Complexity of Counting

Lecture 20

#P

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    - e.g.: Number of satisfying assignments to a boolean formula
    - e.g.: Number of inputs less than x (lexicographically) that are in a language L

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- Easy to see: FP ⊆ #P [Exercise]

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      - HAMILTONICITY(G) ⇔ #CYCLES(G) ≥ $n^{n\hat{}2}$

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    - So if PP = P, then #P = FP (and vice versa)

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        - Permanent (for binary matrices) is #P-complete

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      - $Perm(A) = \Sigma_\sigma W(\sigma)$ over all cycle covers σ of directed graph G_A (with edge-weights from A)

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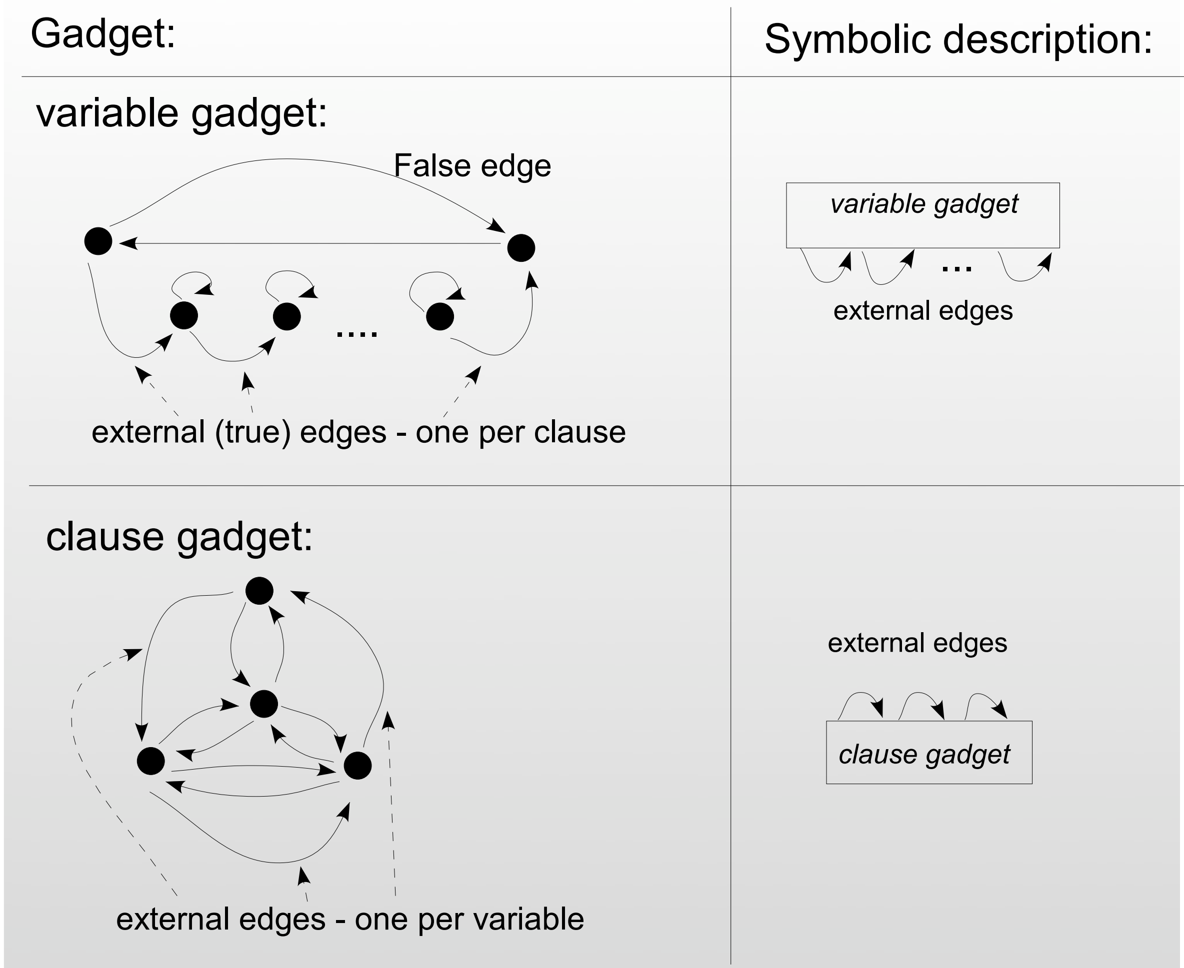

[Figures from the textbook]

