# Interactive Proofs

Lecture 18
AM

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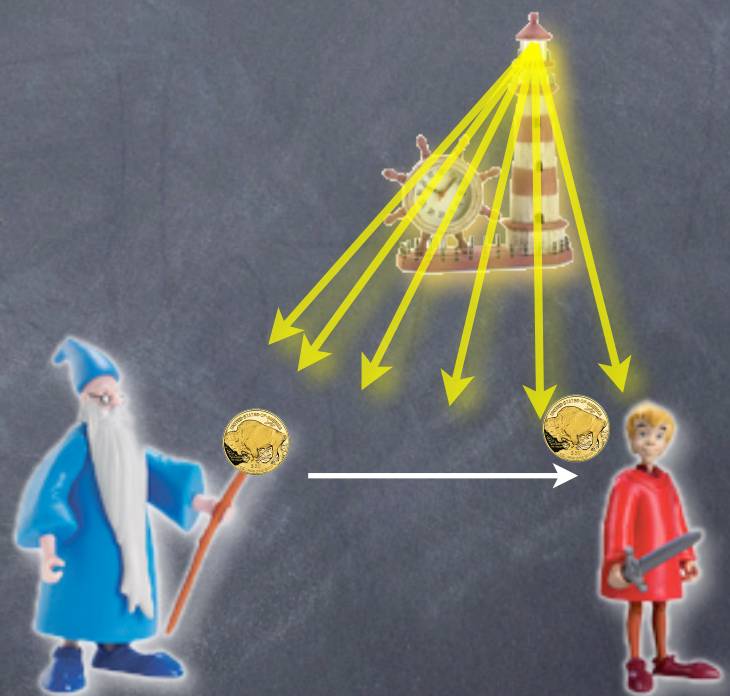

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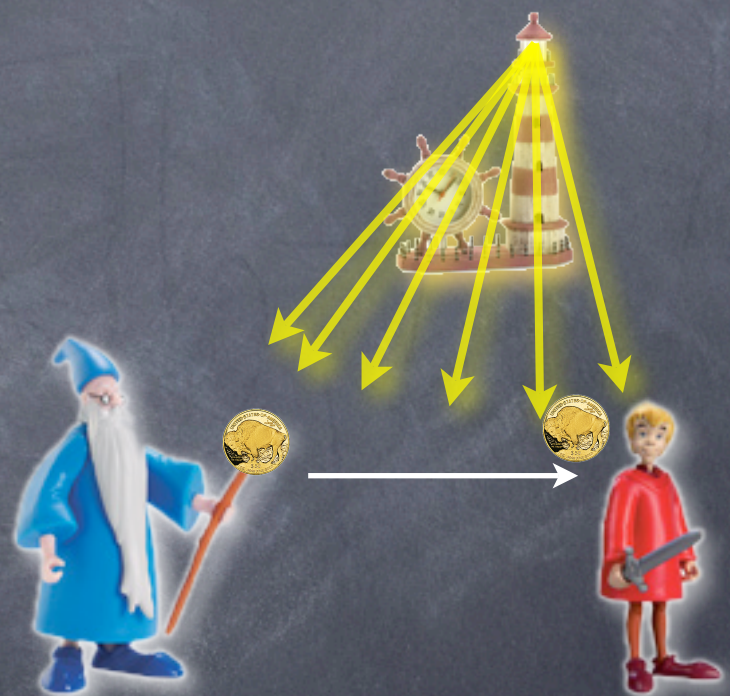

