# Communication Complexity

Lecture 23

Computing with remote inputs

# Communication Complexity

# Communication Complexity

- Setting

# Communication Complexity

- Setting
  - Alice wants to compute $f(x,y)$

# Communication Complexity

- Setting
  - Alice wants to compute f(x,y)
  - Alice is given only x. Her friend Bob gets y.

# Communication Complexity

- Setting
  - Alice wants to compute $f(x,y)$
  - Alice is given only $x$. Her friend Bob gets $y$.
  - Least amount of communication to achieve this

# Communication Complexity

- Setting
  - Alice wants to compute $f(x,y)$
  - Alice is given only $x$. Her friend Bob gets $y$.
  - Least amount of communication to achieve this
- Compare with decision tree complexity

# Communication Complexity

- Setting
  - Alice wants to compute f(x,y)
  - Alice is given only x. Her friend Bob gets y.
  - Least amount of communication to achieve this
- Compare with decision tree complexity
  - Trivial upper-bound of |x|

# Communication Complexity

- Setting
  - Alice wants to compute f(x,y)
  - Alice is given only x. Her friend Bob gets y.
  - Least amount of communication to achieve this
- Compare with decision tree complexity
  - Trivial upper-bound of |x|
- Interested in proving lower bounds for various f

# Examples

# Examples

- $\text{PARITY}(x,y) = \bigoplus_i (x_i \oplus y_i)$

# Examples

- PARITY(x,y) = $\bigoplus_i (x_i \oplus y_i)$
  - CC(PARITY) = 1

# Examples

- PARITY(x,y) = $\bigoplus_i (x_i \oplus y_i)$
  - CC(PARITY) = 1
- EQ(x,y) = 1 iff x=y

# Examples

- PARITY(x,y) = $\bigoplus_i (x_i \oplus y_i)$
  - CC(PARITY) = 1
- EQ(x,y) = 1 iff x=y
  - Lower-bound?

# Examples

- PARITY(x,y) = $\bigoplus_i (x_i \oplus y_i)$
  - CC(PARITY) = 1
- EQ(x,y) = 1 iff x=y
  - Lower-bound?
- DISJ(x,y)=1 if $x \wedge y = 0^n$

# Motivation

# Motivation

- Distributed computing

# Motivation

- Distributed computing
- Lower-bounds for Circuit complexity

# Motivation

- Distributed computing
- Lower-bounds for Circuit complexity
  - Amount of communication across a cut in the circuit

# Motivation

- Distributed computing
- Lower-bounds for Circuit complexity
    - Amount of communication across a cut in the circuit
- Proving optimality of algorithms and data-structures

# Protocol

# Protocol

- We'll consider deterministic protocols

# Protocol

- We'll consider deterministic protocols
- Fixed number of rounds (Alice to Bob, then Bob to Alice), each party sends a fixed number of bits in each round

# Protocol

- We'll consider deterministic protocols
- Fixed number of rounds (Alice to Bob, then Bob to Alice), each party sends a fixed number of bits in each round
  - Can even consider protocol to have Alice and Bob alternately exchanging single bits (since not considering number of rounds)

# Protocol

- We'll consider deterministic protocols
- Fixed number of rounds (Alice to Bob, then Bob to Alice), each party sends a fixed number of bits in each round
  - Can even consider protocol to have Alice and Bob alternately exchanging single bits (since not considering number of rounds)
    - At most doubles the communication complexity

# Protocol Execution

# Protocol Execution

- $i^{th}$ message from Alice is a function of her input and previous messages

# Protocol Execution

- $i^{th}$ message from Alice is a function of her input and previous messages
- Her output is a function of the final "transcript" and her own input (her "view")

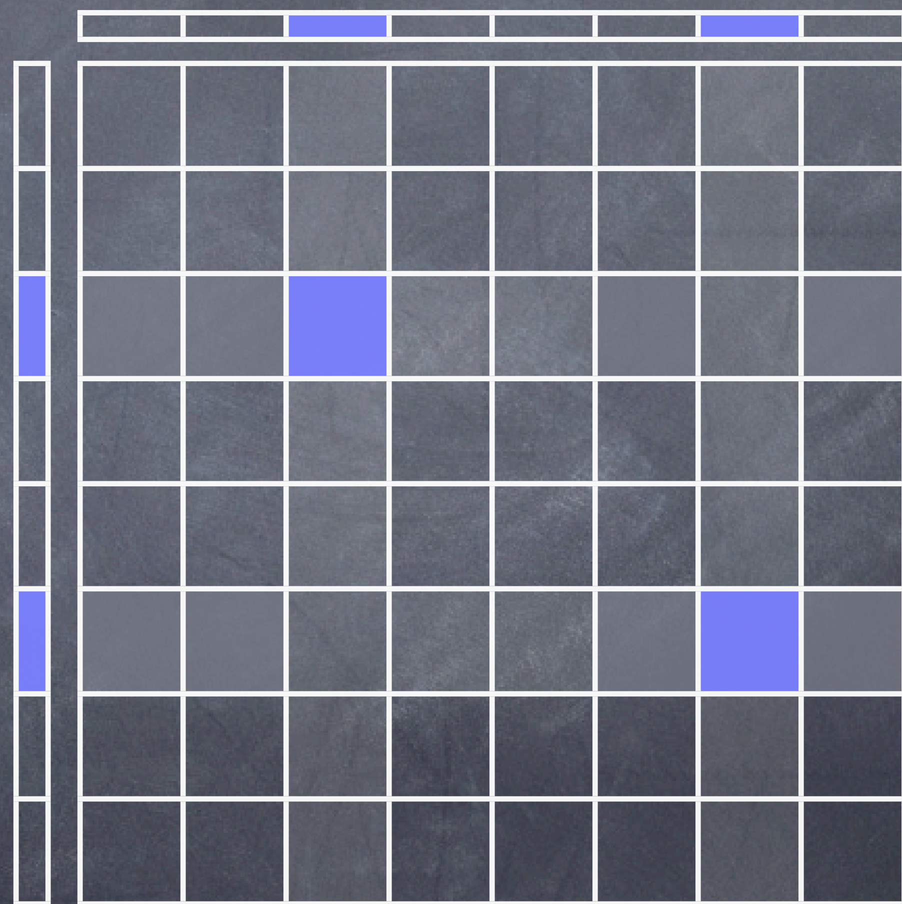
# Protocol Execution

- $i^{th}$ message from Alice is a function of her input and previous messages
- Her output is a function of the final "transcript" and her own input (her "view")
  - Similarly for Bob. His view = transcript + his input

# Protocol Execution

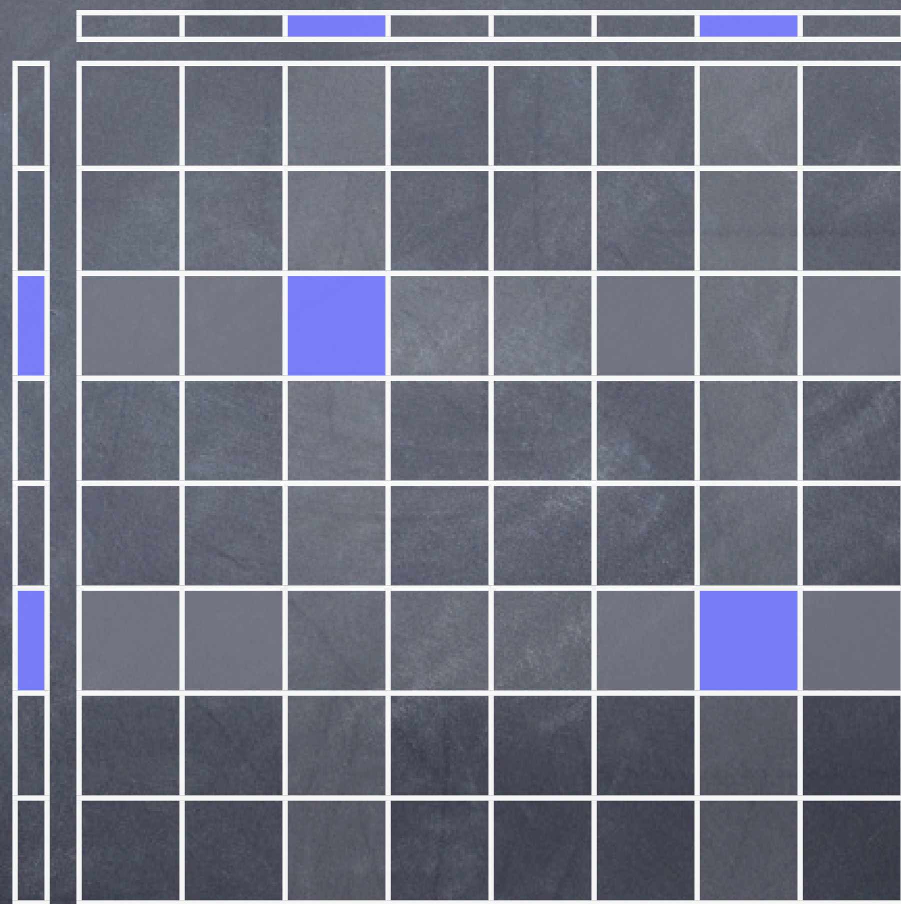
- $i^{th}$ message from Alice is a function of her input and previous messages
- Her output is a function of the final "transcript" and her own input (her "view")
    - Similarly for Bob. His view = transcript + his input
- #transcripts $\leq 2^{CC}$. i.e. $CC \geq \log(\#transcripts)$

# Transcript Table



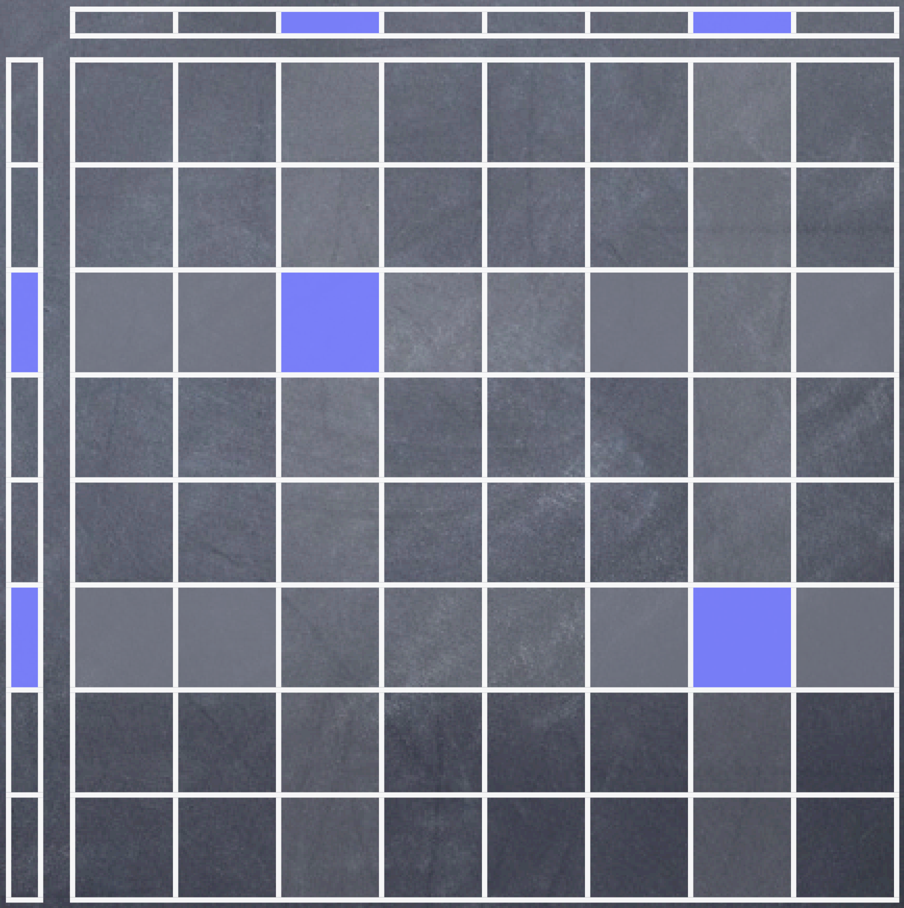
# Transcript Table

- Consider the transcript table



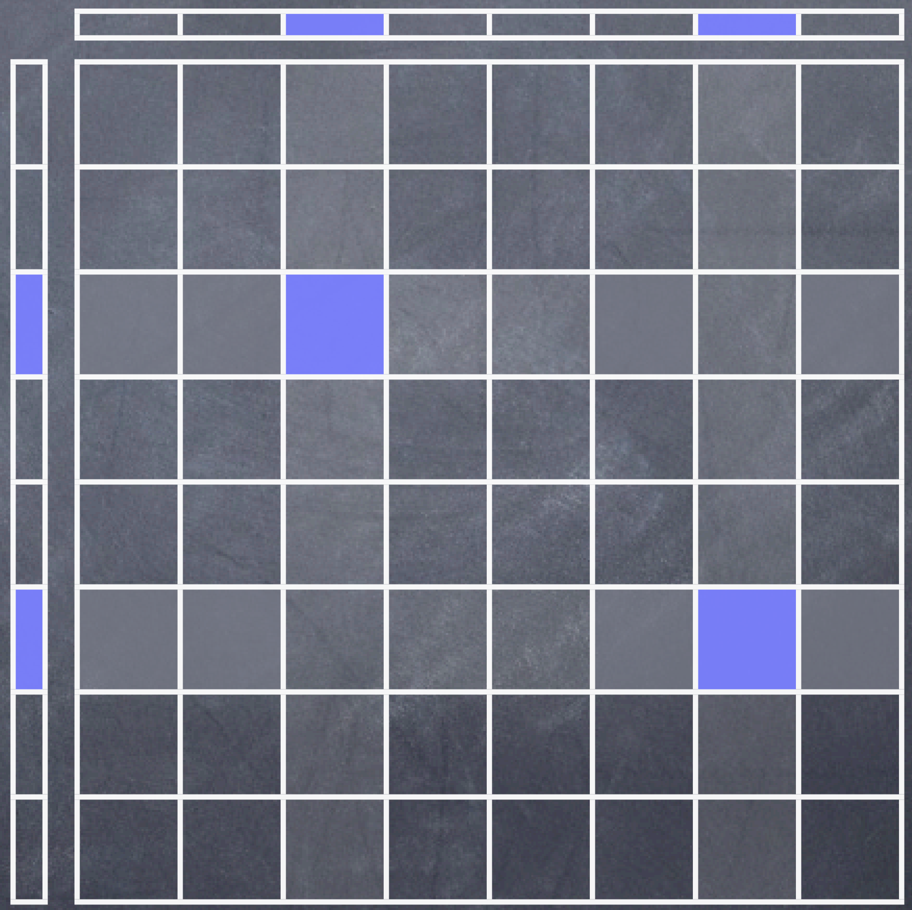
# Transcript Table

- Consider the transcript table
  - If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript