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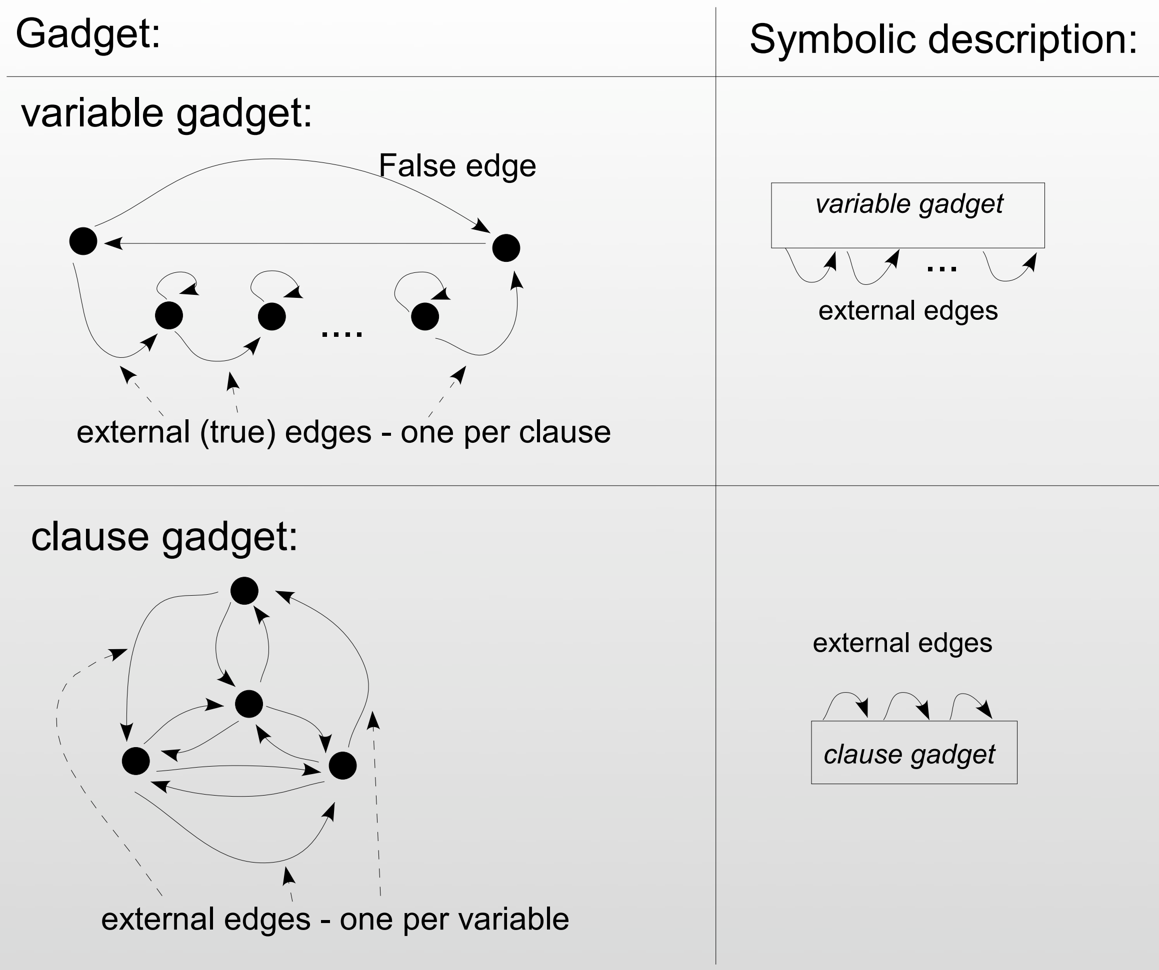