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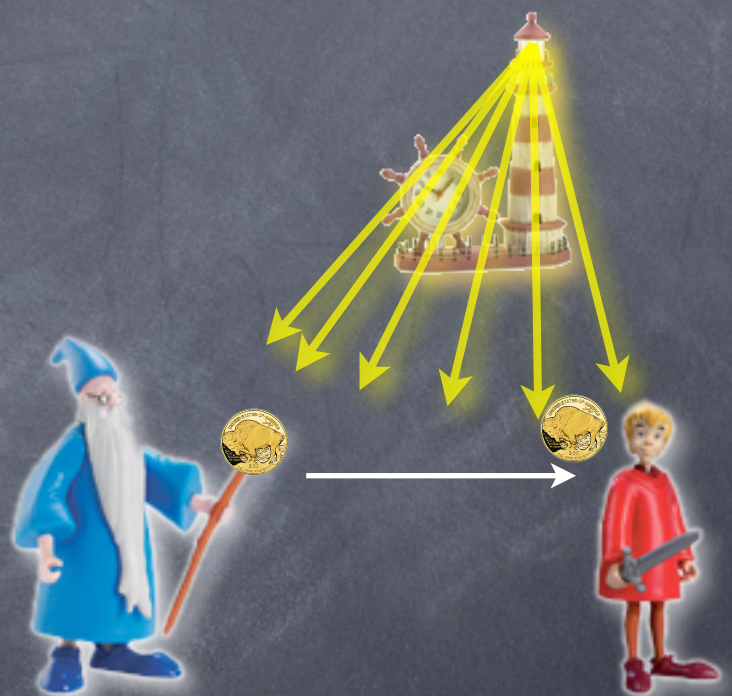

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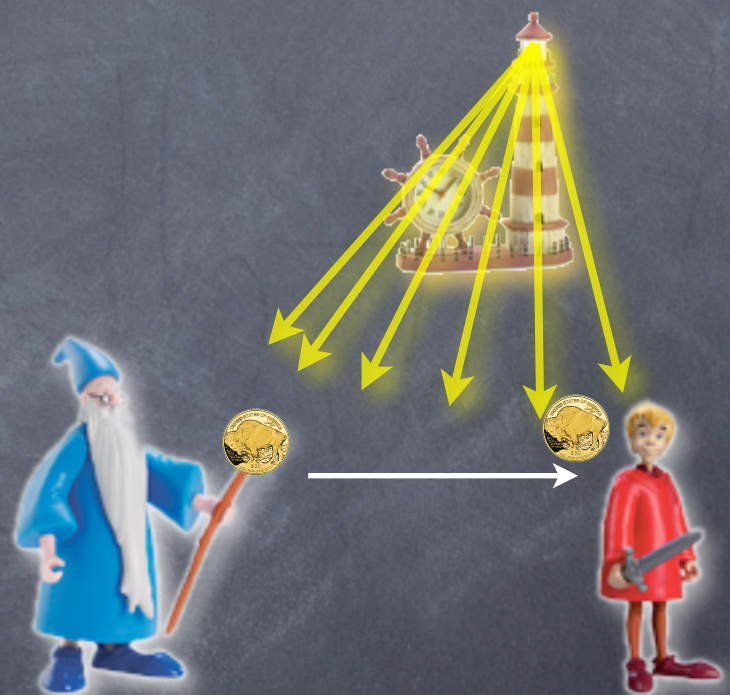

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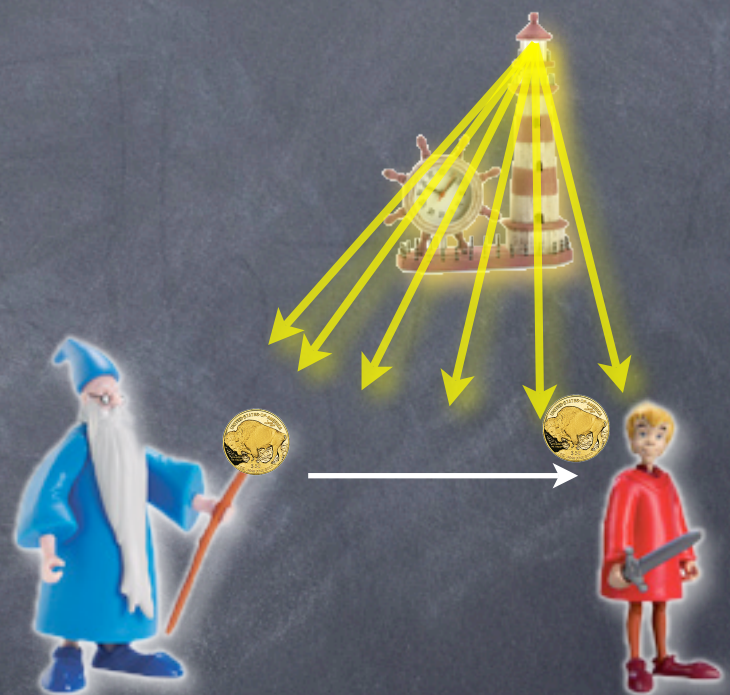

