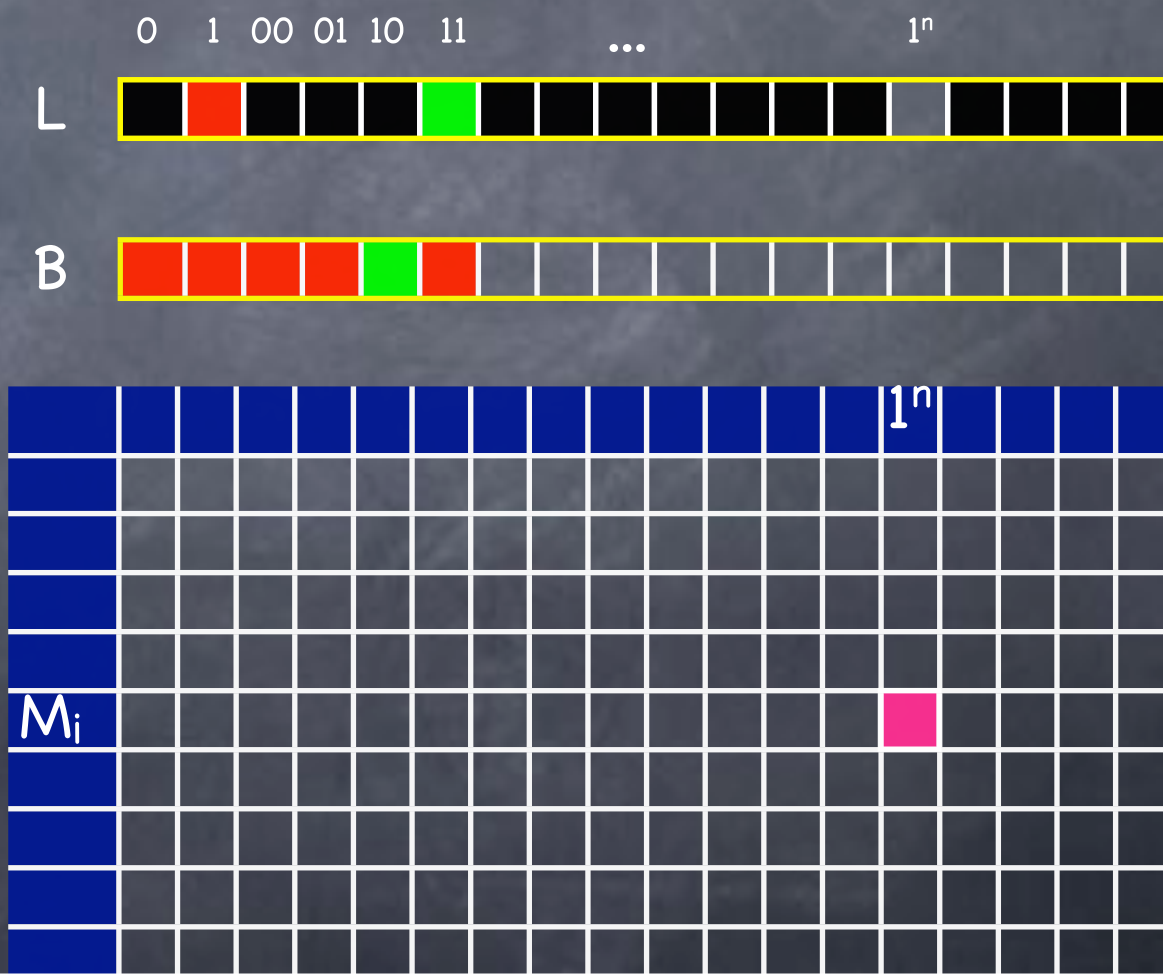

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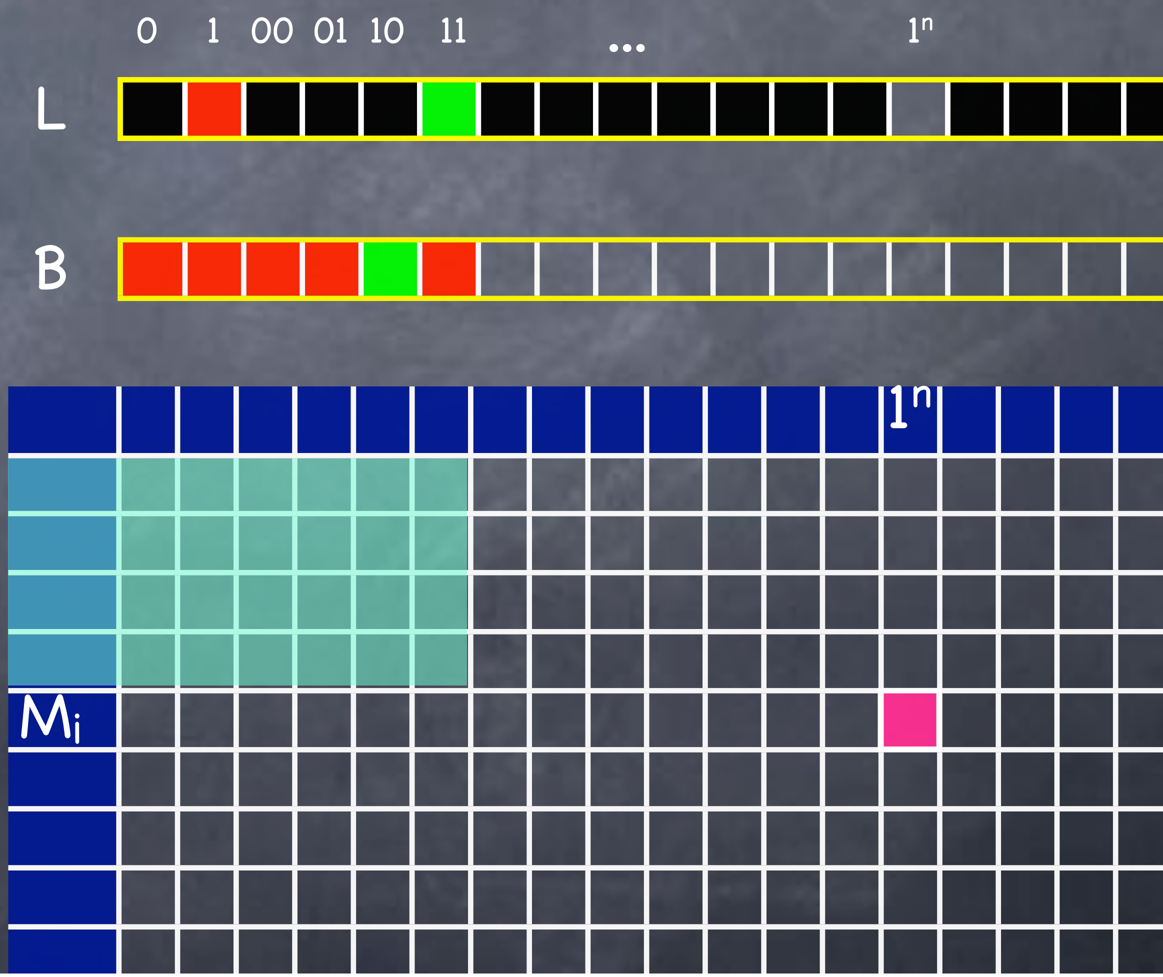

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