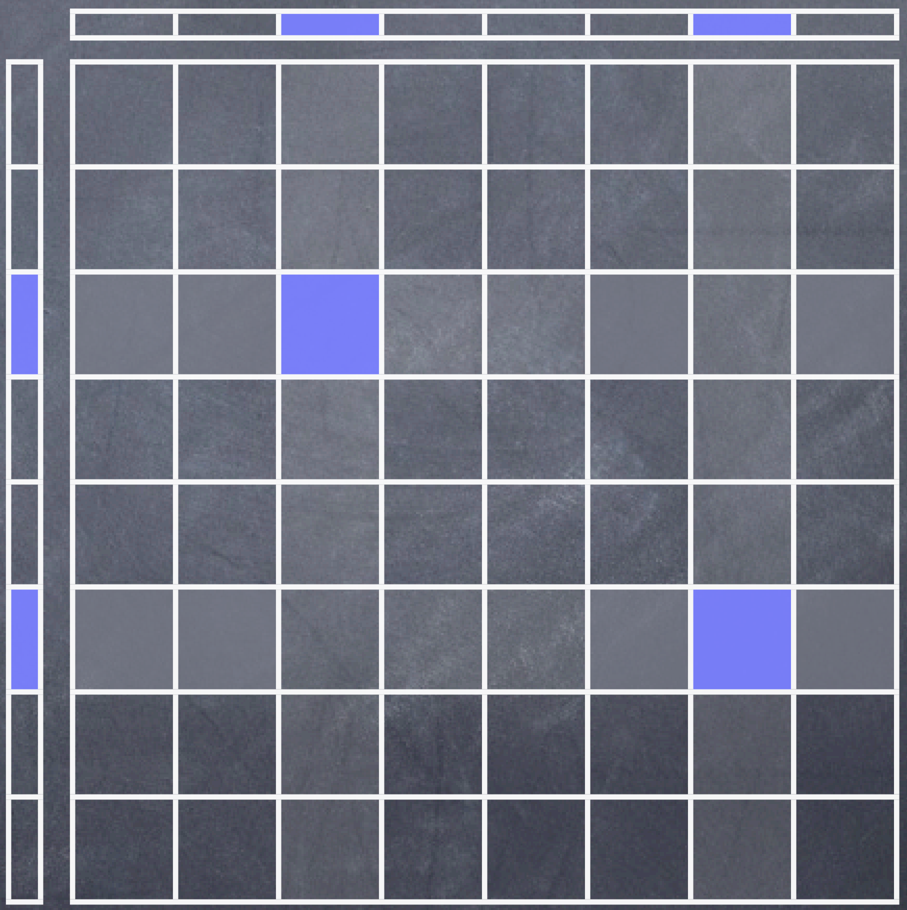
# Transcript Table

- Consider the transcript table
  - If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript
    - Then same transcript on $(a_1,b_2)$ also!

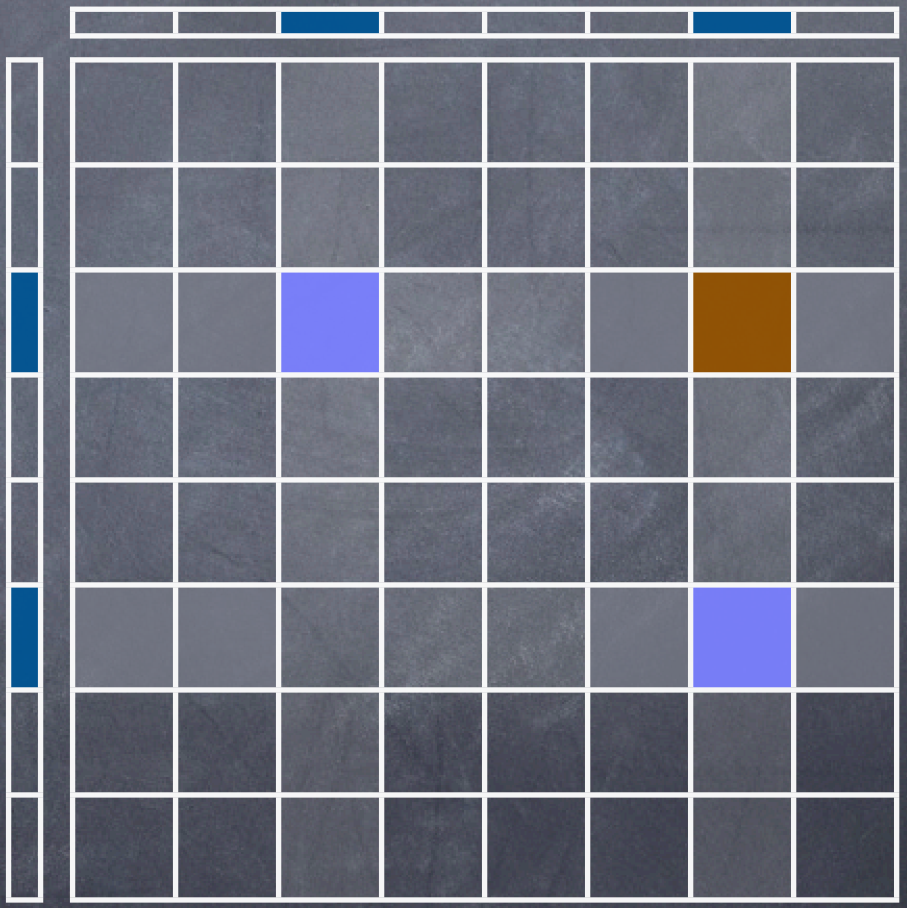


# Transcript Table

- Consider the transcript table
  - If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript
    - Then same transcript on $(a_1,b_2)$ also!
    - Alice and Bob never realize the difference through out the protocol

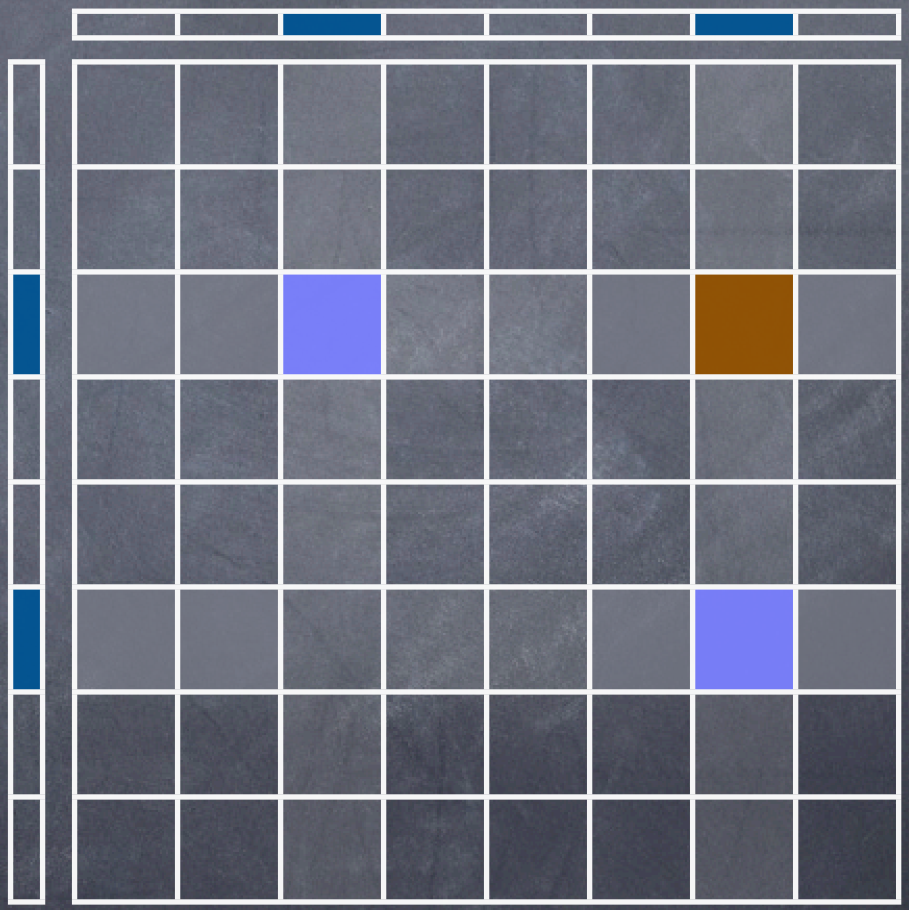


# Fooling Set



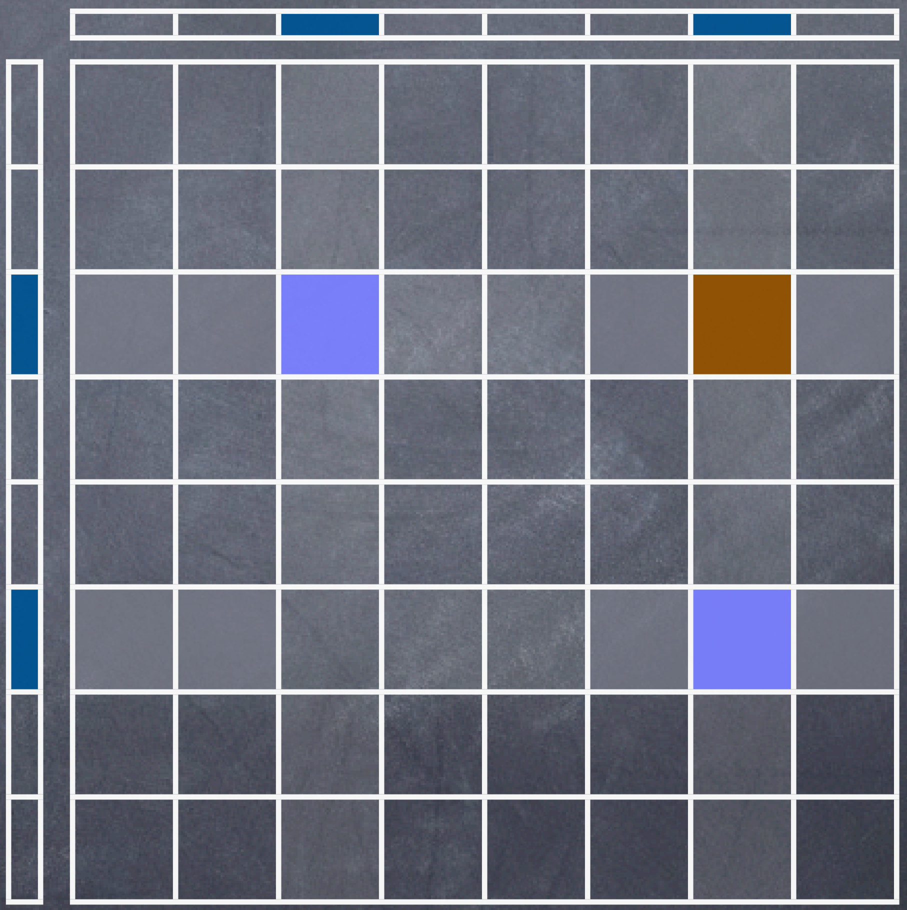
# Fooling Set

- If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript, then same transcript on $(a_1,b_2)$ also



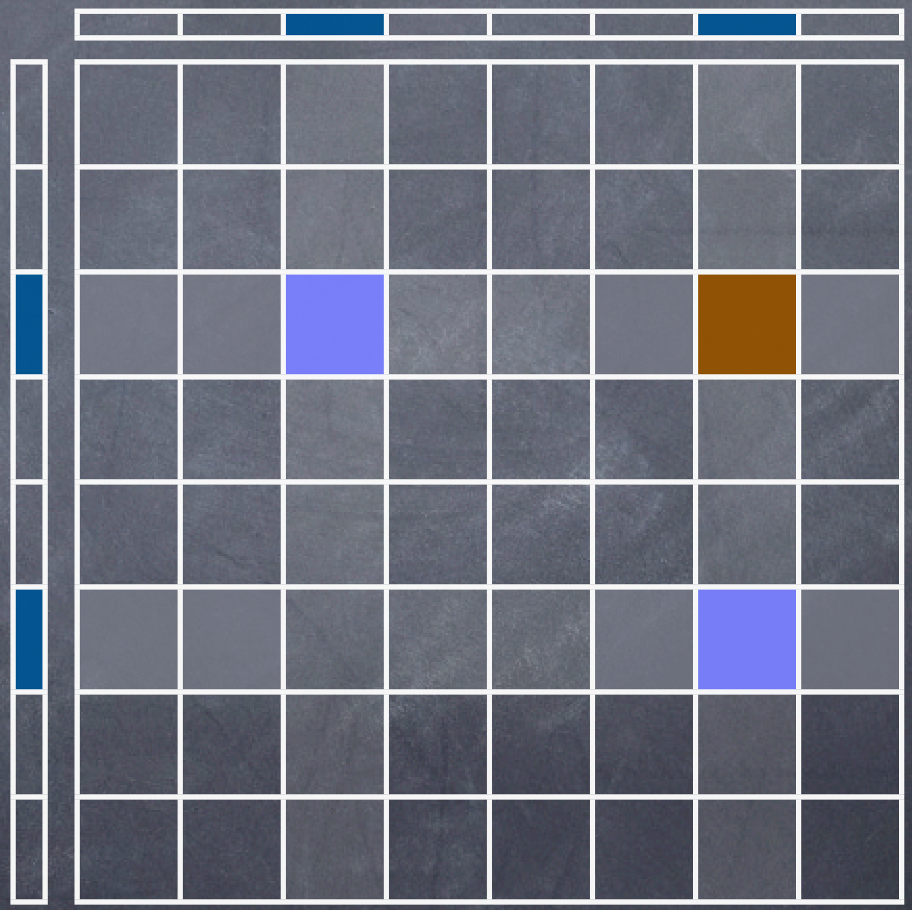
# Fooling Set

- If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript, then same transcript on $(a_1,b_2)$ also
- Showing a set S of input-pairs that must have distinct transcripts