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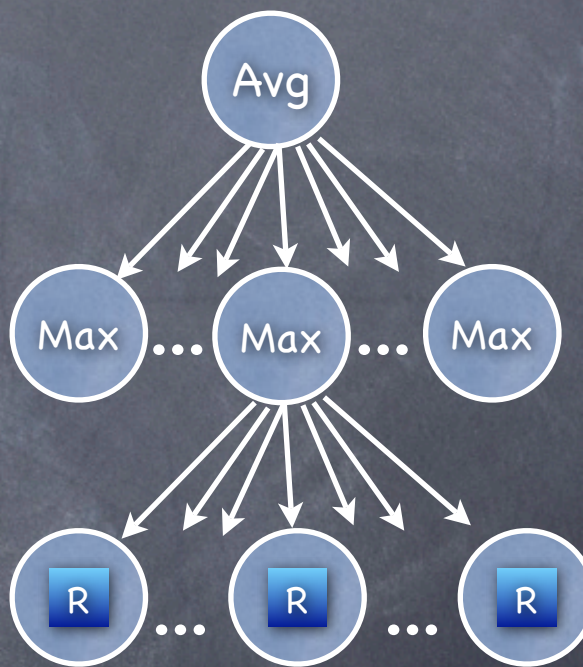

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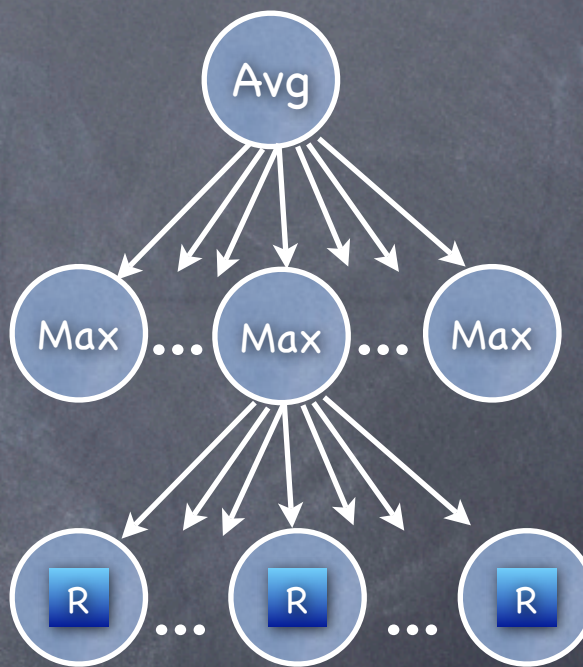

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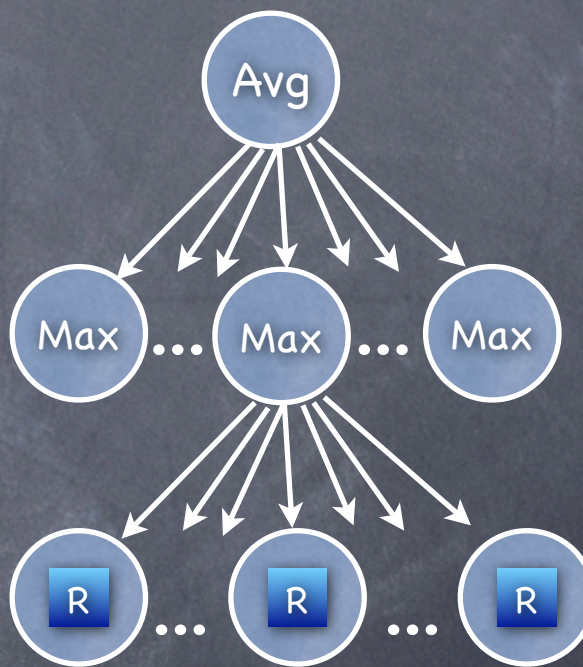