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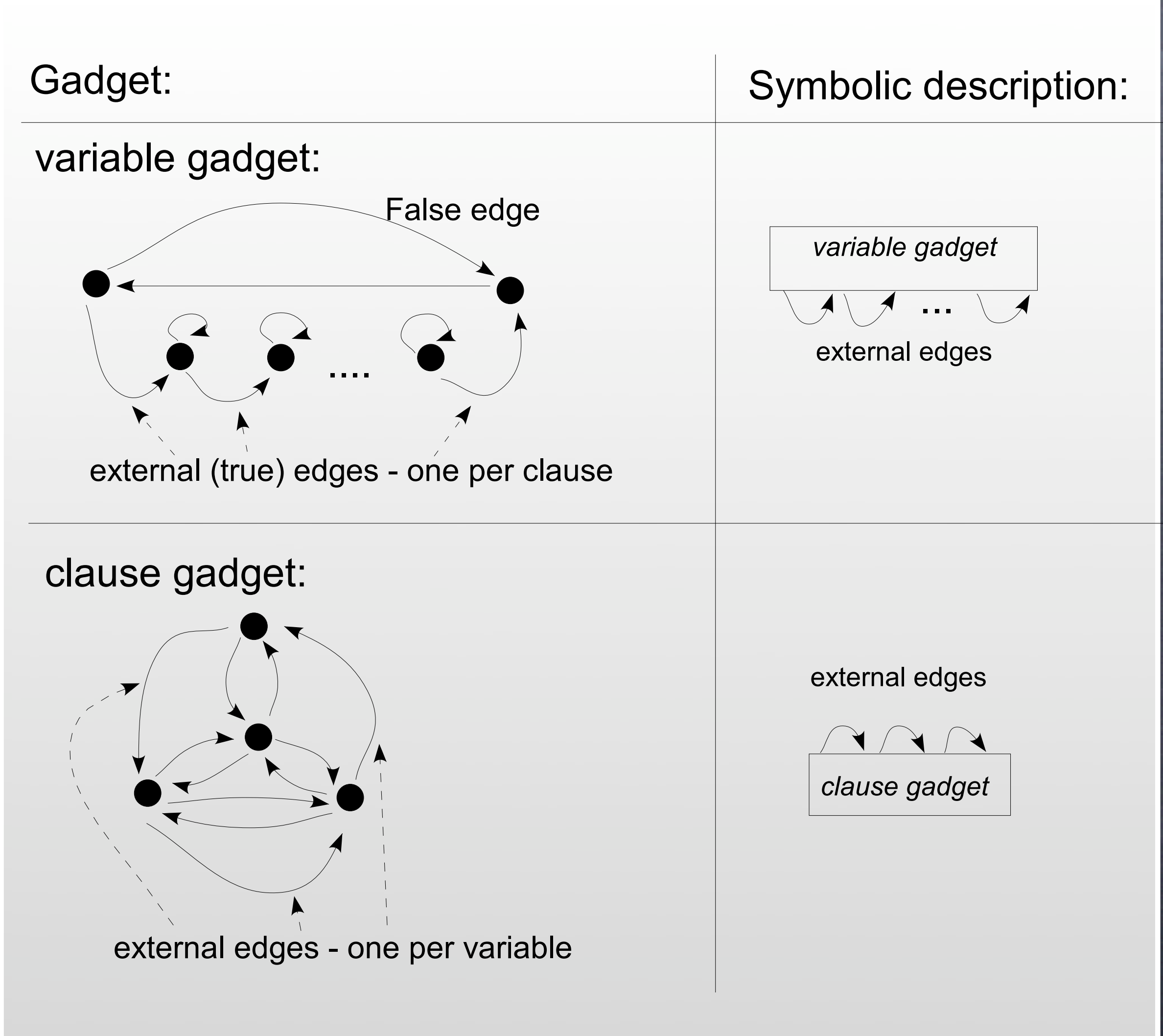

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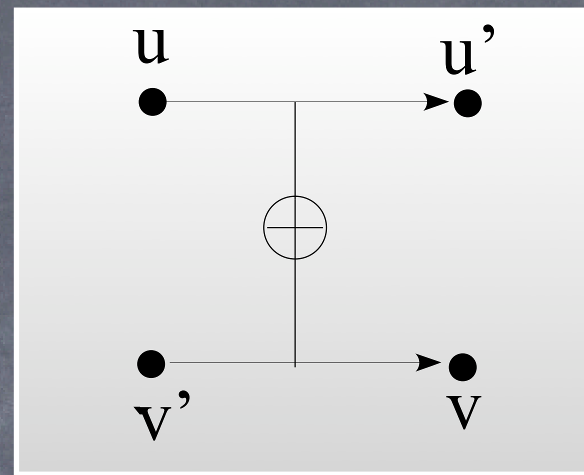