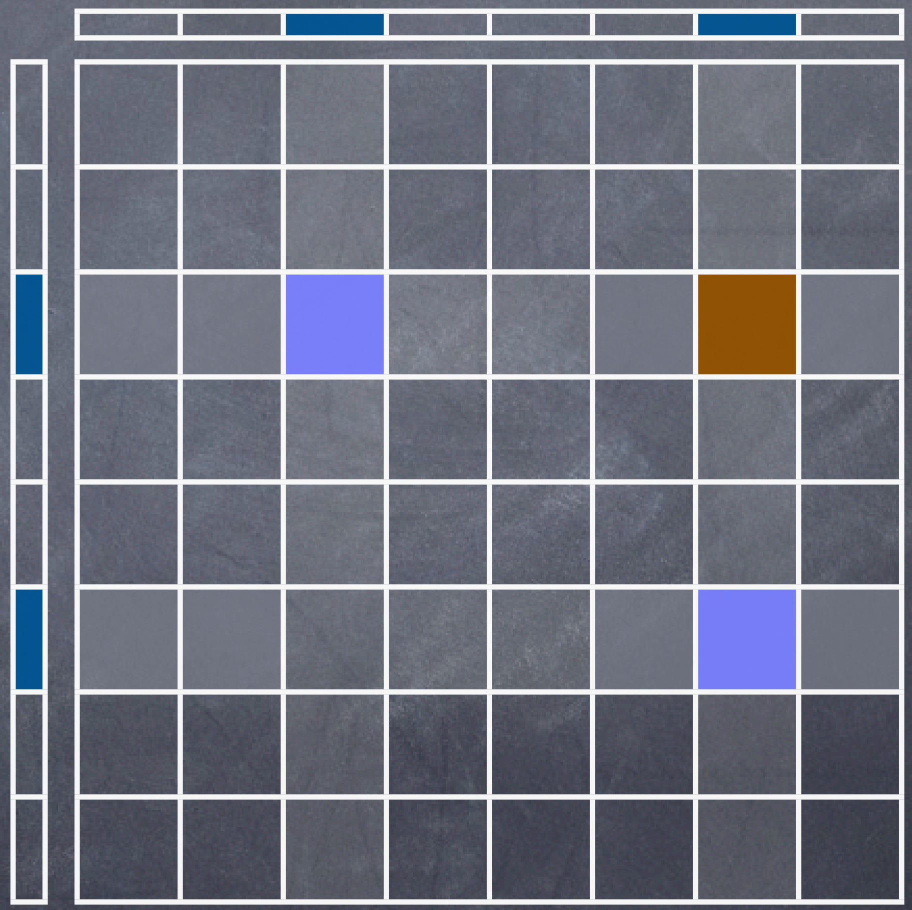
# Fooling Set

- If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript, then same transcript on $(a_1,b_2)$ also
- Showing a set S of input-pairs that must have distinct transcripts
    - All pairs have same output



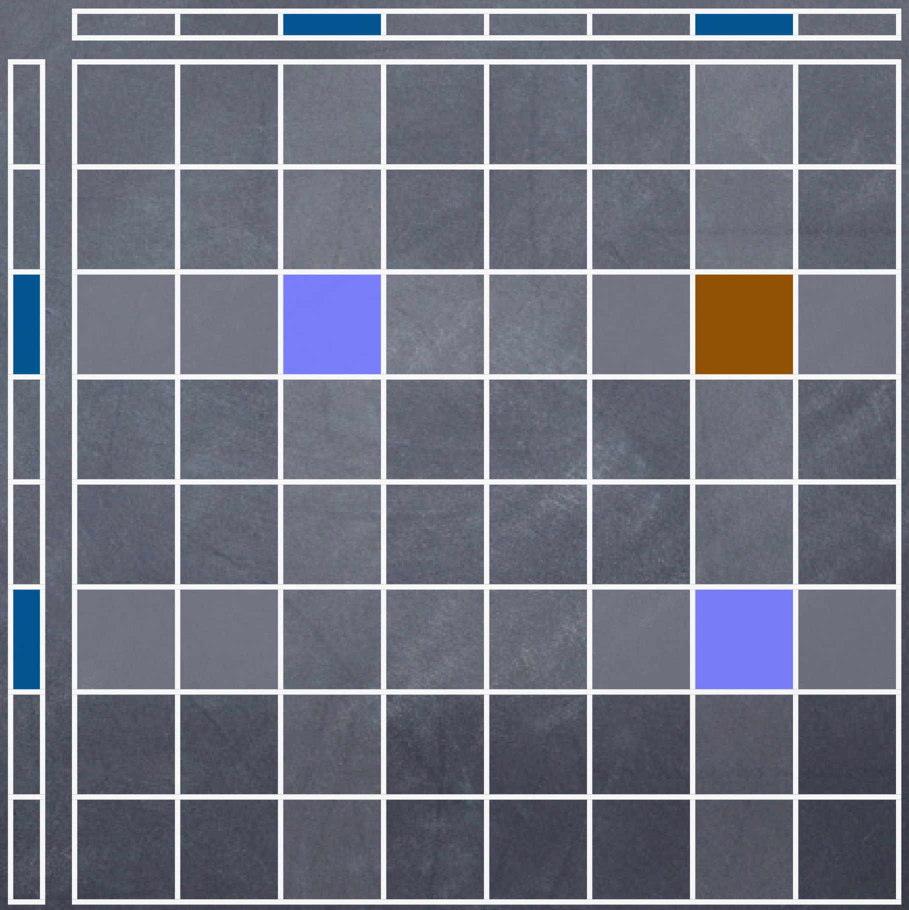
# Fooling Set

- If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript, then same transcript on $(a_1,b_2)$ also
- Showing a set S of input-pairs that must have distinct transcripts
  - All pairs have same output
  - "Cross" of no two pairs has the same output

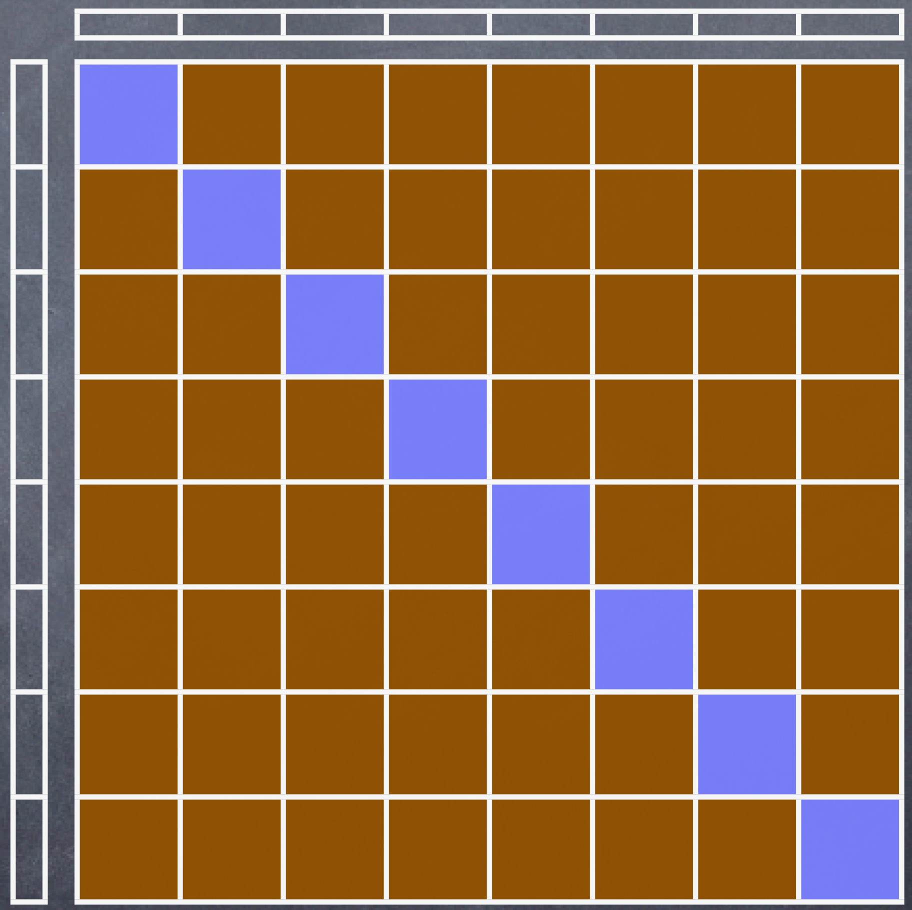


# Fooling Set

- If on $(a_1,b_1)$ and $(a_2,b_2)$ same transcript, then same transcript on $(a_1,b_2)$ also
- Showing a set S of input-pairs that must have distinct transcripts
  - All pairs have same output
  - "Cross" of no two pairs has the same output
- If S is a set of such pairs, $CC \geq \log(|S|)$

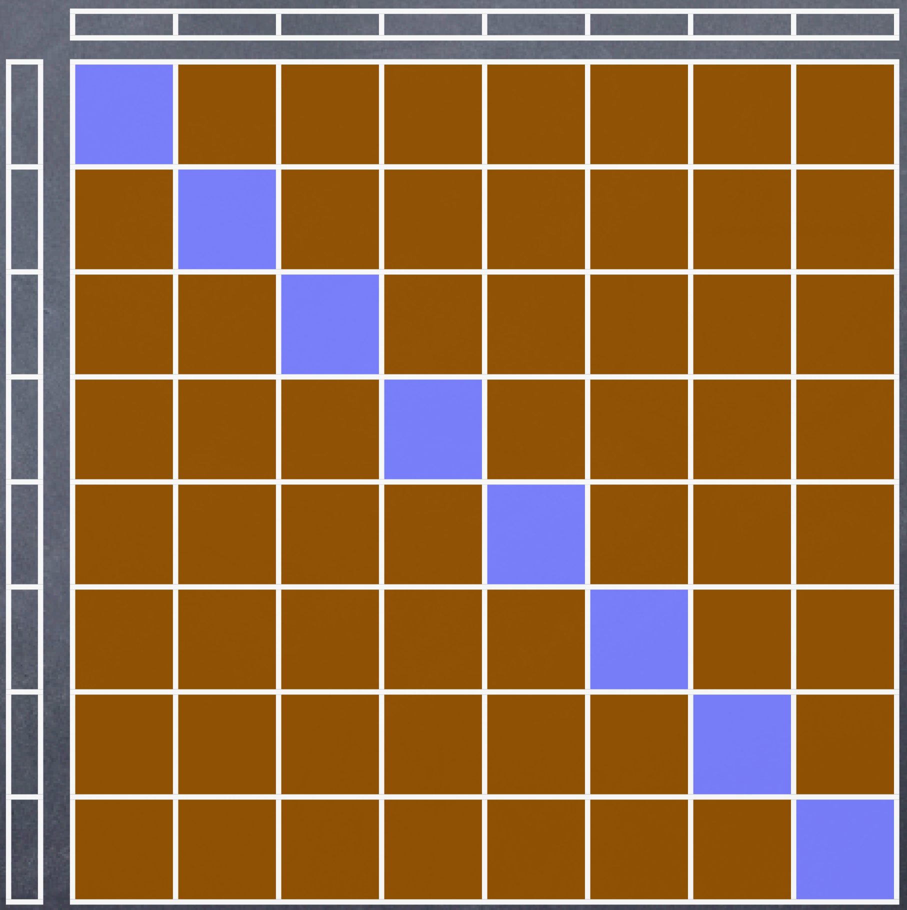


# Fooling Set for EQ



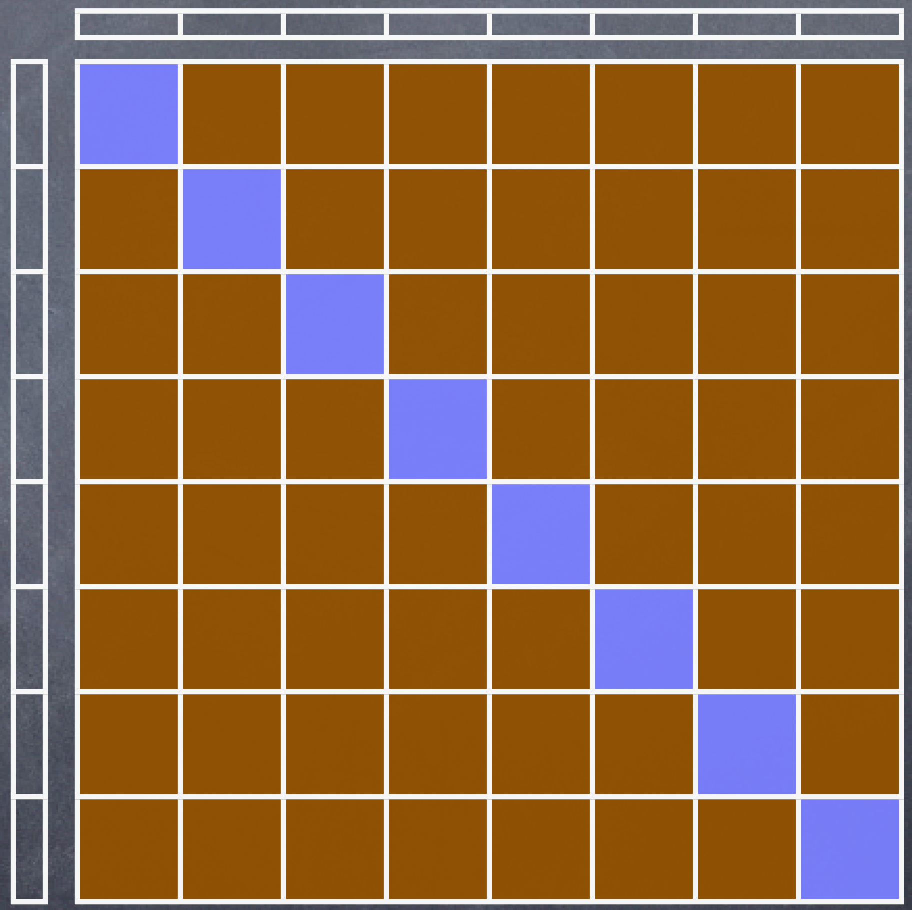
# Fooling Set for EQ

- S = set of all pairs (x,x)