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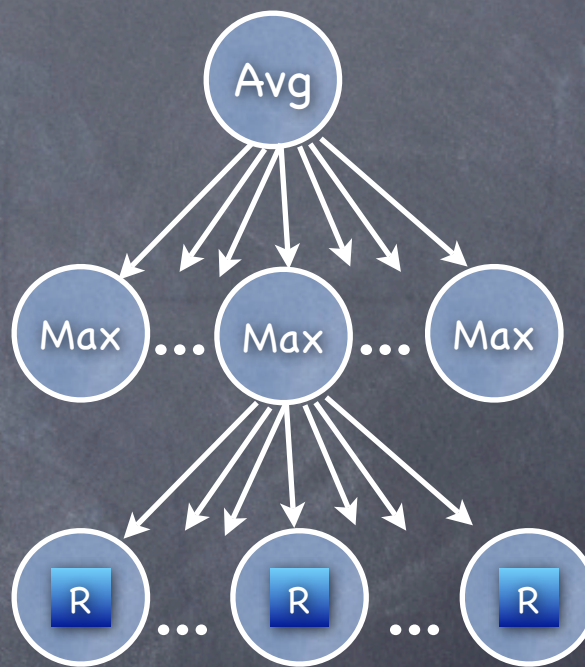

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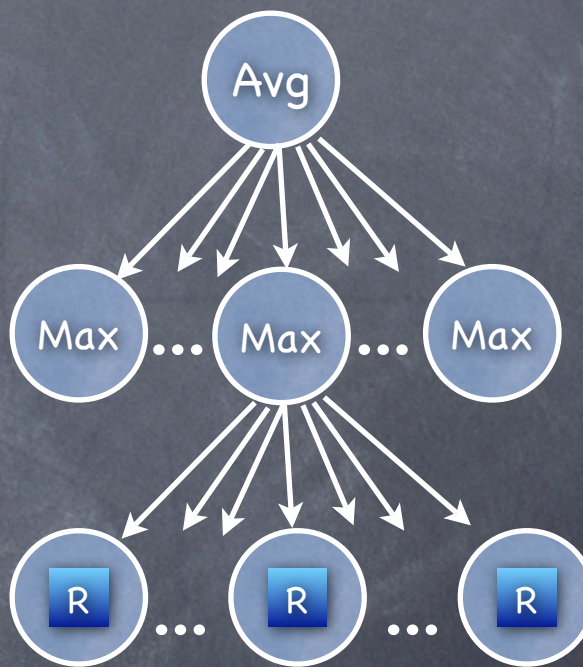
  - Extends to AM[k], with k alternating levels



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  - Still turns out to give exponential amplification

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  - So MAM = AM

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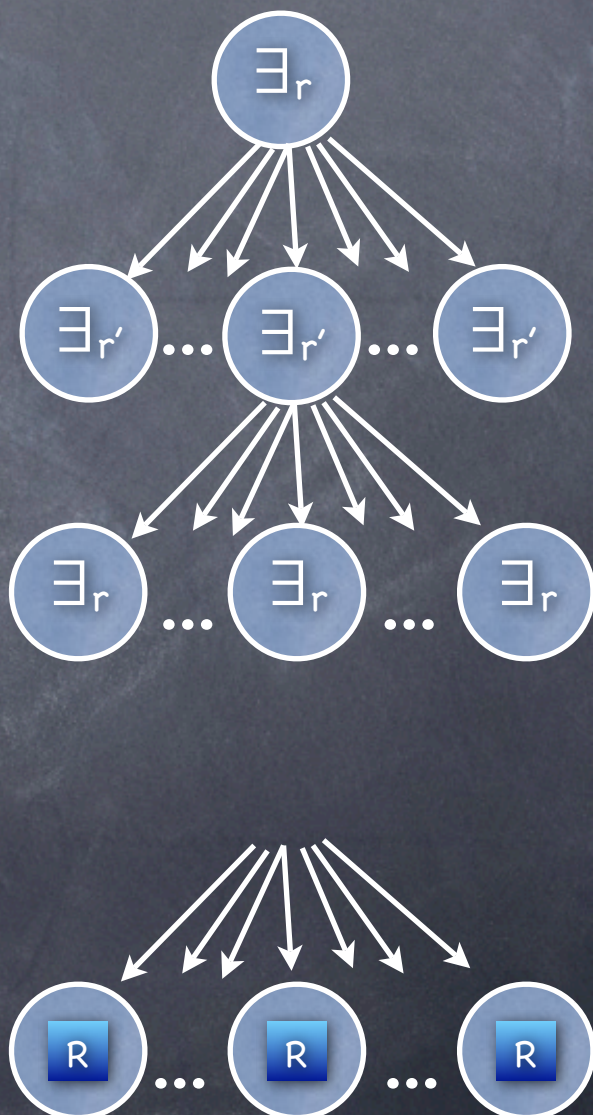