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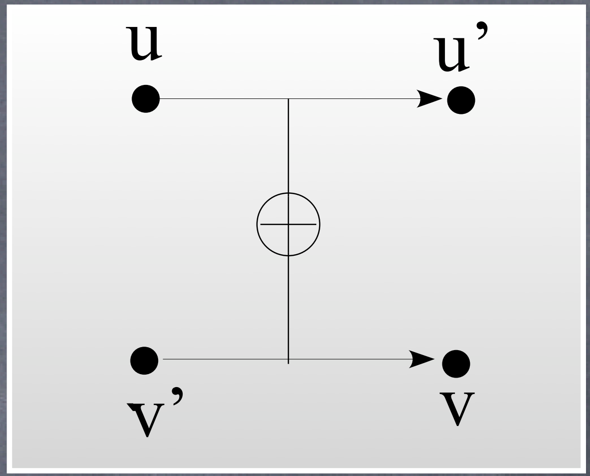

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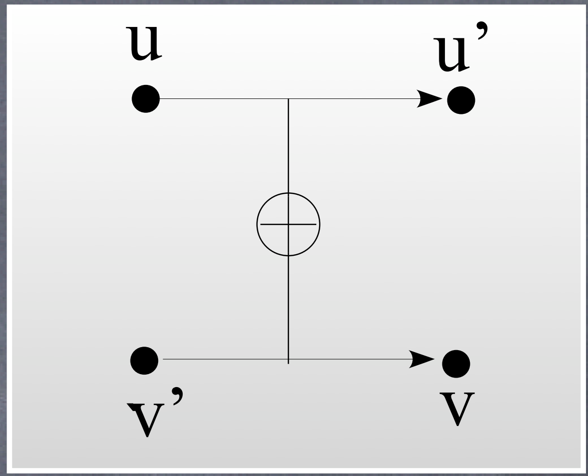

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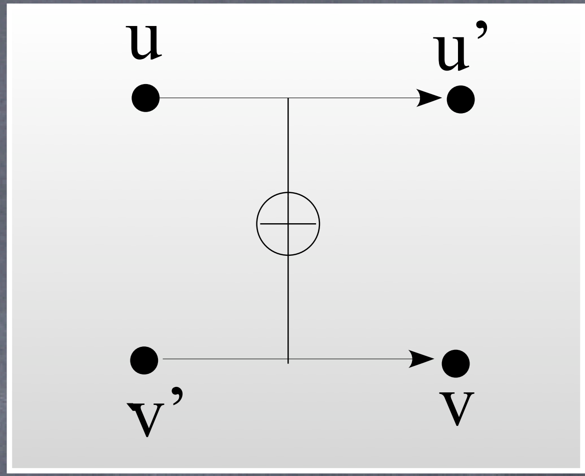

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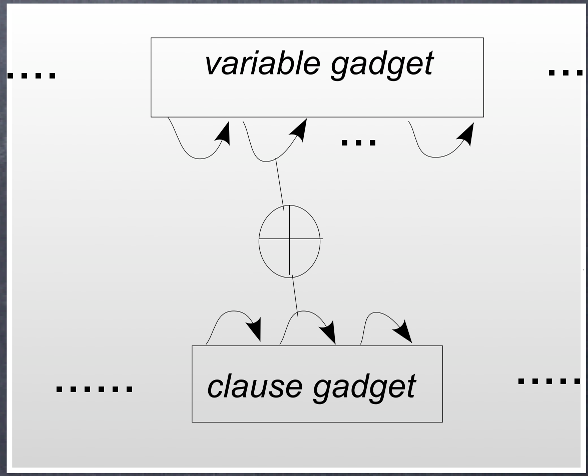

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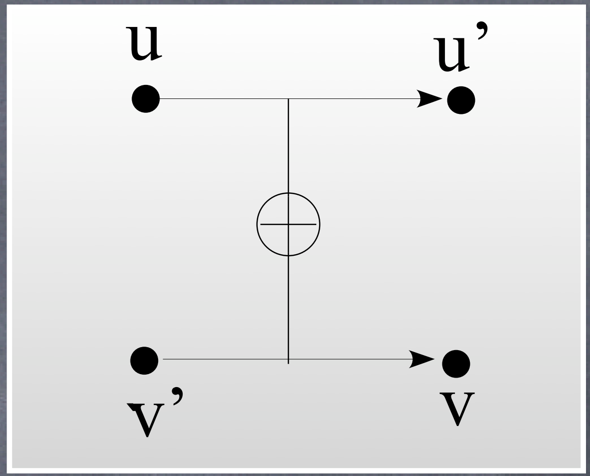