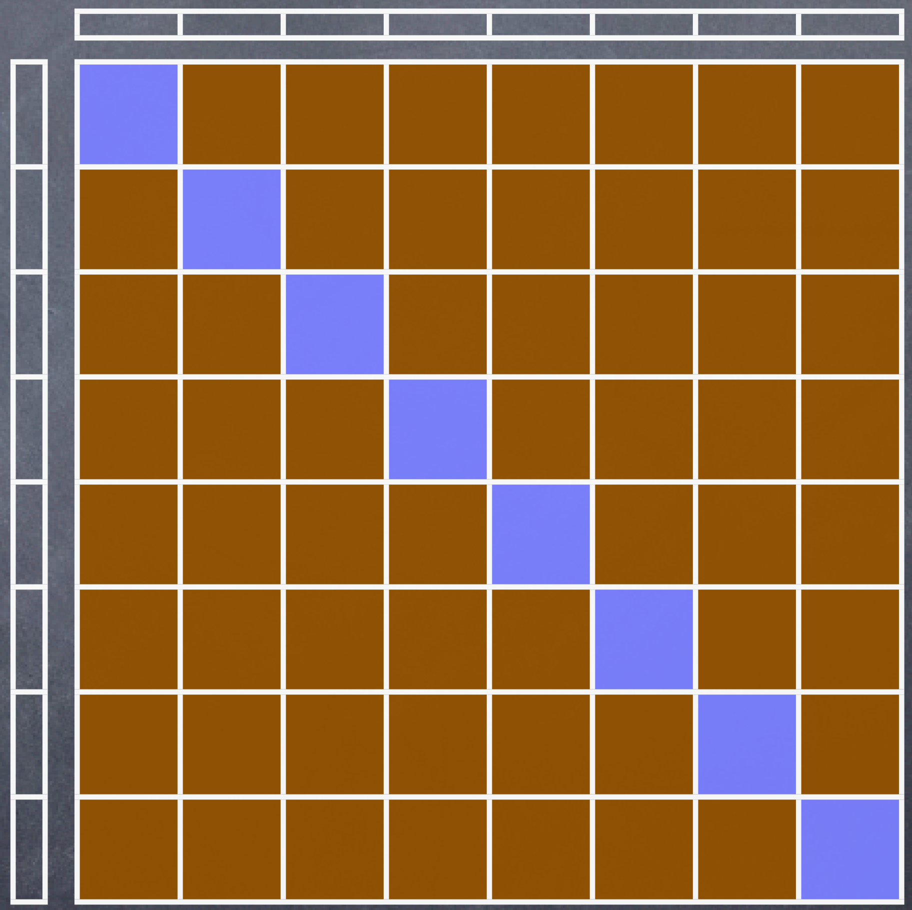
# Fooling Set for EQ

- S = set of all pairs (x,x)
  - CC(EQ) ≥ log(|S|) ≥ n



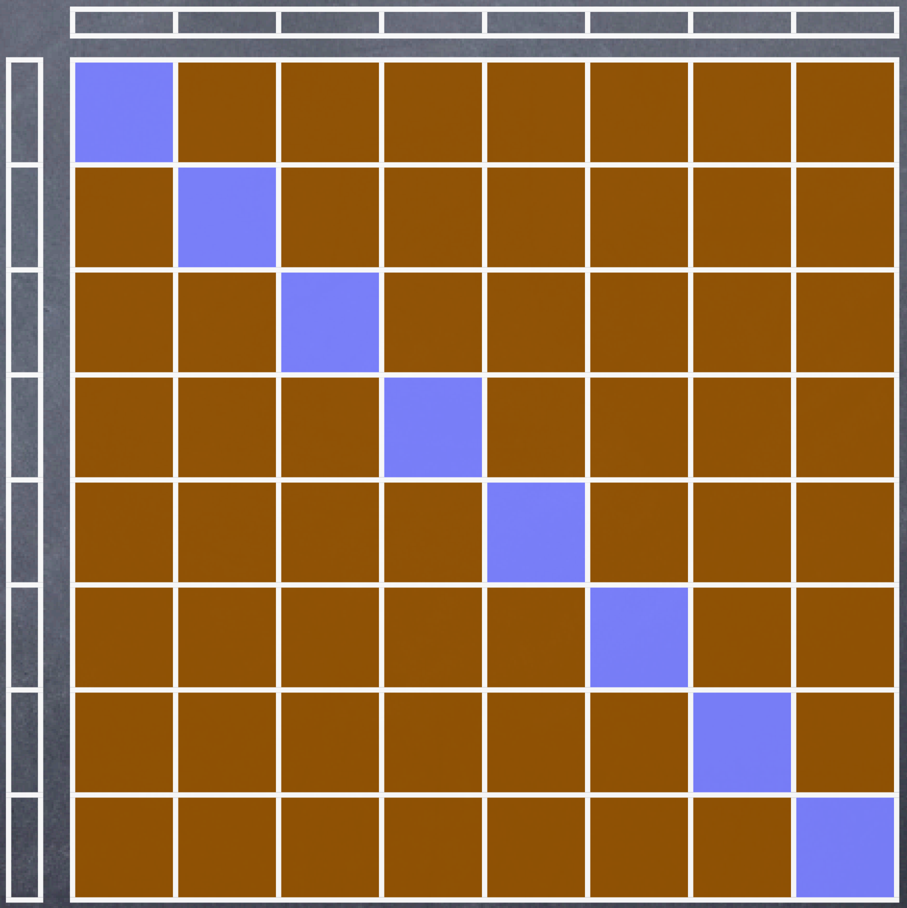
# Fooling Set for EQ

- S = set of all pairs (x,x)
  - CC(EQ) ≥ log(|S|) ≥ n
- True for any function in which each row and column has exactly one 1



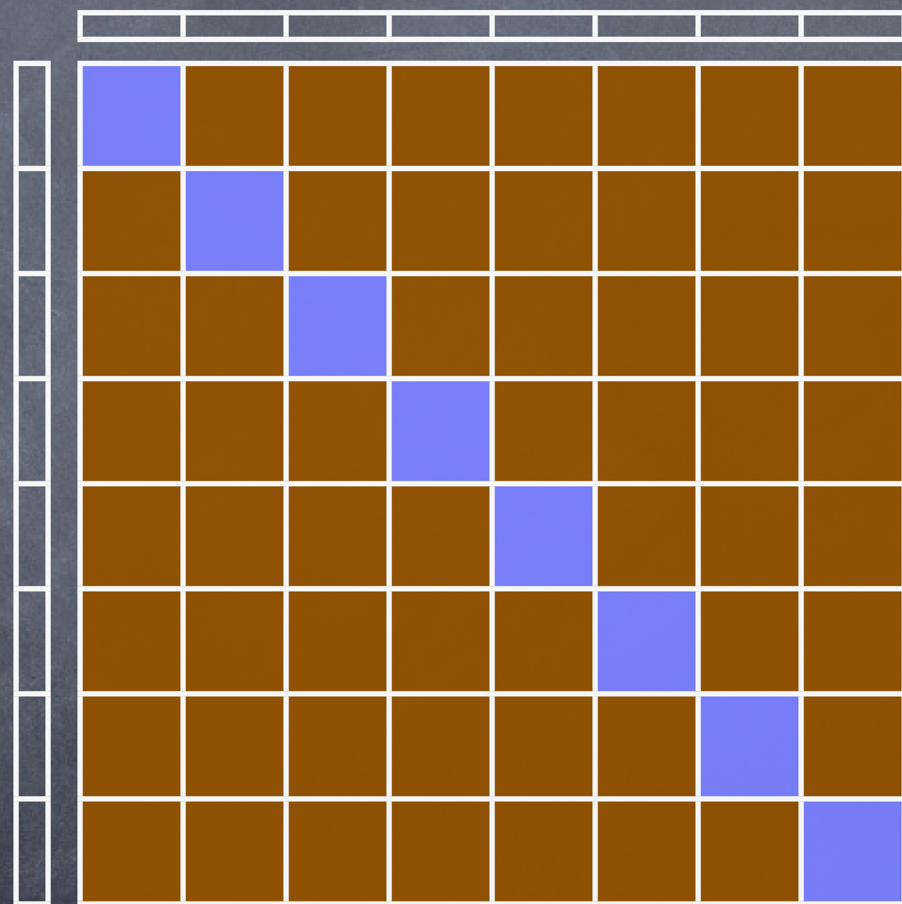
# Fooling Set for EQ

- S = set of all pairs (x,x)
  - CC(EQ) ≥ log(|S|) ≥ n
- True for any function in which each row and column has exactly one 1
- Other functions too



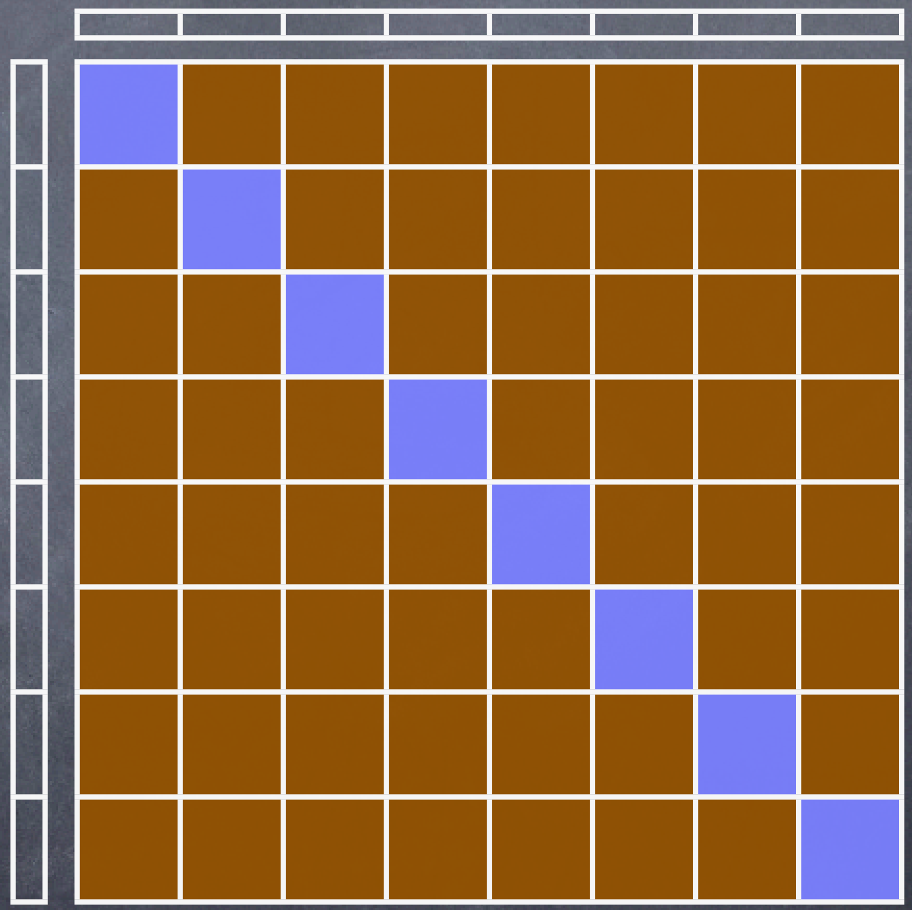
# Fooling Set for EQ

- S = set of all pairs (x,x)
  - CC(EQ) ≥ log(|S|) ≥ n
- True for any function in which each row and column has exactly one 1
- Other functions too
  - e.g.: DISJ(x,y) if $x \wedge y = 0^n$

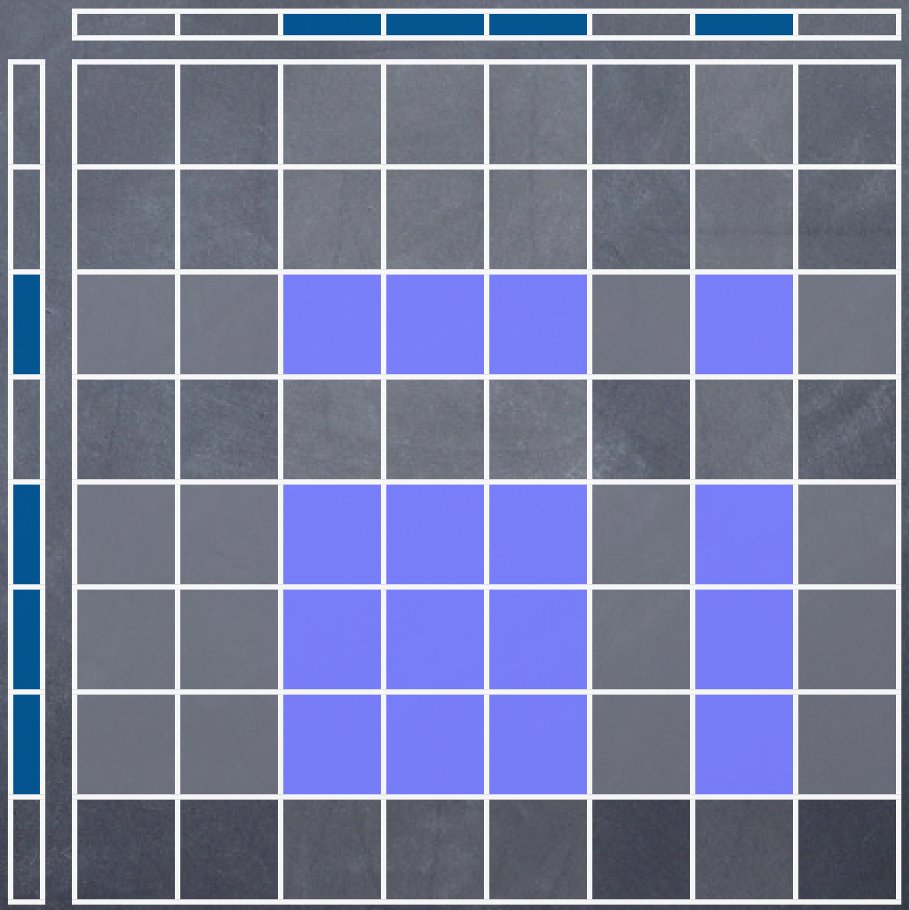


# Fooling Set for EQ

- S = set of all pairs (x,x)
  - CC(EQ) ≥ log(|S|) ≥ n
- True for any function in which each row and column has exactly one 1
- Other functions too
  - e.g.: DISJ(x,y) if $x \wedge y = 0^n$
  - S = set of complementary pairs, $(x, \neg x)$