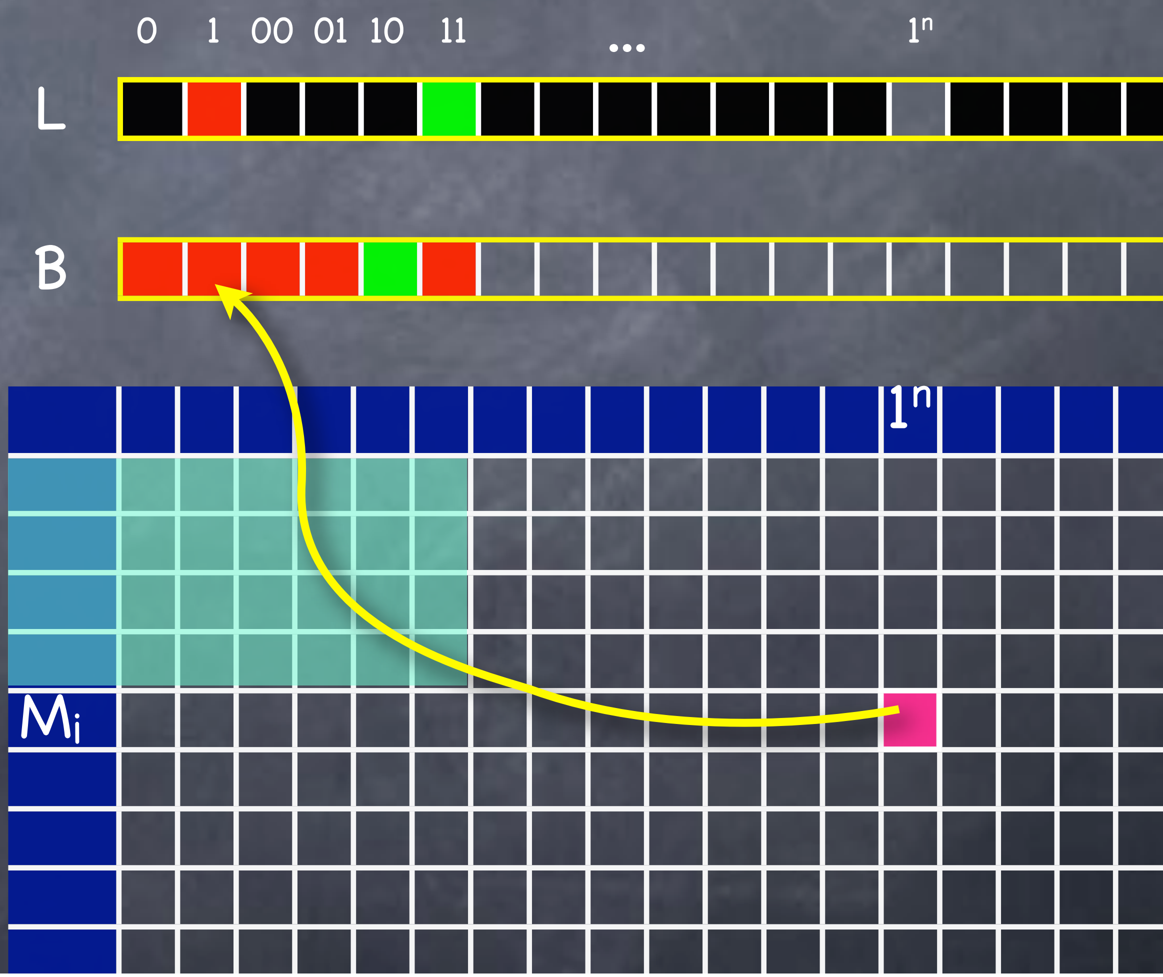

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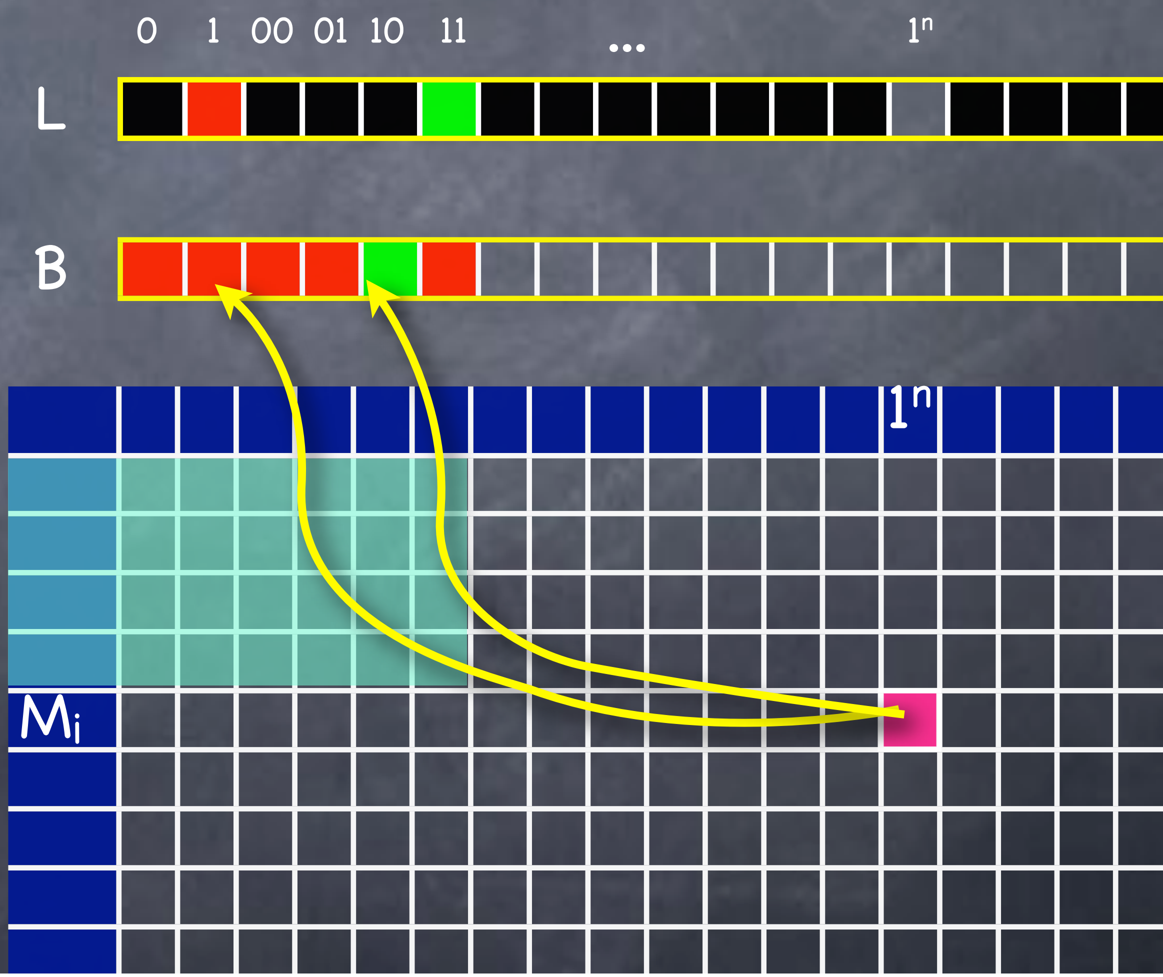

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