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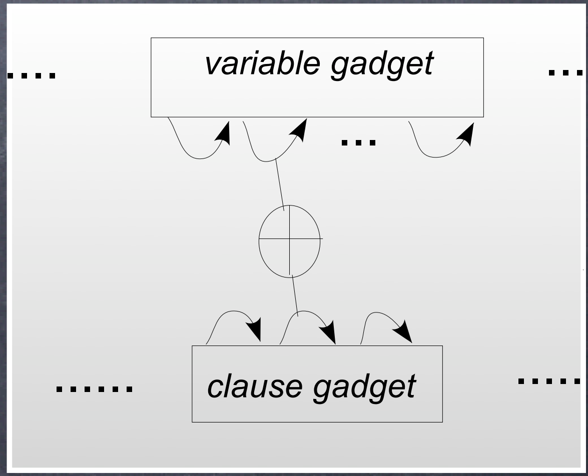

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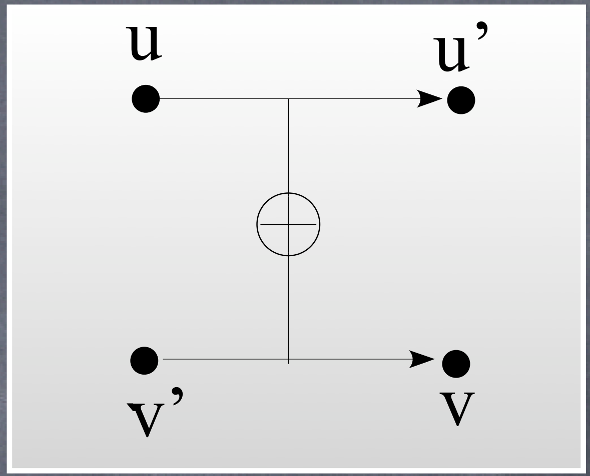

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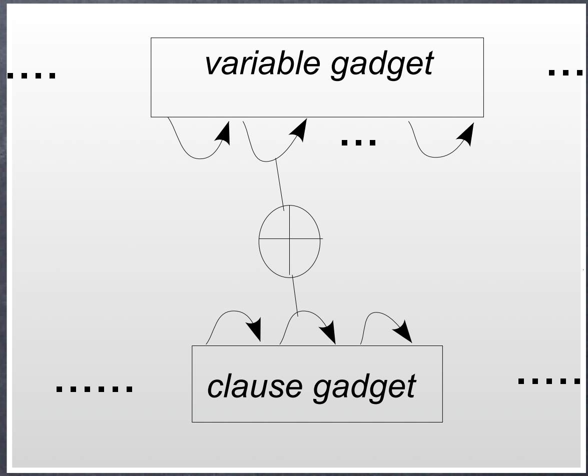

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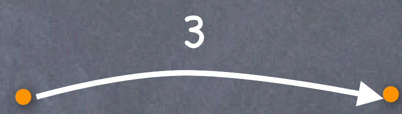

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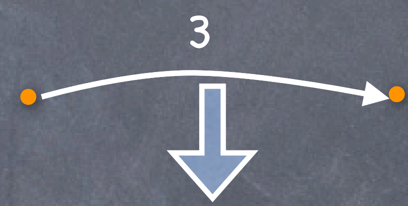

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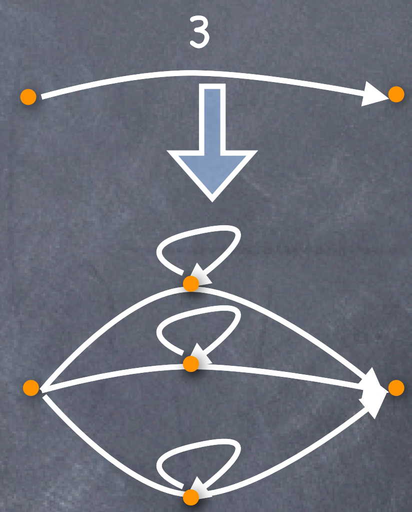

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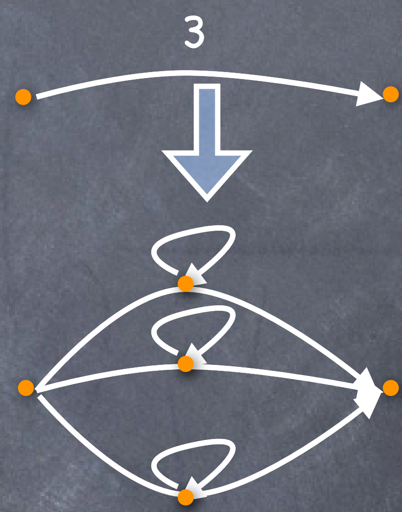