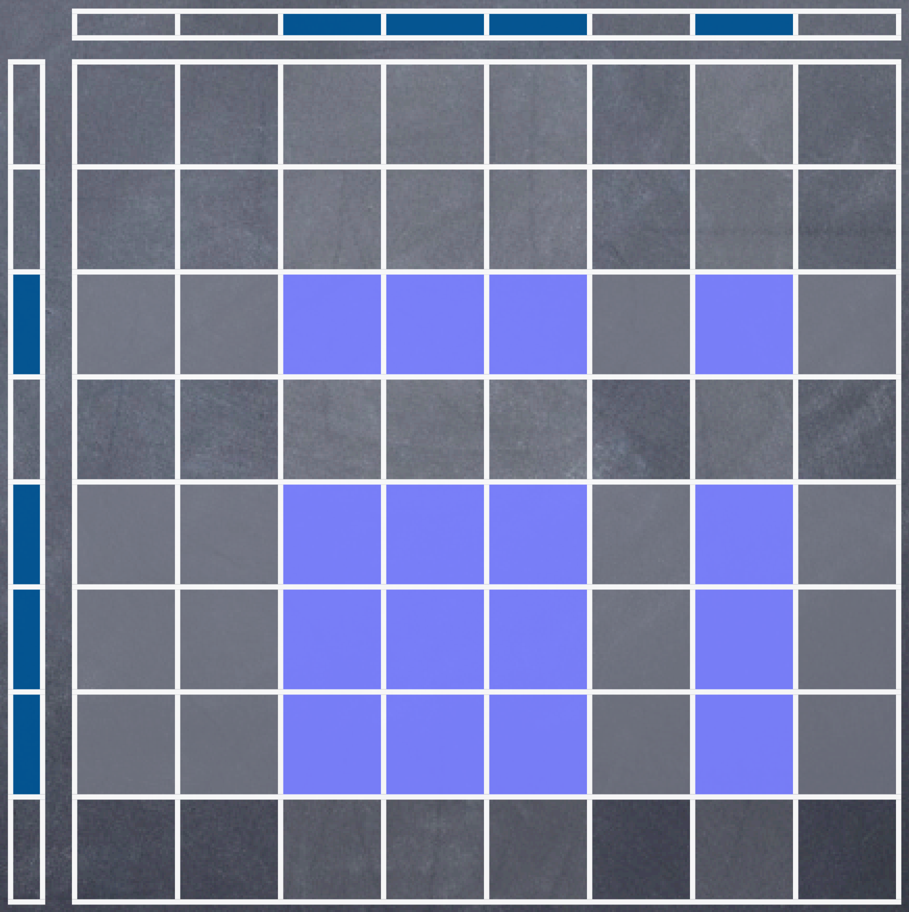


# Monochromatic Rectangles



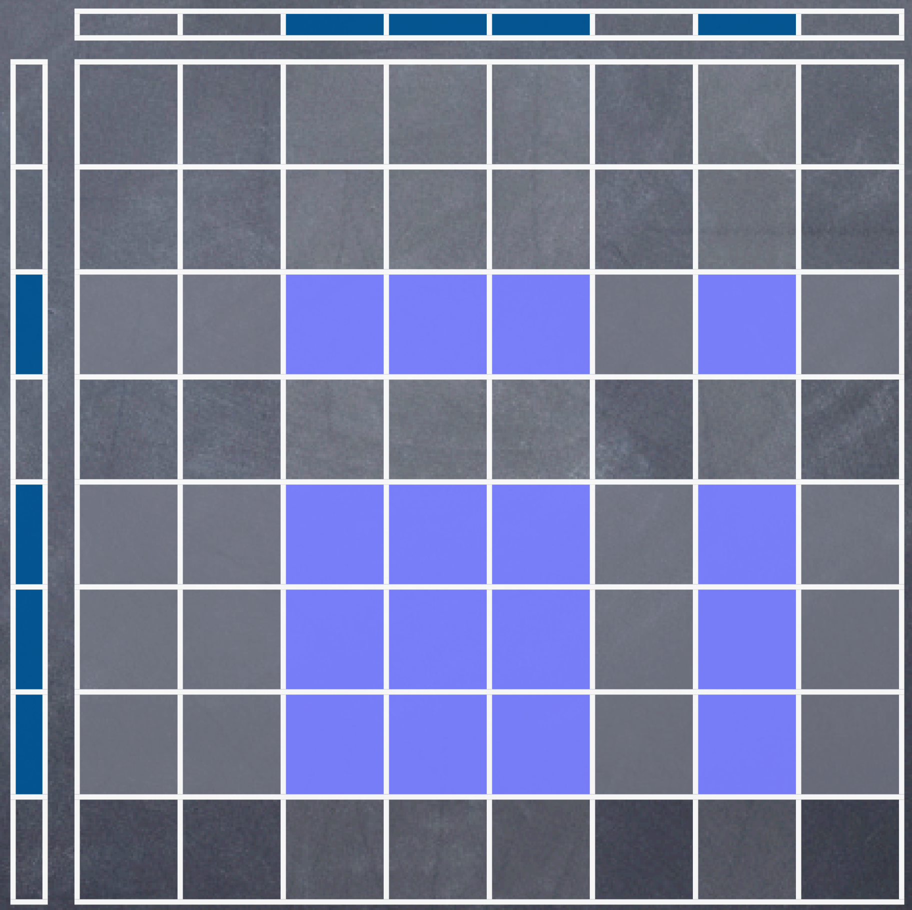
# Monochromatic Rectangles

- Rectangle: a subset of $D_1 \times D_2$ of the form $S_1 \times S_2$



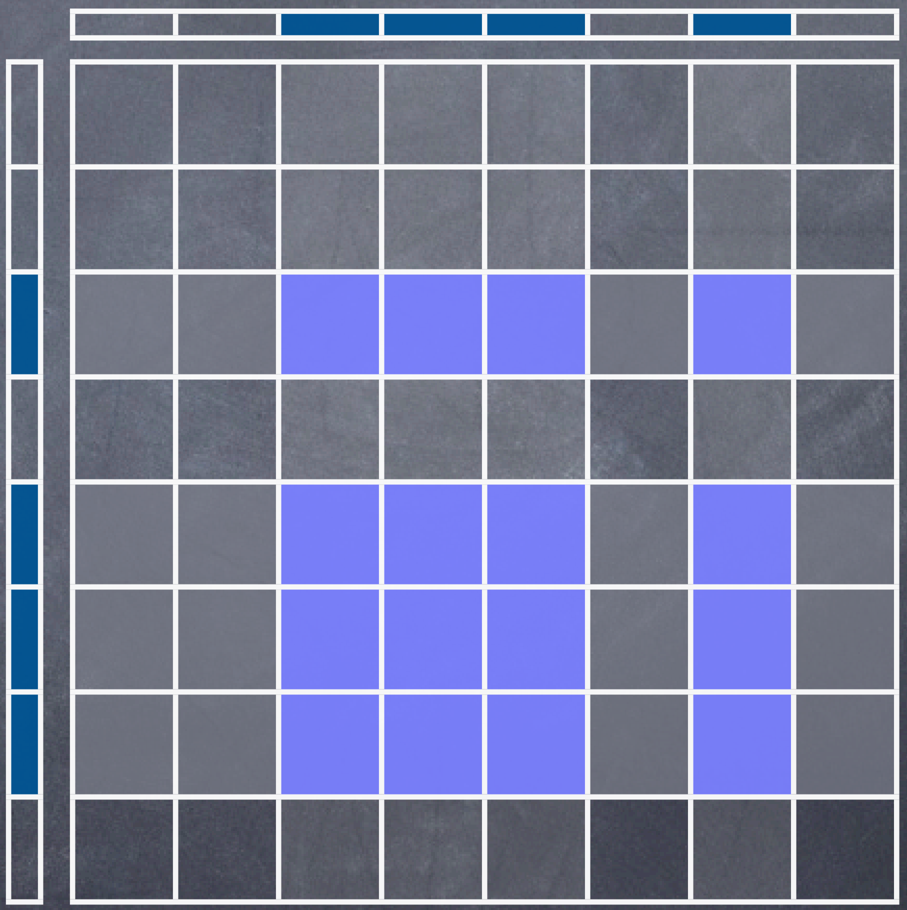
# Monochromatic Rectangles

- Rectangle: a subset of $D_1 \times D_2$ of the form $S_1 \times S_2$
- Monochromatic: same f-value



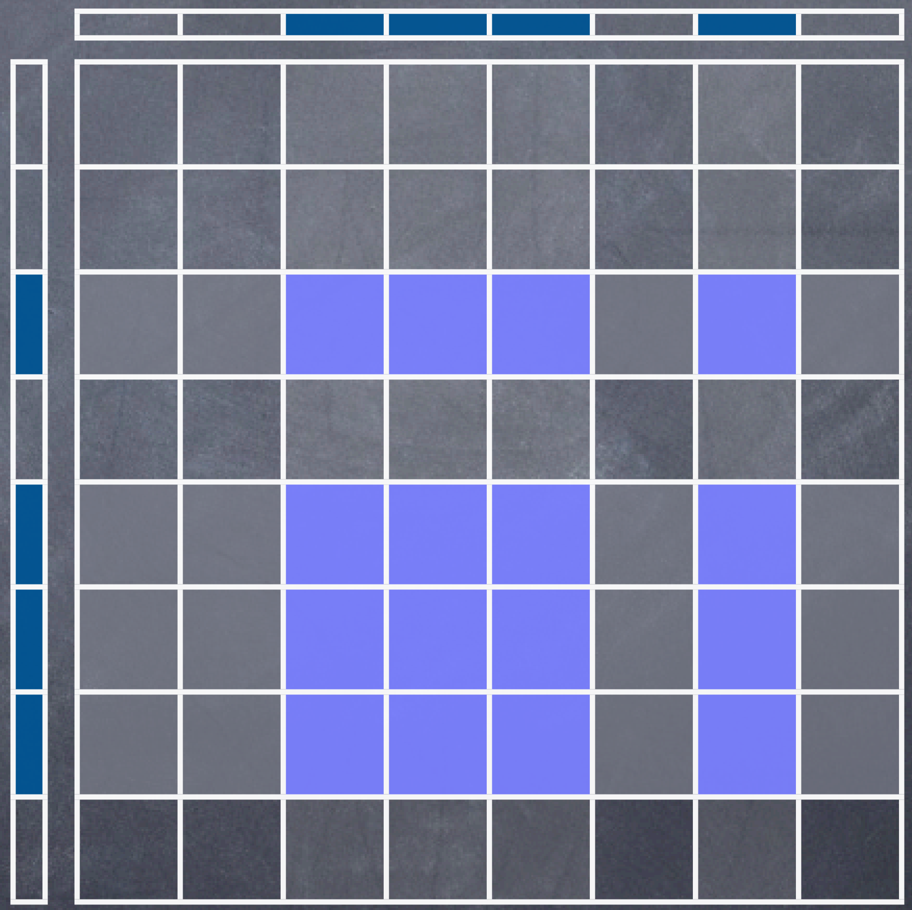
# Monochromatic Rectangles

- Rectangle: a subset of $D_1 \times D_2$ of the form $S_1 \times S_2$
- Monochromatic: same f-value
- Recall: for any protocol, set of all input-pairs with the same transcript is a rectangle

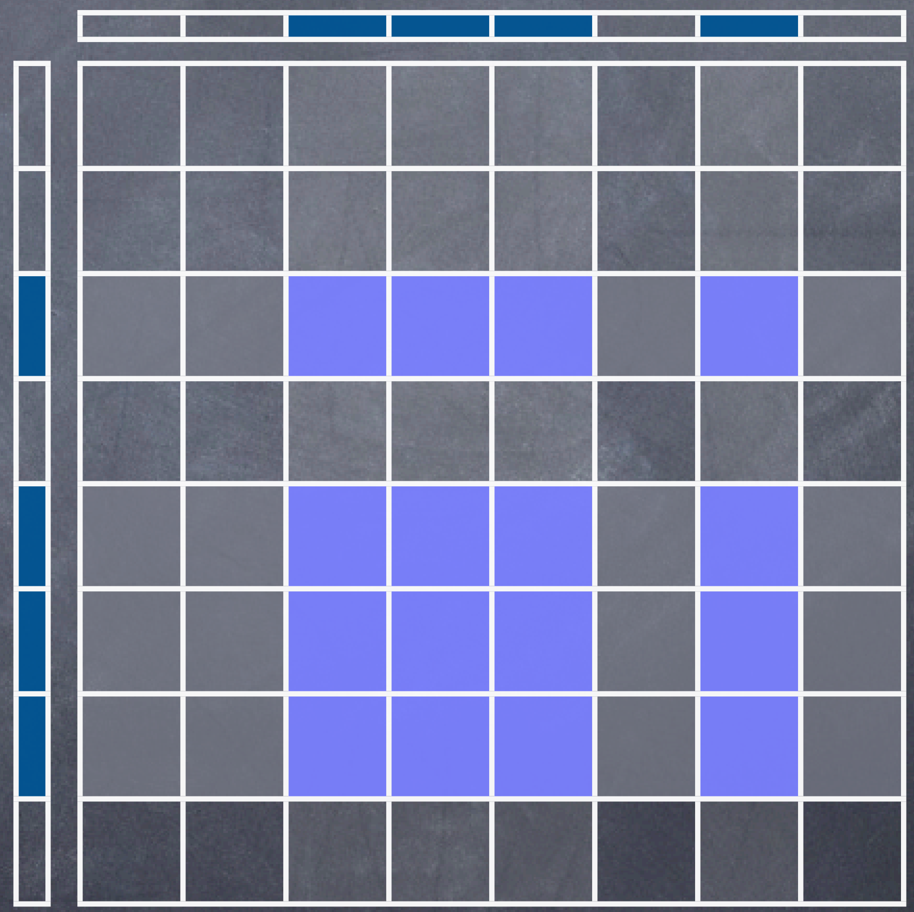


# Monochromatic Rectangles

- Rectangle: a subset of $D_1 \times D_2$ of the form $S_1 \times S_2$
- Monochromatic: same f-value
- Recall: for any protocol, set of all input-pairs with the same transcript is a rectangle
- For protocol to be correct, the rectangles should be monochromatic