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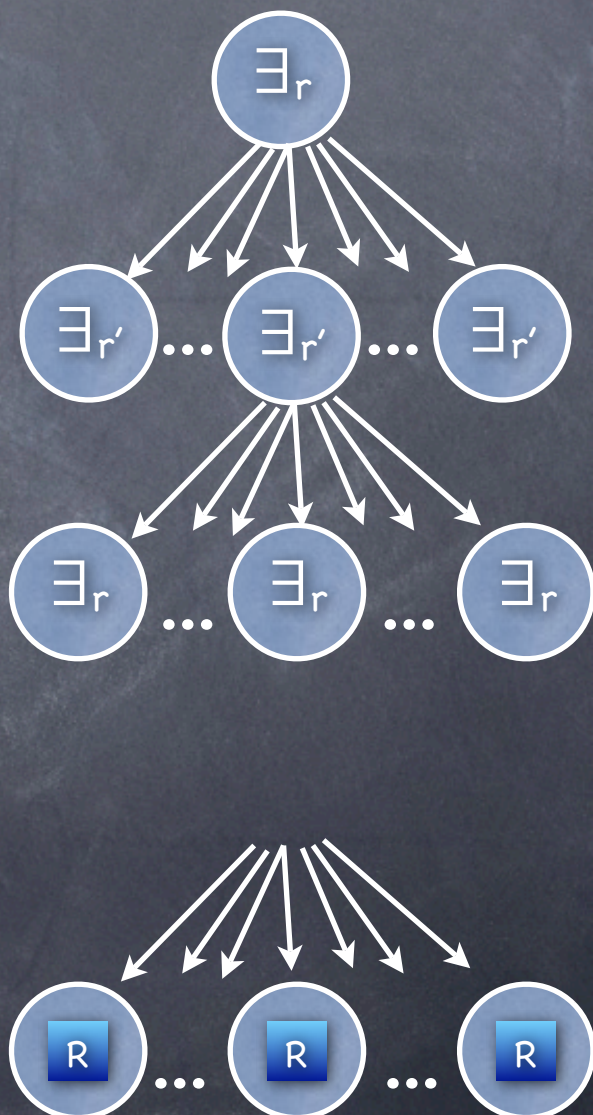

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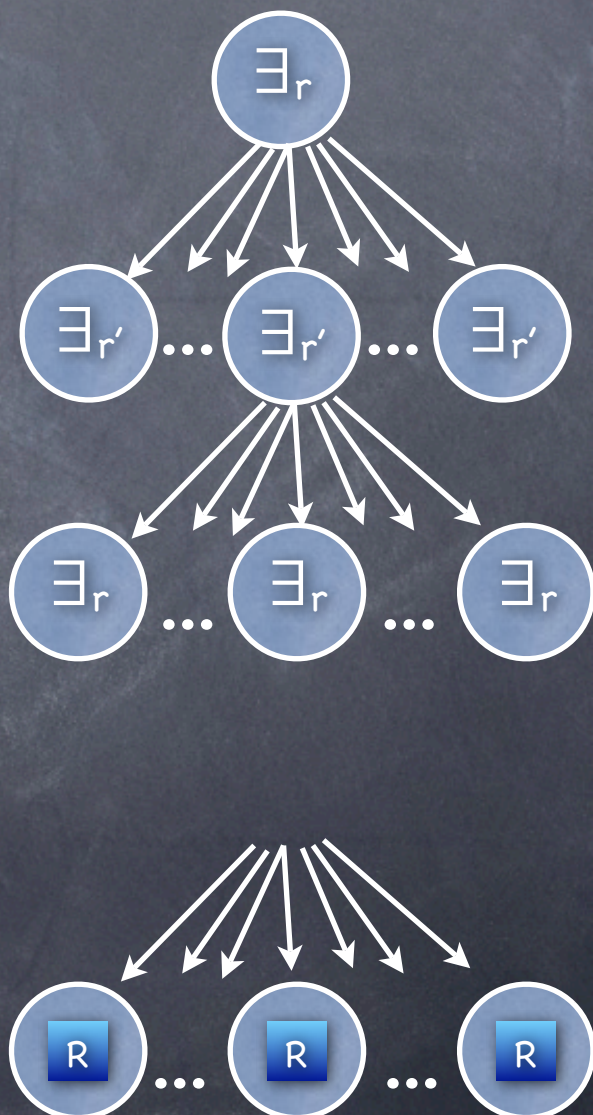