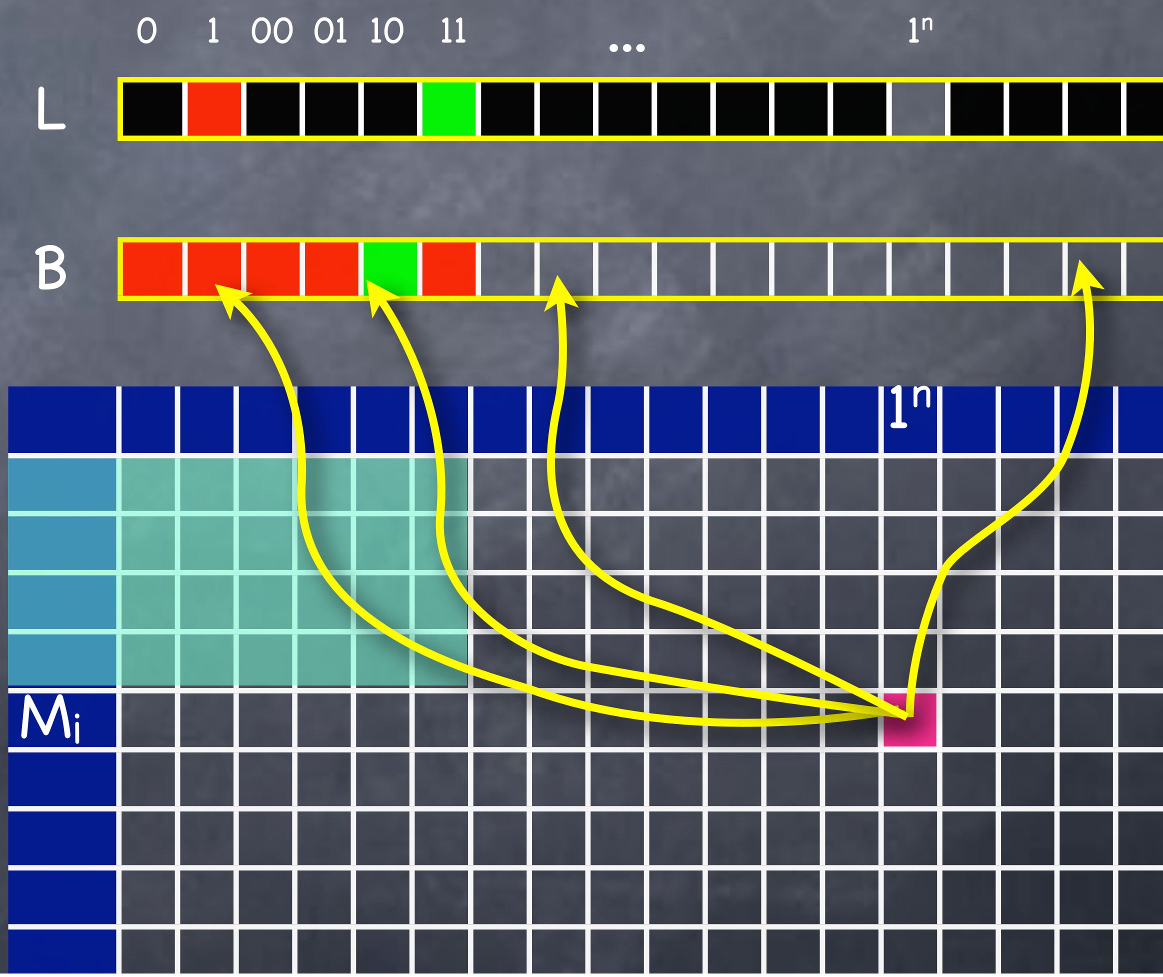

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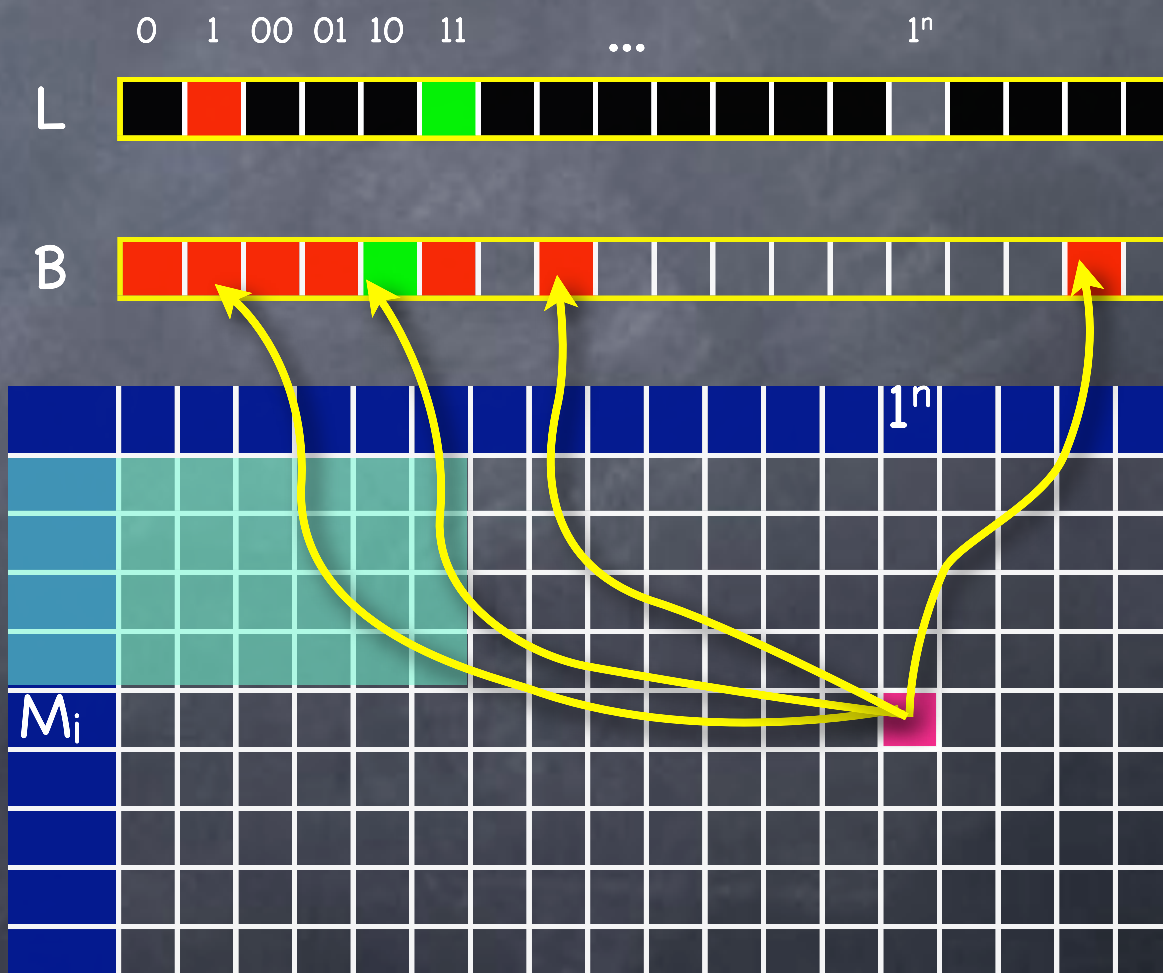