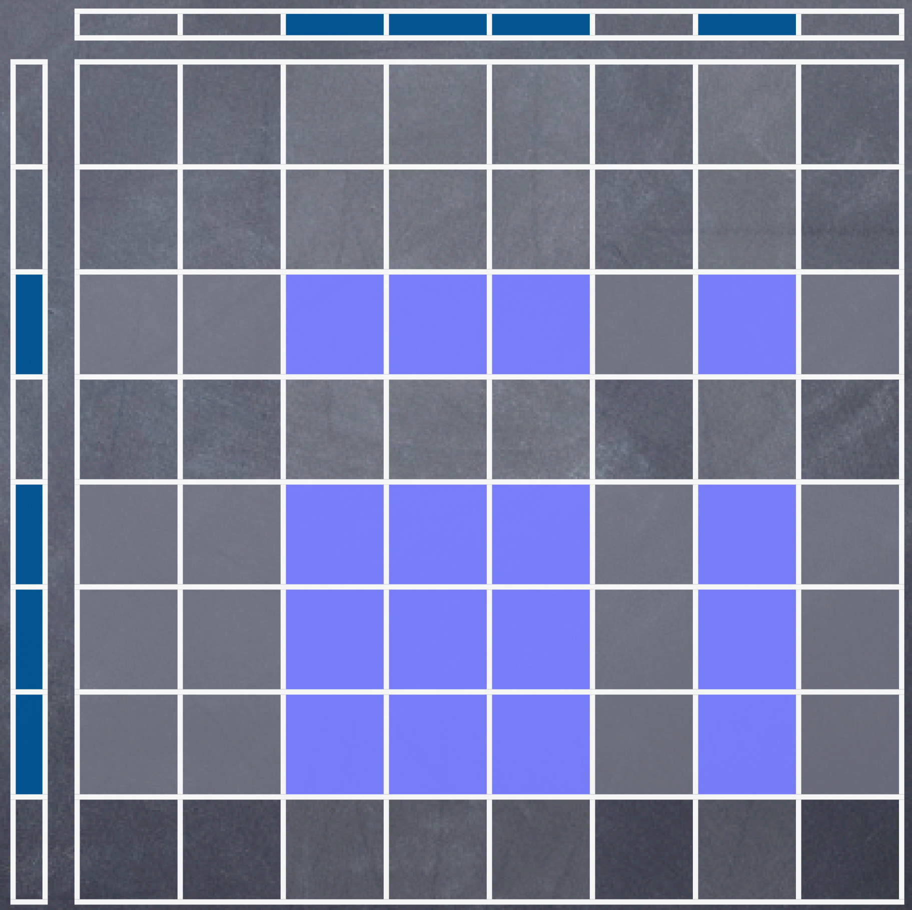


# Tiling Lower-Bound



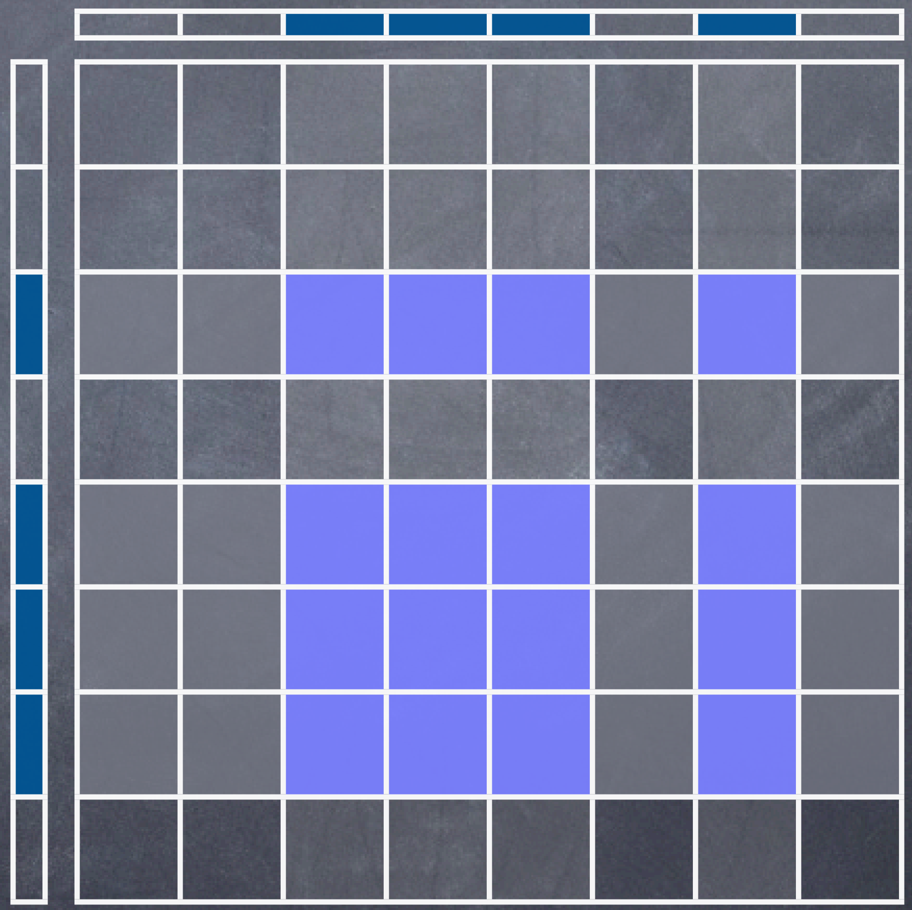
# Tiling Lower-Bound

- For protocol to be correct, same-transcript rectangles should be monochromatic



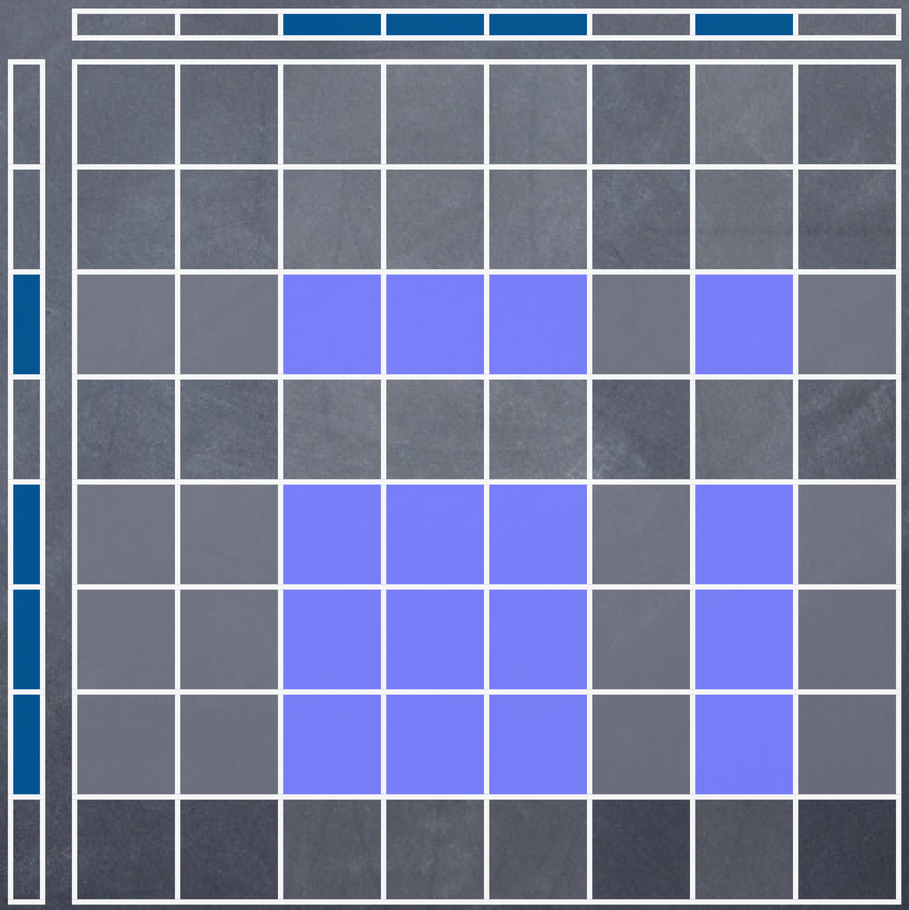
# Tiling Lower-Bound

- For protocol to be correct, same-transcript rectangles should be monochromatic
- Find the least number of monochromatic rectangles that can tile the function, $\chi(f)$



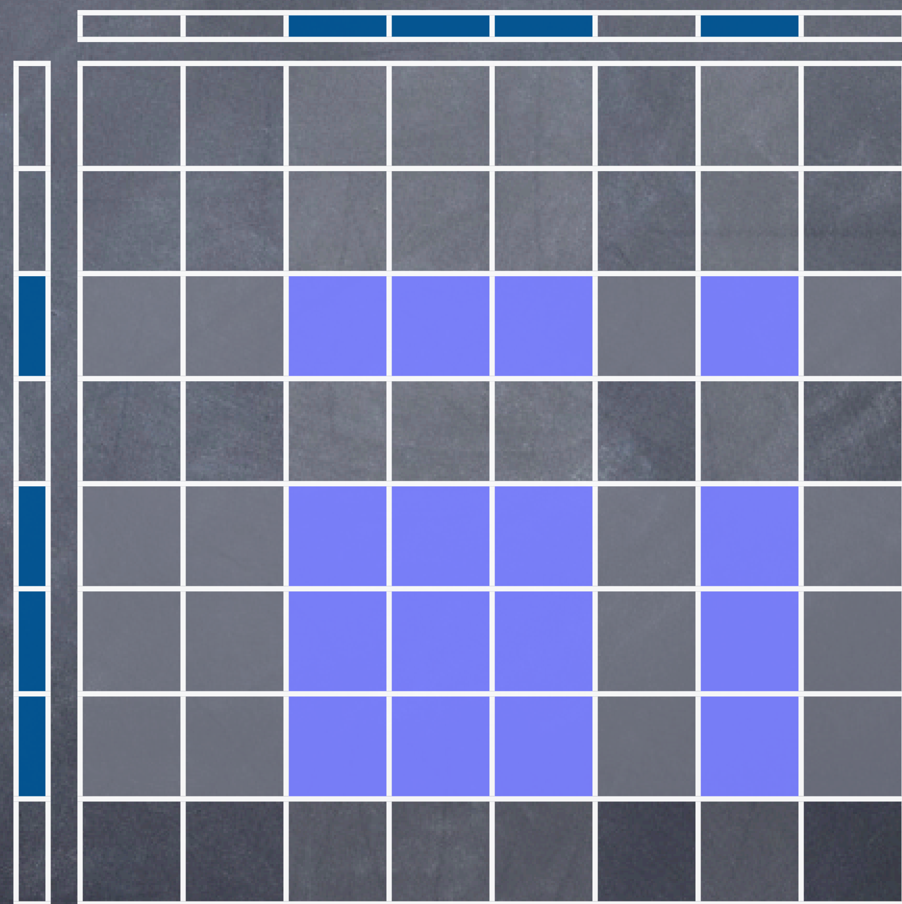
# Tiling Lower-Bound

- For protocol to be correct, same-transcript rectangles should be monochromatic
- Find the least number of monochromatic rectangles that can tile the function, $\chi(f)$
  - #transcripts $\geq \chi(f)$



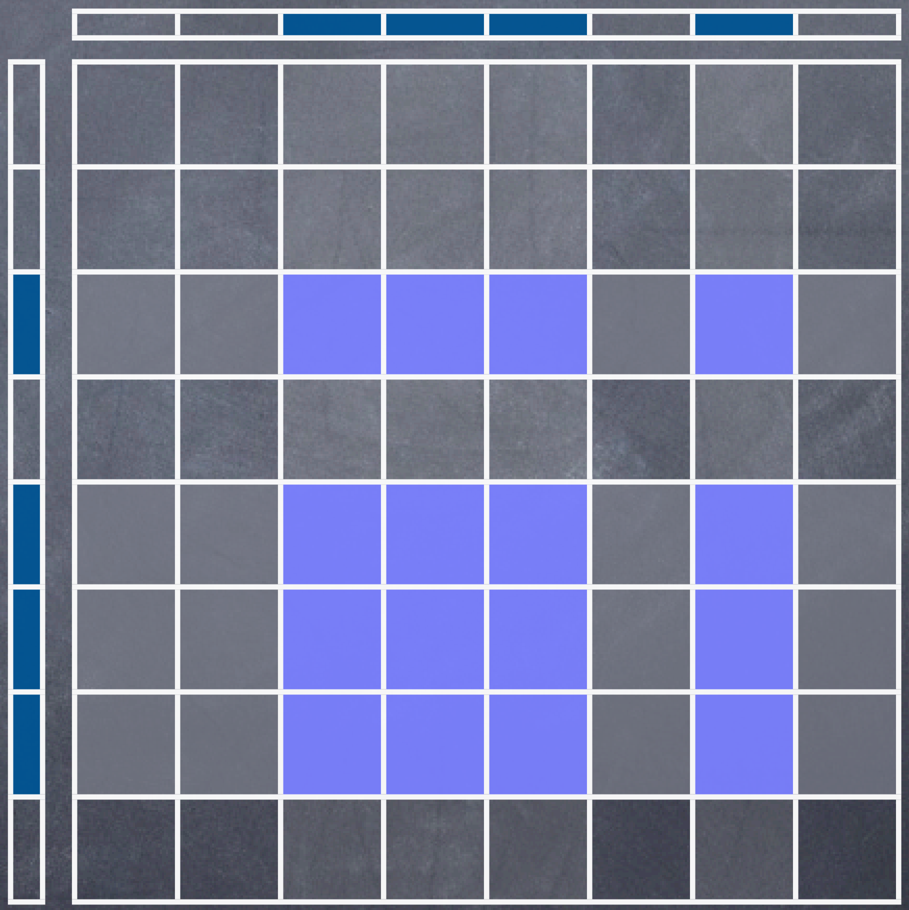
# Tiling Lower-Bound

- For protocol to be correct, same-transcript rectangles should be monochromatic
- Find the least number of monochromatic rectangles that can tile the function, $\chi(f)$
  - #transcripts $\geq \chi(f)$
  - $CC(f) \geq \log(\chi(f))$

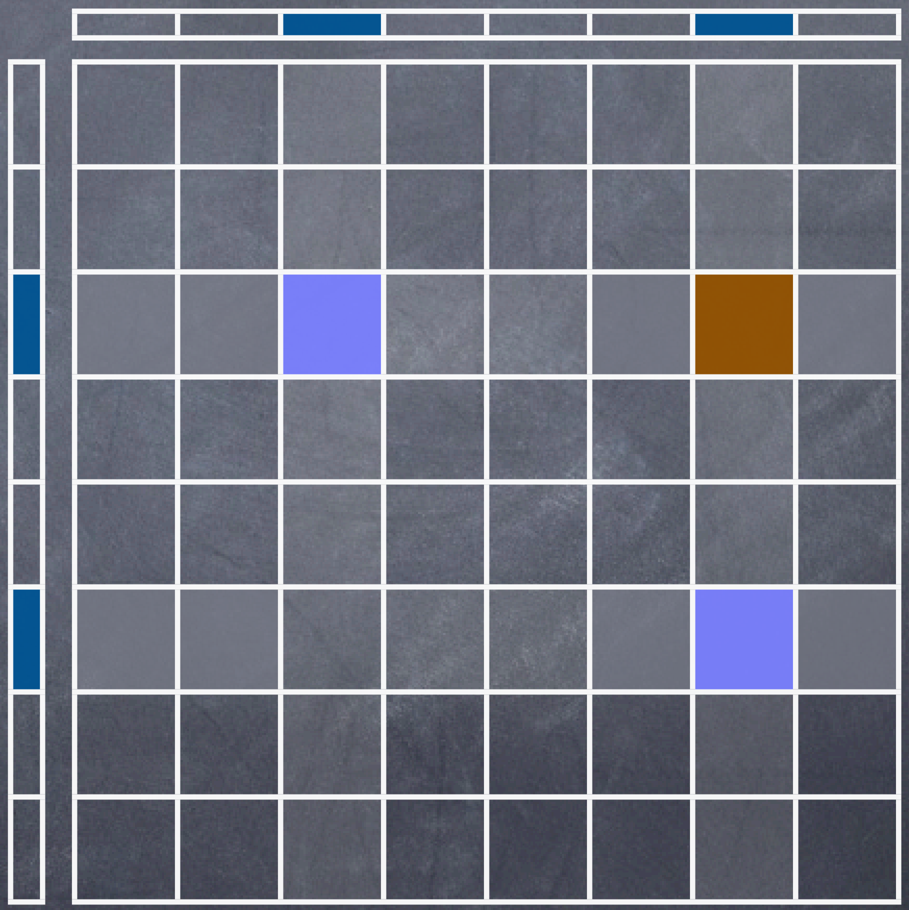


# Tiling Lower-Bound

- For protocol to be correct, same-transcript rectangles should be monochromatic
- Find the least number of monochromatic rectangles that can tile the function, $\chi(f)$
  - #transcripts $\geq \chi(f)$
  - $CC(f) \geq \log(\chi(f))$
- How to lower-bound $\chi(f)$?