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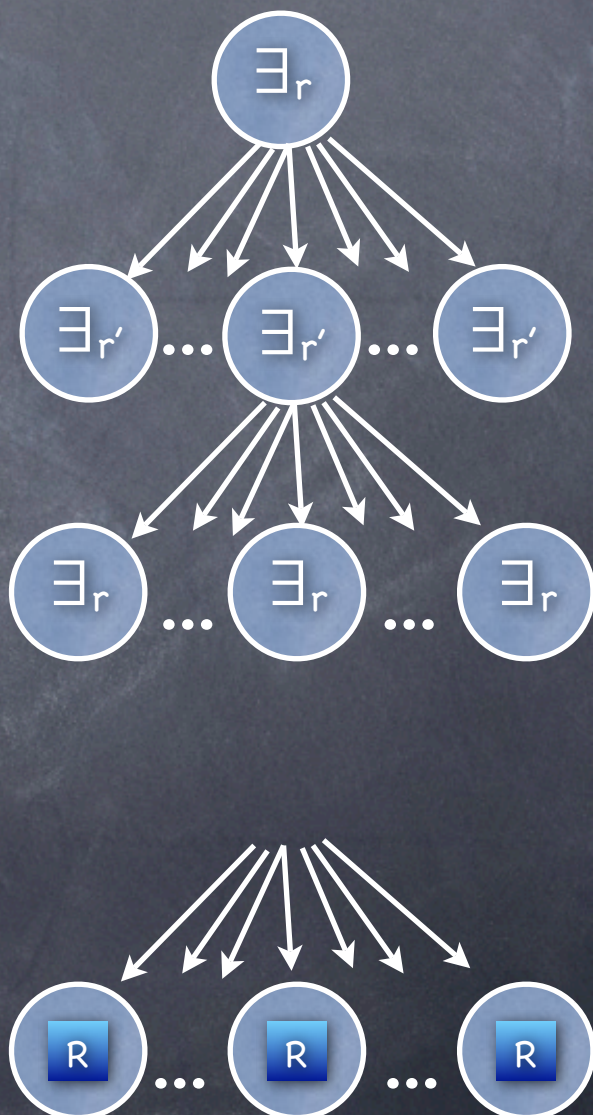

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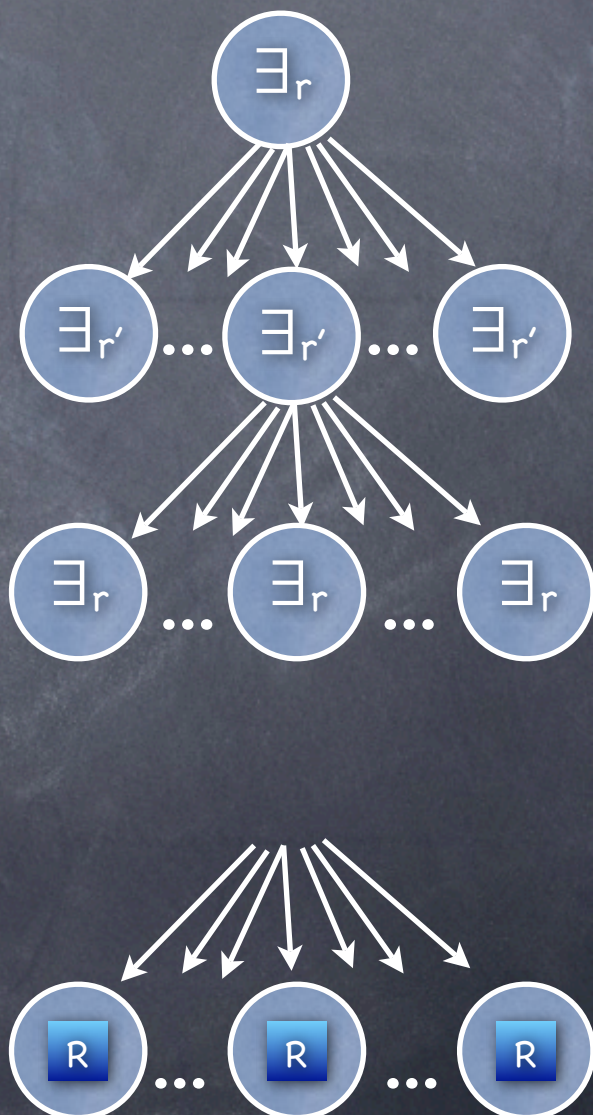