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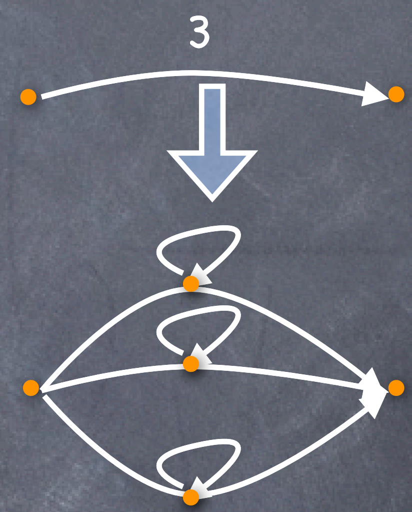

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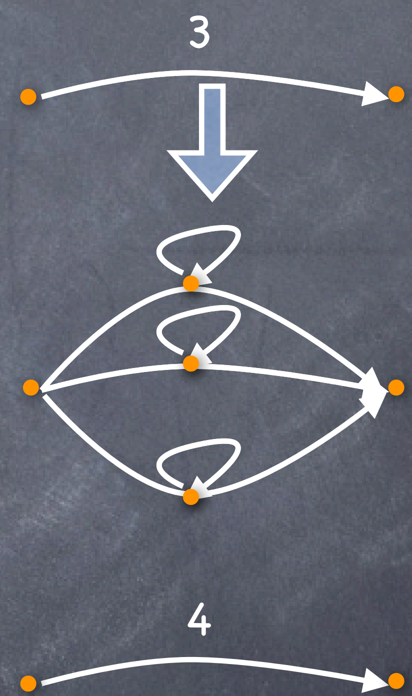

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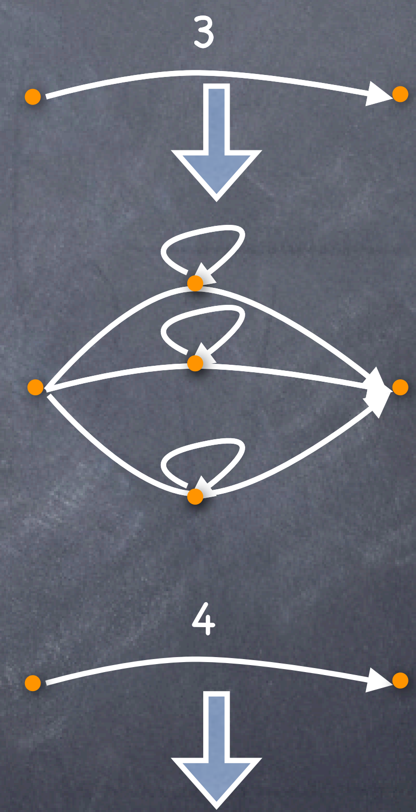

Permanent is #P-complete

- Can use binary matrix instead of integer matrix
  - First change to +1/-1 weights (adding vertices)
  - To replace -1: working modulo M+1 (for say $M=2^{n \log n} > n!$) does not change positive values. $M = 2^k$.
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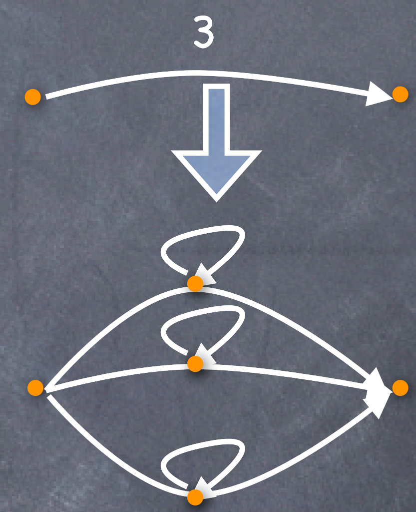

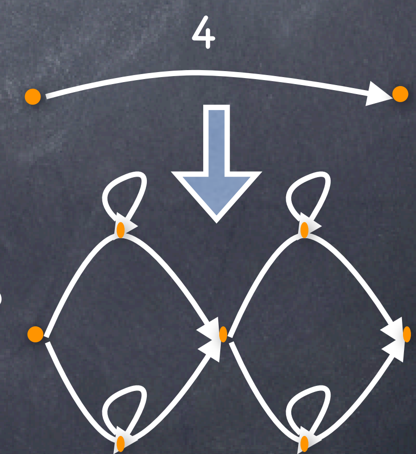

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- Next: Toda's Theorem: $PH \subseteq P^{\#P} = P^{PP}$