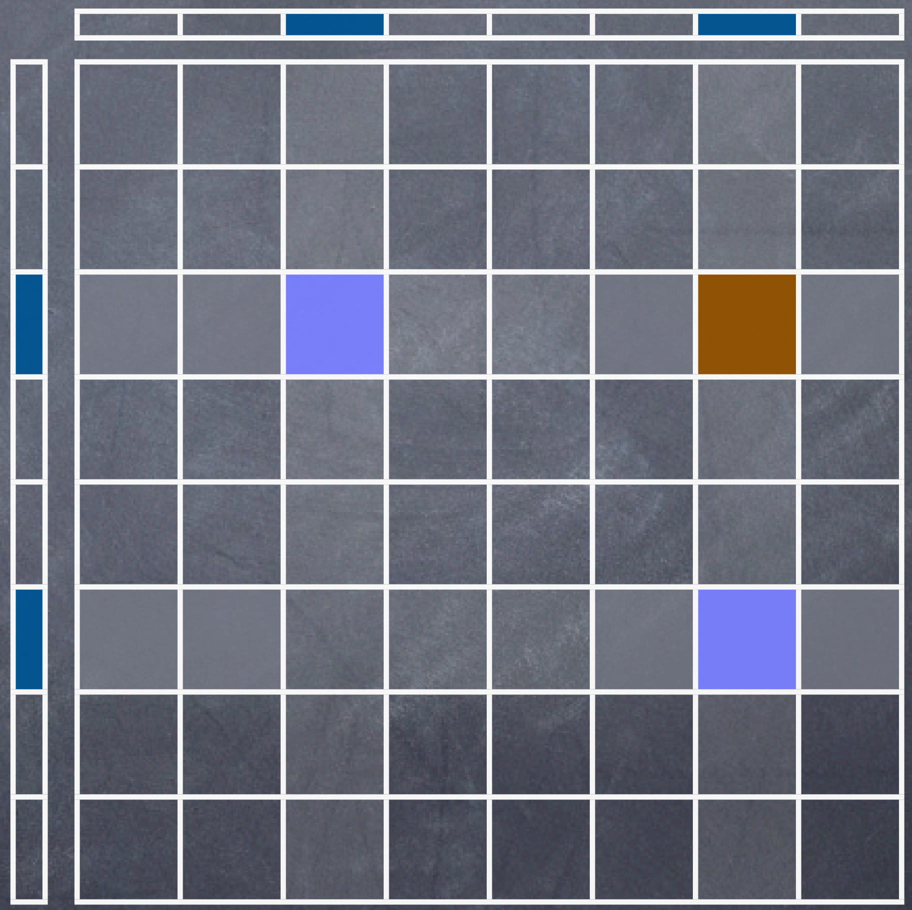


# Lower-Bounding χ(f)



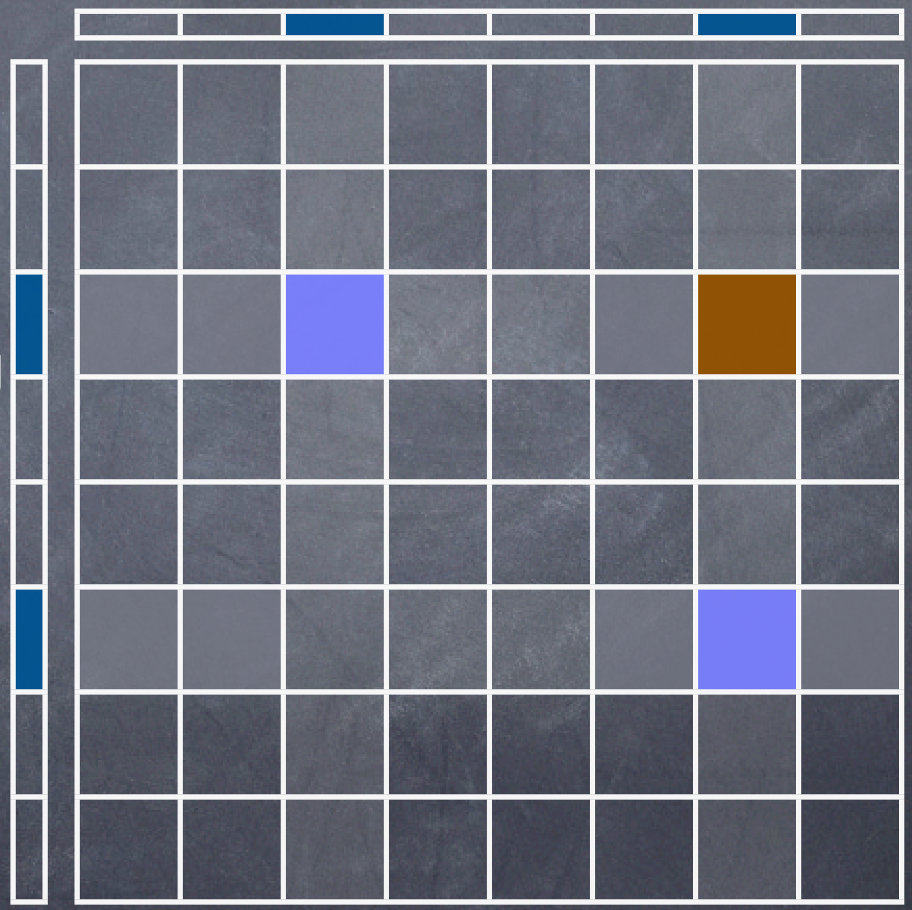
# Lower-Bounding χ(f)

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling



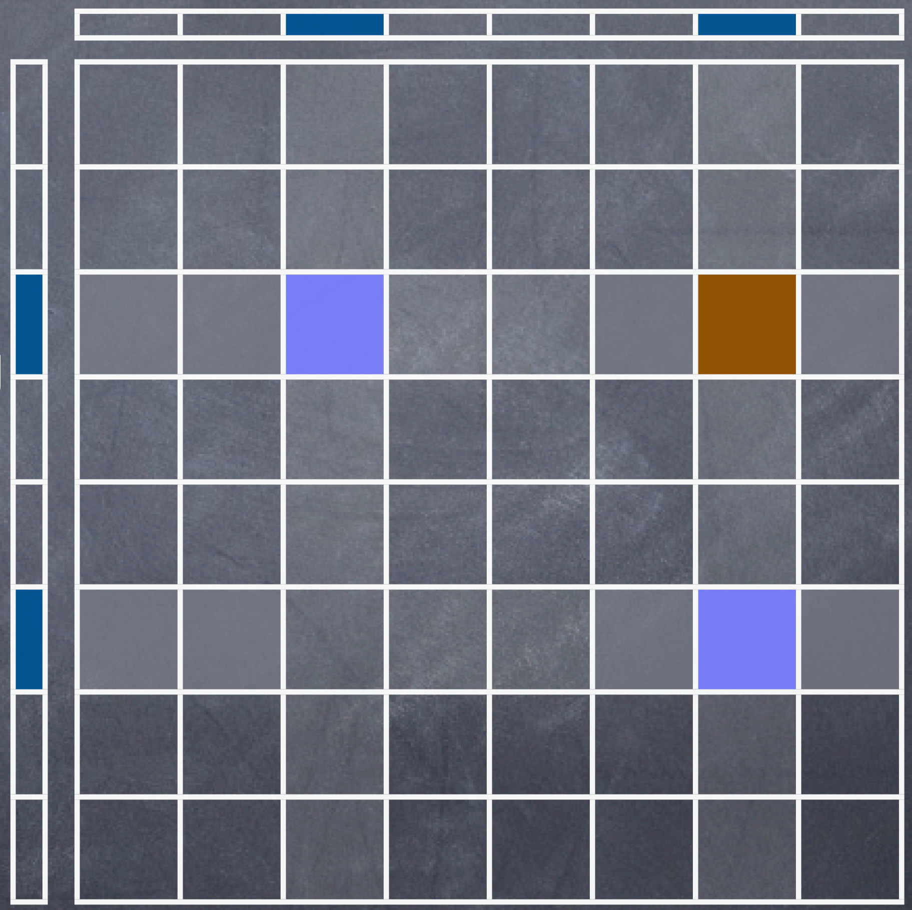
# Lower-Bounding $\chi(f)$

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling
  - $\chi(f) \geq |S|$ for every fooling set S



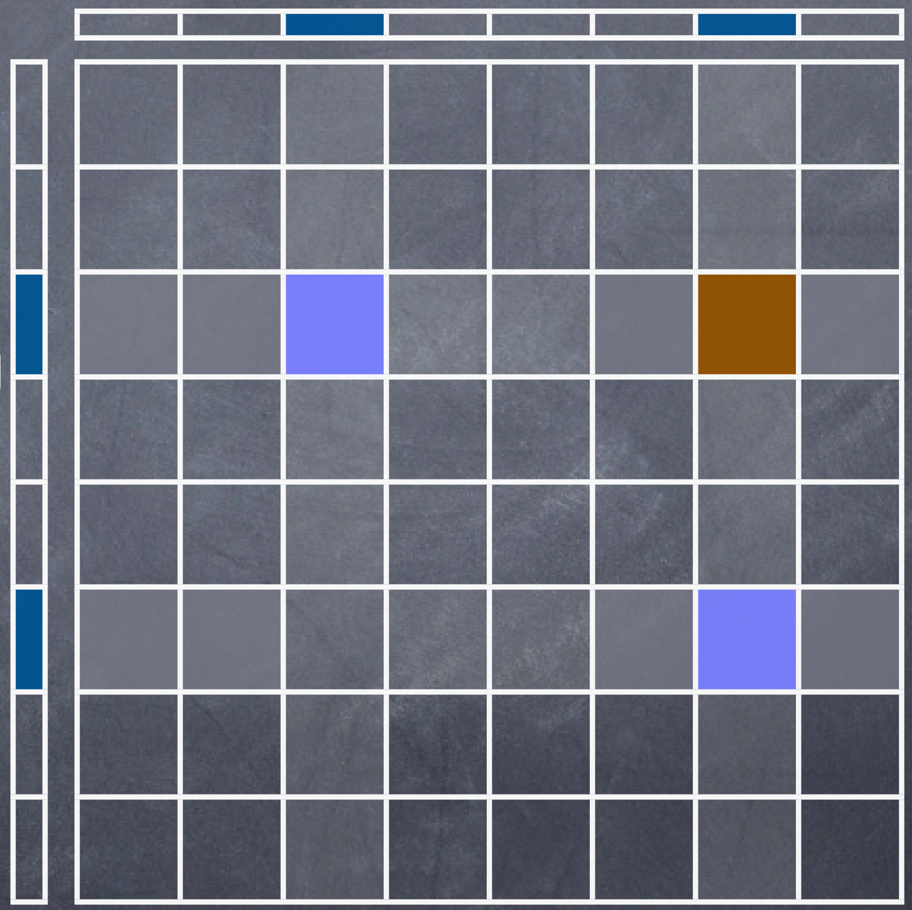
# Lower-Bounding $\chi(f)$

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling
    - $\chi(f) \geq |S|$ for every fooling set S
- Rank lower-bound



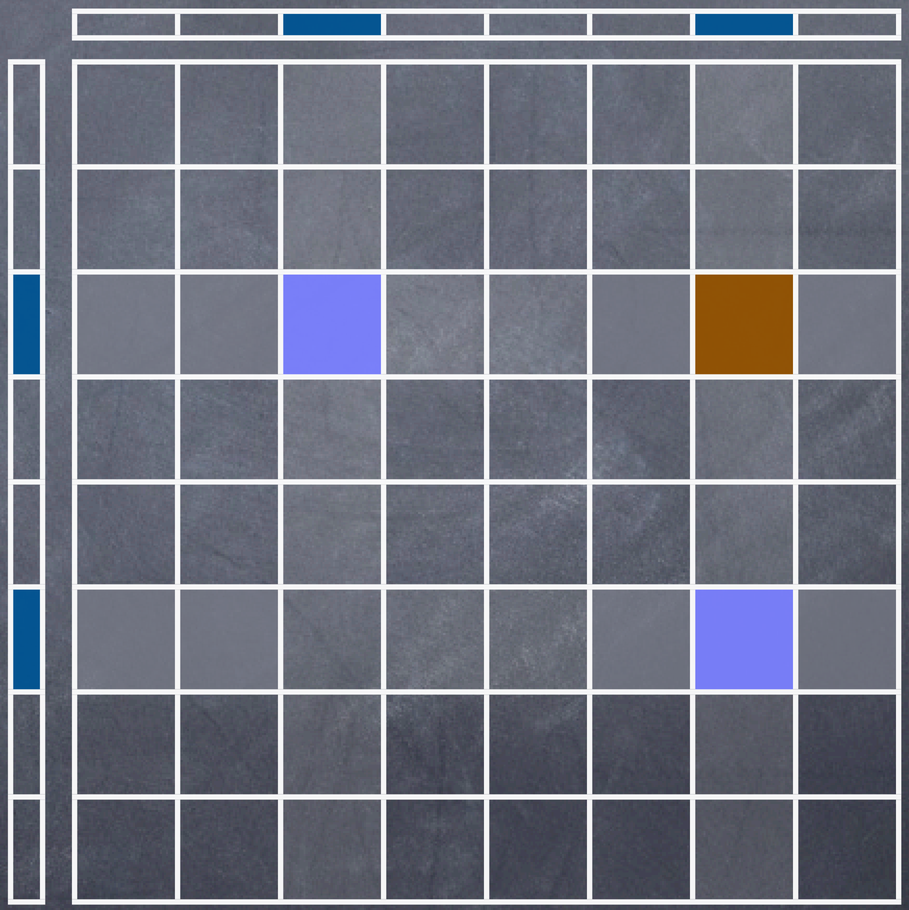
# Lower-Bounding $\chi(f)$

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling
  - $\chi(f) \geq |S|$ for every fooling set S
- Rank lower-bound
  - $\chi(f) \geq \mathrm{Rank}(M_f)$