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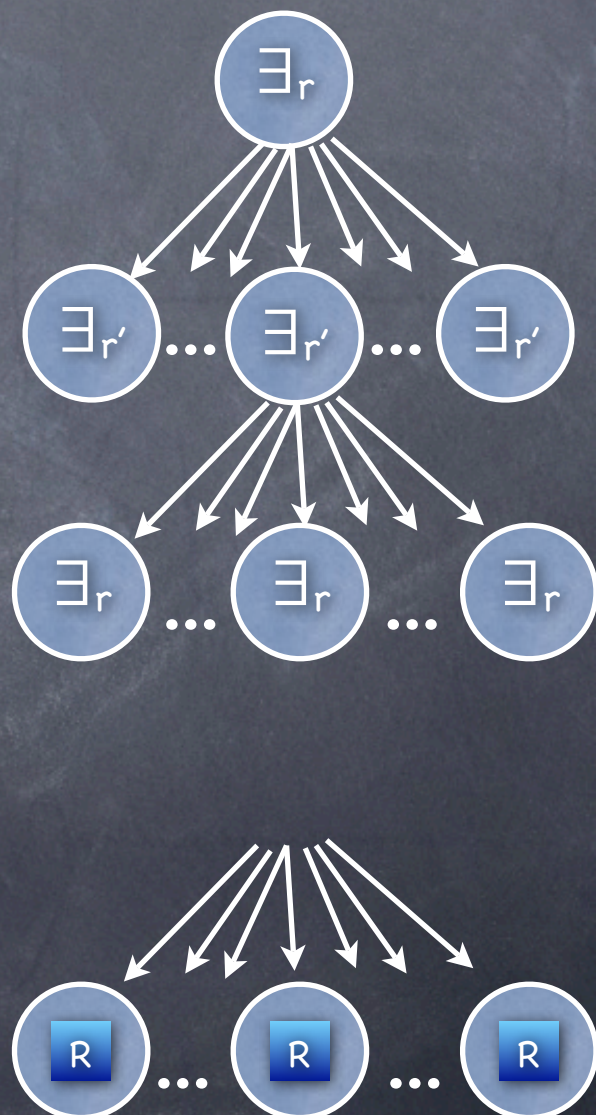

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  - k unchanged, size increases by a polynomial factor

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    - Reducing threshold enlarges $\{x|\ M_+(x)=1\}$ and $\{x|\ M_-(x)=1\}$
    - And they stay disjoint
    - So they do not change (as they were already a partitioning)
  - By parallel repetition, can increase threshold to exponentially close to 1, starting from $1/2 + c$

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  - Enough, because we can reduce error (increase thresholds) for both AM protocols and ATTMs

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  - Repeat ~k/2 times to reduce to AM[2]