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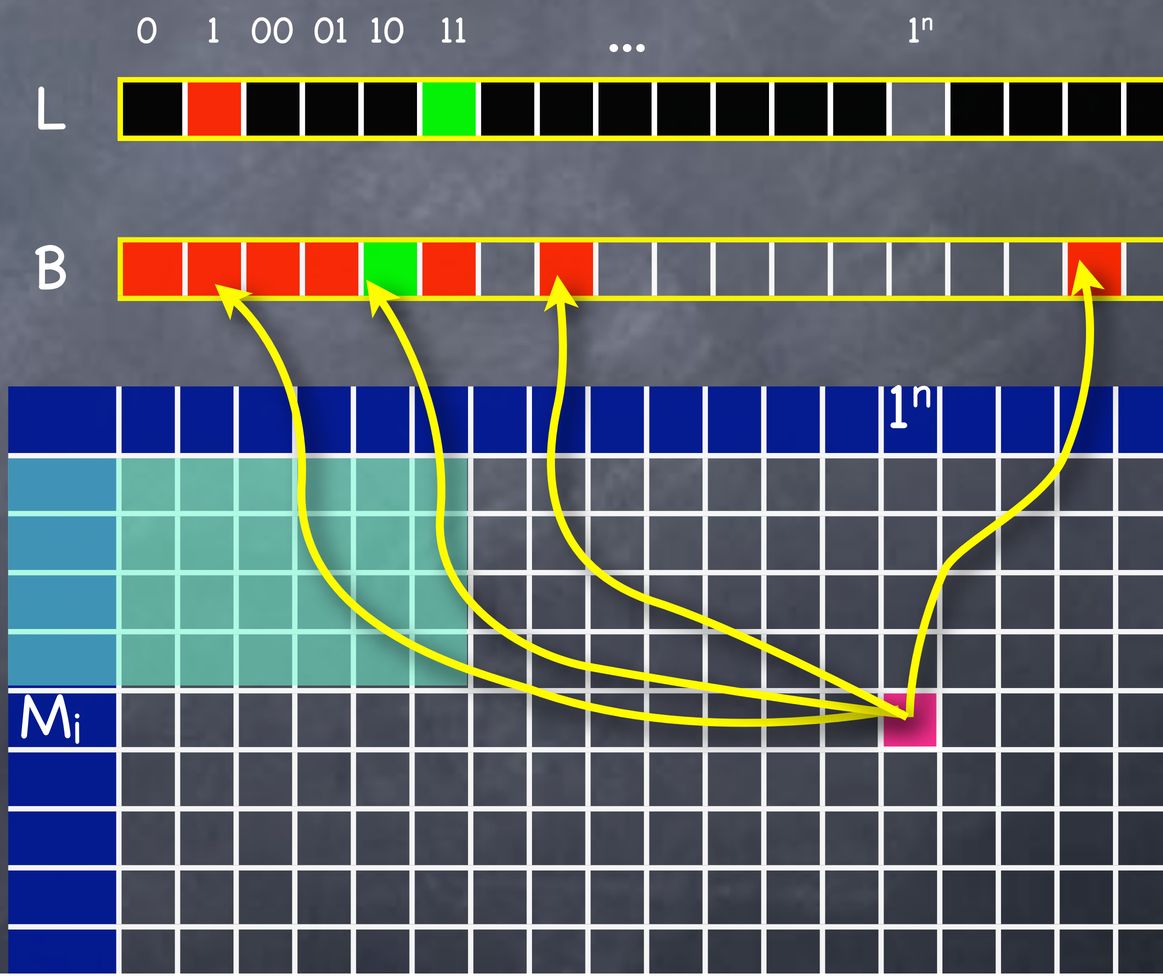

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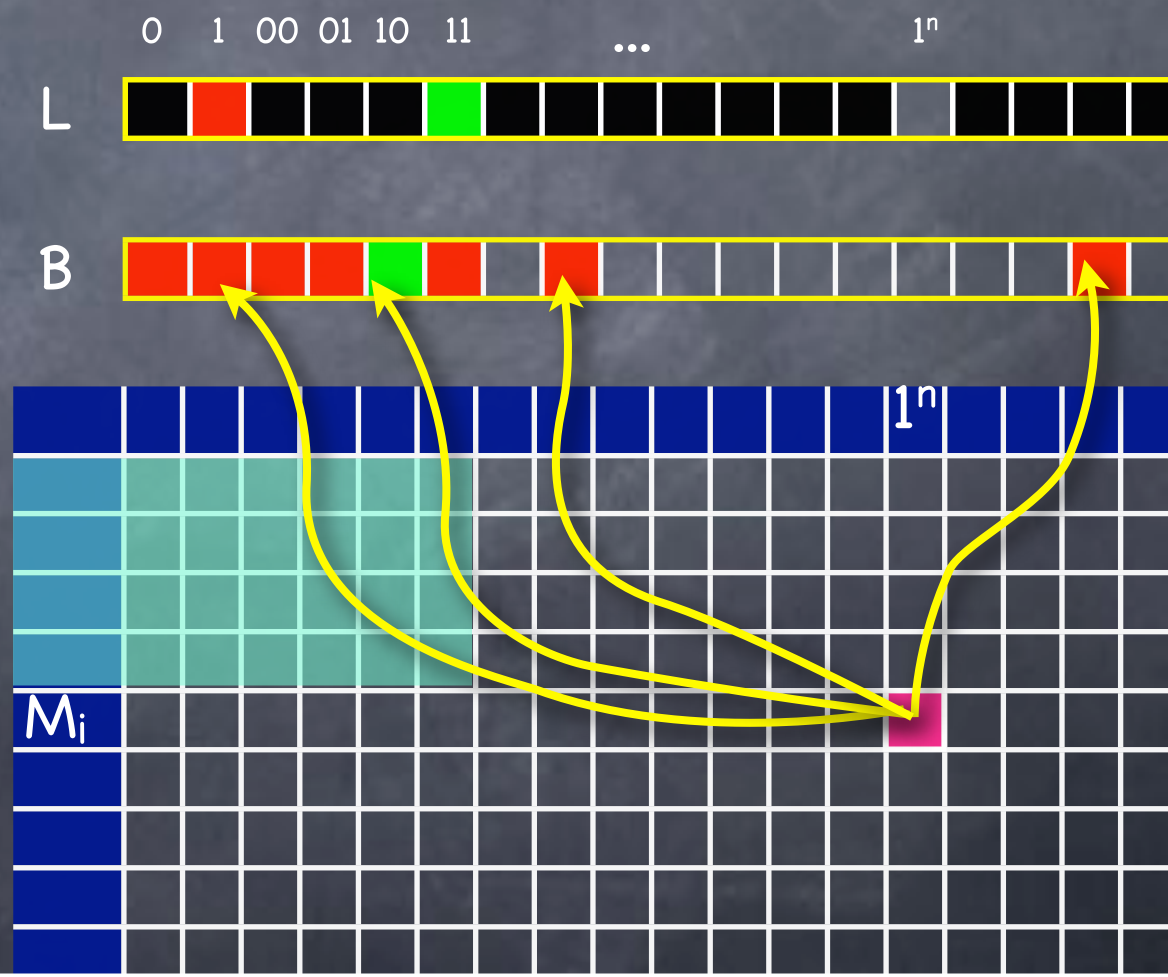

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  - e.g. of non-relativizing proof: that of Cook-Levin theorem

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    - Turns out, often a little memory can go a long way (if we can spare the time)

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- "L and NL are to space what P and NP are to time"

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      - No "delayed flip," because, as we will see later, NSPACE(O(S)) = co-NSPACE(O(S))!

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