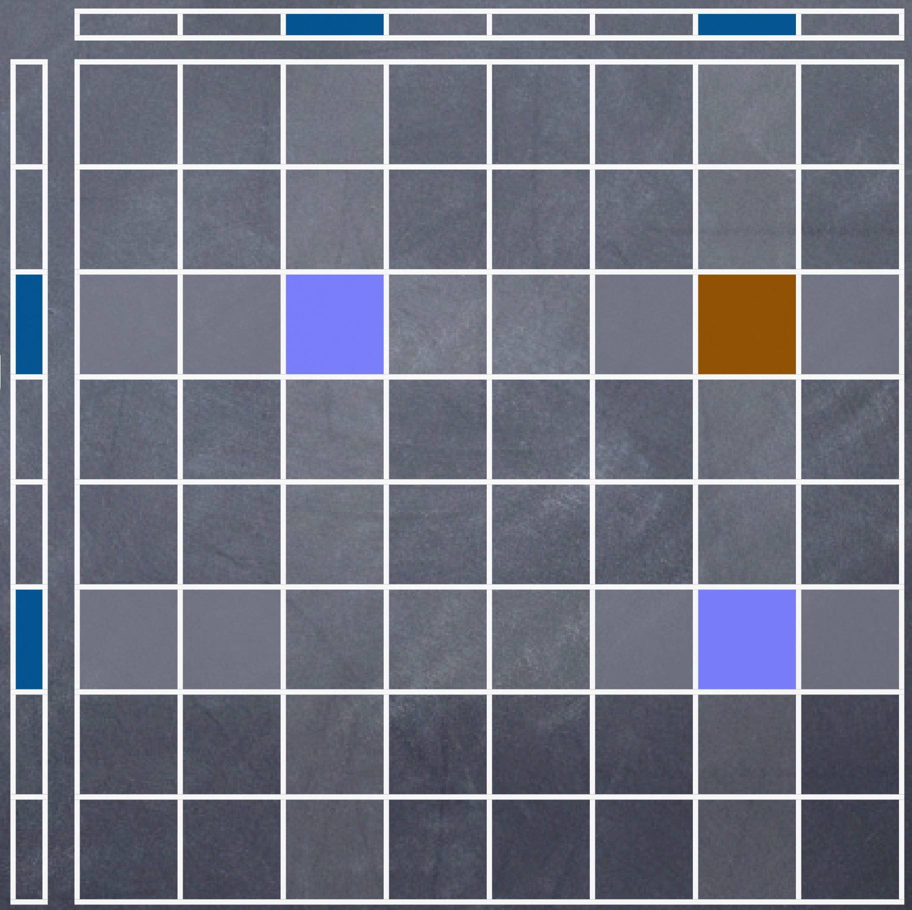
# Lower-Bounding $\chi(f)$

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling
  - $\chi(f) \geq |S|$ for every fooling set S
- Rank lower-bound
  - $\chi(f) \geq \text{Rank}(M_f)$
- Discrepancy lower-bound



# Lower-Bounding $\chi(f)$

- If a fooling set of size S, no two input-pairs from S can be on the same tile in a monochromatic tiling
  - $\chi(f) \geq |S|$ for every fooling set S
- Rank lower-bound
  - $\chi(f) \geq \text{Rank}(M_f)$
- Discrepancy lower-bound
  - $\chi(f) \geq \text{Discrepancy}(f)$



# Rank(M)

# Rank(M)

- Rank of a matrix

# Rank(M)

- Rank of a matrix
  - Maximum number of linearly independent rows (or equivalently, columns)

# Rank(M)

- Rank of a matrix
    - Maximum number of linearly independent rows (or equivalently, columns)
    - Linear independence: operations in a field

# Rank(M)

- Rank of a matrix
  - Maximum number of linearly independent rows (or equivalently, columns)
  - Linear independence: operations in a field
- Rank-r matrix: after row & column reductions $D_{(m \times n)}$ diagonal, with r 1's, rest 0's. M = UDV

# Rank(M)

- Rank of a matrix
  - Maximum number of linearly independent rows (or equivalently, columns)
  - Linear independence: operations in a field
- Rank-r matrix: after row & column reductions $D_{(m \times n)}$ diagonal, with r 1's, rest 0's. M = UDV
- Rank(M) $\leq$ r, iff M can be written as sum of $\leq$ r rank 1 matrices

# Rank(M)

- Rank of a matrix
  - Maximum number of linearly independent rows (or equivalently, columns)
  - Linear independence: operations in a field
- Rank-r matrix: after row & column reductions $D_{(m \times n)}$ diagonal, with r 1's, rest 0's. M = UDV
- Rank(M) $\leq$ r, iff M can be written as sum of $\leq$ r rank 1 matrices
  - $M = UDV = \Sigma_{i \leq r} D_{ii} U_{i(m \times 1)} V_{i(1 \times n)} = \Sigma_{i \leq r} B_i$, where $\mathrm{Rank}(B_i)=1$

# Rank(M)

- Rank of a matrix
  - Maximum number of linearly independent rows (or equivalently, columns)
  - Linear independence: operations in a field
- Rank-r matrix: after row & column reductions $D_{(m \times n)}$ diagonal, with r 1's, rest 0's. M = UDV
- Rank(M) $\leq$ r, iff M can be written as sum of $\leq$ r rank 1 matrices
  - $M = UDV = \sum_{i \leq r} D_{ii}\ U_{i(m \times 1)}\ V_{i(1 \times n)} = \sum_{i \leq r} B_i$, where $\mathrm{Rank}(B_i)=1$
  - If $M = \sum_{i \leq r} B_i = UDV$, $\mathrm{Rank}(M) \leq \min\{\mathrm{Rank}(U),\mathrm{Rank}(D),\mathrm{Rank}(V)\} \leq \mathrm{Rank}(D) = r$

# $\chi(f) \geq \mathrm{Rank}(M_f)$

# $\chi(f) \geq \text{Rank}(M_f)$

- If $M = \Sigma_{i \leq r} B_i$ with $\text{Rank}(B_i)=1$, then $\text{Rank}(M) \leq r$

# $\chi(f) \geq \mathrm{Rank}(M_f)$

- If $M = \Sigma_{i \leq r} B_i$ with $\mathrm{Rank}(B_i)=1$, then $\mathrm{Rank}(M) \leq r$
- $M_f = \Sigma_{i \leq \chi(f)} \mathrm{Tile}_i$, where $\mathrm{Tile}_i$ has a monochromatic rectangle and 0's elsewhere

# $\chi(f) \geq \mathrm{Rank}(M_f)$

- If $M = \Sigma_{i \leq r} B_i$ with $\mathrm{Rank}(B_i)=1$, then $\mathrm{Rank}(M) \leq r$
- $M_f = \Sigma_{i \leq \chi(f)} \mathrm{Tile}_i$, where $\mathrm{Tile}_i$ has a monochromatic rectangle and 0's elsewhere
  - $\mathrm{Rank}(\mathrm{Tile}_i)=1$

# $\chi(f) \geq \mathrm{Rank}(M_f)$

- If $M = \Sigma_{i \leq r} B_i$ with $\mathrm{Rank}(B_i)=1$, then $\mathrm{Rank}(M) \leq r$
- $M_f = \Sigma_{i \leq \chi(f)} \mathrm{Tile}_i$, where $\mathrm{Tile}_i$ has a monochromatic rectangle and 0's elsewhere
  - $\mathrm{Rank}(\mathrm{Tile}_i)=1$
  - $\mathrm{Rank}(M_f) \leq \chi(f)$

# $\chi(f) \geq \text{Rank}(M_f)$

- If $M = \Sigma_{i \leq r} B_i$ with $\text{Rank}(B_i)=1$, then $\text{Rank}(M) \leq r$
- $M_f = \Sigma_{i \leq \chi(f)} \text{Tile}_i$, where $\text{Tile}_i$ has a monochromatic rectangle and 0's elsewhere
  - $\text{Rank}(\text{Tile}_i)=1$
  - $\text{Rank}(M_f) \leq \chi(f)$
- $CC(f) \geq \log(\chi(f)) \geq \log(\text{Rank}(M_f))$

# Discrepancy

# Discrepancy

- Discrepancy of a 0-1 matrix

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max “imbalance” in any rectangle

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max "imbalance" in any rectangle
  - Imbalance = | #1's - #0's |

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max "imbalance" in any rectangle
  - Imbalance = | #1's - #0's |
  - Disc(M) = 1/(mn) $\max_{rect}$ imbalance(rect)

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max "imbalance" in any rectangle
  - Imbalance = | #1's - #0's |
  - Disc(M) = 1/(mn) $\max_{rect}$ imbalance(rect)
- $\chi(f) \geq 1/\mathrm{Disc}(M_f)$

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max “imbalance” in any rectangle
  - Imbalance = | #1’s - #0’s |
  - Disc(M) = 1/(mn) $\max_{rect}$ imbalance(rect)
- $\chi(f) \geq 1/\mathrm{Disc}(M_f)$
  - $\mathrm{Disc}(M_f) \geq 1/(mn)$ (size of largest monochromatic tile)

# Discrepancy

- Discrepancy of a 0-1 matrix
  - max "imbalance" in any rectangle
  - Imbalance = | #1's - #0's |
  - $\mathrm{Disc}(M) = 1/(mn)\ \max_{rect}\ \mathrm{imbalance}(rect)$
- $\chi(f) \geq 1/\mathrm{Disc}(M_f)$
  - $\mathrm{Disc}(M_f) \geq 1/(mn)$ (size of largest monochromatic tile)
  - $\chi(f) \geq (mn)/$(size of largest monochromatic tile)

# CC Lower-bounds Summary

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$
  - Tiling Lower-bound: $\#\text{transcripts} \geq \chi(f)$

# CC Lower-bounds Summary

- CC(f) ≥ log(#transcripts)
  - Tiling Lower-bound: #transcripts ≥ $\chi$(f)
  - Both fairly tight: CC(f) = O( $\log^2(\chi(f))$ )

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$
  - Tiling Lower-bound: $\#\text{transcripts} \geq \chi(f)$
  - Both fairly tight: $CC(f) = O(\ \log^2(\chi(f))\ )$
- To lower-bound $\chi(f)$: fooling-set, rank, 1/Disc

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$
  - Tiling Lower-bound: $\#\text{transcripts} \geq \chi(f)$
  - Both fairly tight: $CC(f) = O(\log^2(\chi(f)))$
- To lower-bound $\chi(f)$: fooling-set, rank, 1/Disc
  - $\chi(f) \geq |\text{max fooling-set}| \geq (\text{Rank}(M_f))^2$

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#transcripts)$
  - Tiling Lower-bound: $\#transcripts \geq \chi(f)$
  - Both fairly tight: $CC(f) = O(\log^2(\chi(f)))$
- To lower-bound $\chi(f)$: fooling-set, rank, 1/Disc
  - $\chi(f) \geq |\text{max fooling-set}| \geq (\mathrm{Rank}(M_f))^2$
  - 1/Discrepancy lower-bounds can be very loose

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$
  - Tiling Lower-bound: $\#\text{transcripts} \geq \chi(f)$
  - Both fairly tight: $CC(f) = O(\log^2(\chi(f)))$
- To lower-bound $\chi(f)$: fooling-set, rank, 1/Disc
  - $\chi(f) \geq |\text{max fooling-set}| \geq (\text{Rank}(M_f))^2$
  - 1/Discrepancy lower-bounds can be very loose
  - Conjecture: $\text{Rank}(M_f)$ (and hence fooling set) is fairly tight

# CC Lower-bounds Summary

- $CC(f) \geq \log(\#\text{transcripts})$
  - Tiling Lower-bound: $\#\text{transcripts} \geq \chi(f)$
  - Both fairly tight: $CC(f) = O(\log^2(\chi(f)))$
- To lower-bound $\chi(f)$: fooling-set, rank, 1/Disc
  - $\chi(f) \geq |\text{max fooling-set}| \geq (\text{Rank}(M_f))^2$
  - 1/Discrepancy lower-bounds can be very loose
  - Conjecture: $\text{Rank}(M_f)$ (and hence fooling set) is fairly tight
    - i.e., $CC(f) = O(\text{polylog}(\text{Rank}(M_f)))$

# Many Variants

# Many Variants

- Randomized protocols: significant savings in expectation

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. "Communication" now includes shared guess

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. “Communication” now includes shared guess
- Multi-party: Input split across multiple parties. Broadcast channels for communication.

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. "Communication" now includes shared guess
- Multi-party: Input split across multiple parties. Broadcast channels for communication.
    - Number on the forehead version

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. "Communication" now includes shared guess
- Multi-party: Input split across multiple parties. Broadcast channels for communication.
    - Number on the forehead version
- Non-boolean output

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. "Communication" now includes shared guess
- Multi-party: Input split across multiple parties. Broadcast channels for communication.
  - Number on the forehead version
- Non-boolean output
- Multi-valued functions: agree on one value

# Many Variants

- Randomized protocols: significant savings in expectation
- Non-deterministic: Alice and Bob are non-deterministic. "Communication" now includes shared guess
- Multi-party: Input split across multiple parties. Broadcast channels for communication.
    - Number on the forehead version
- Non-boolean output
- Multi-valued functions: agree on one value
- Different costs: asymmetric communication, average-case complexity