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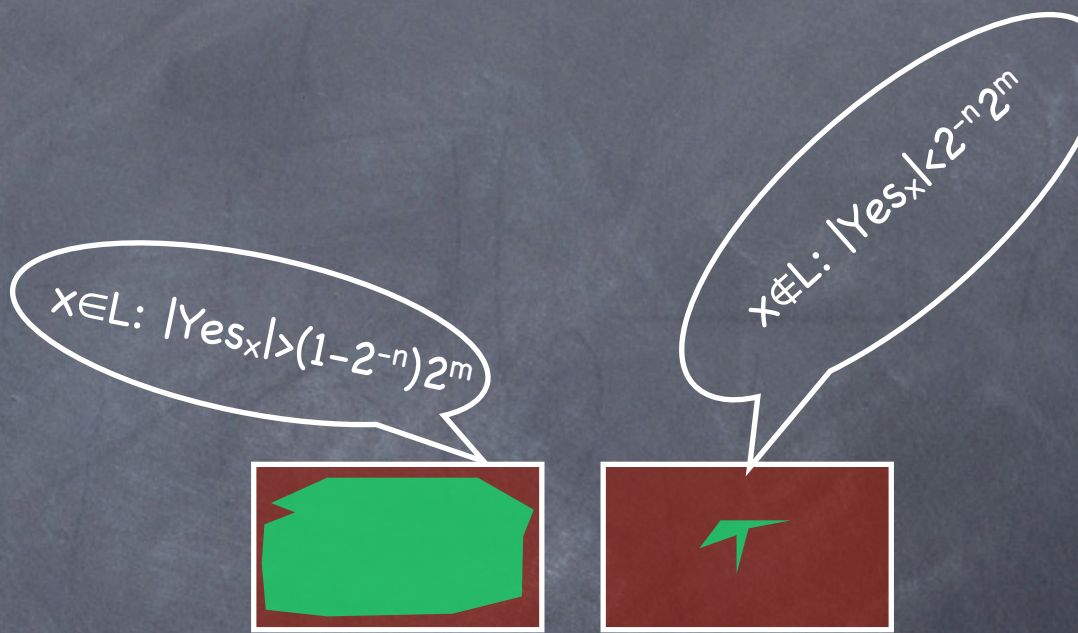


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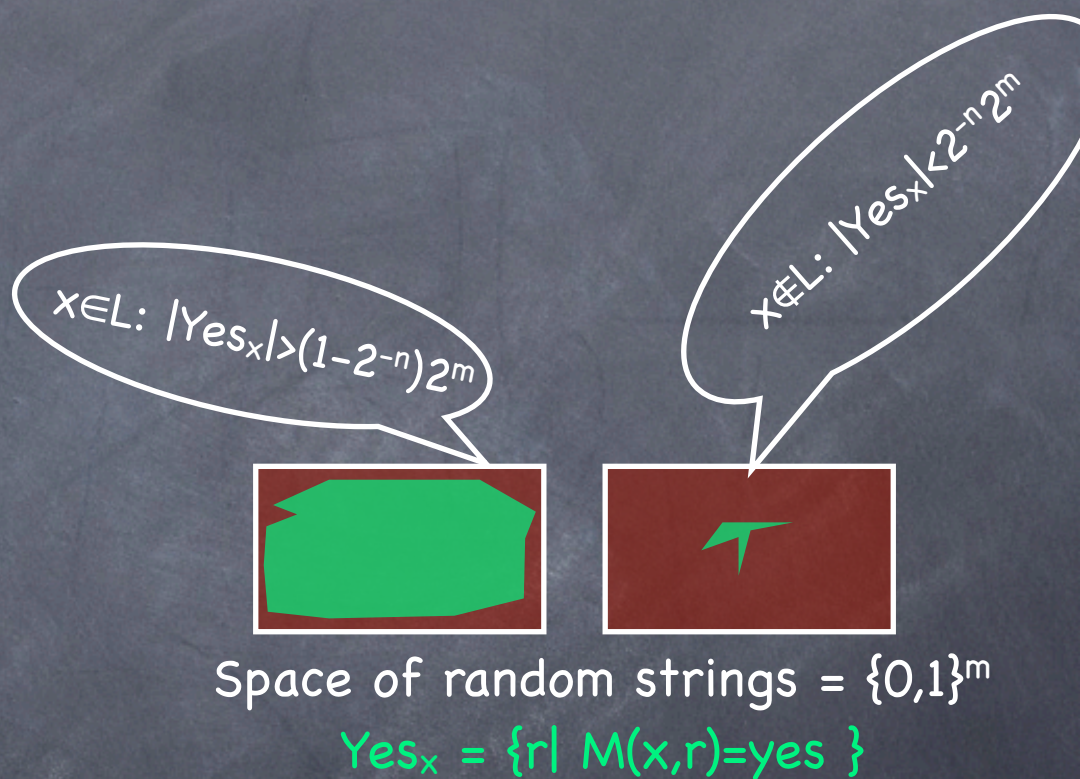

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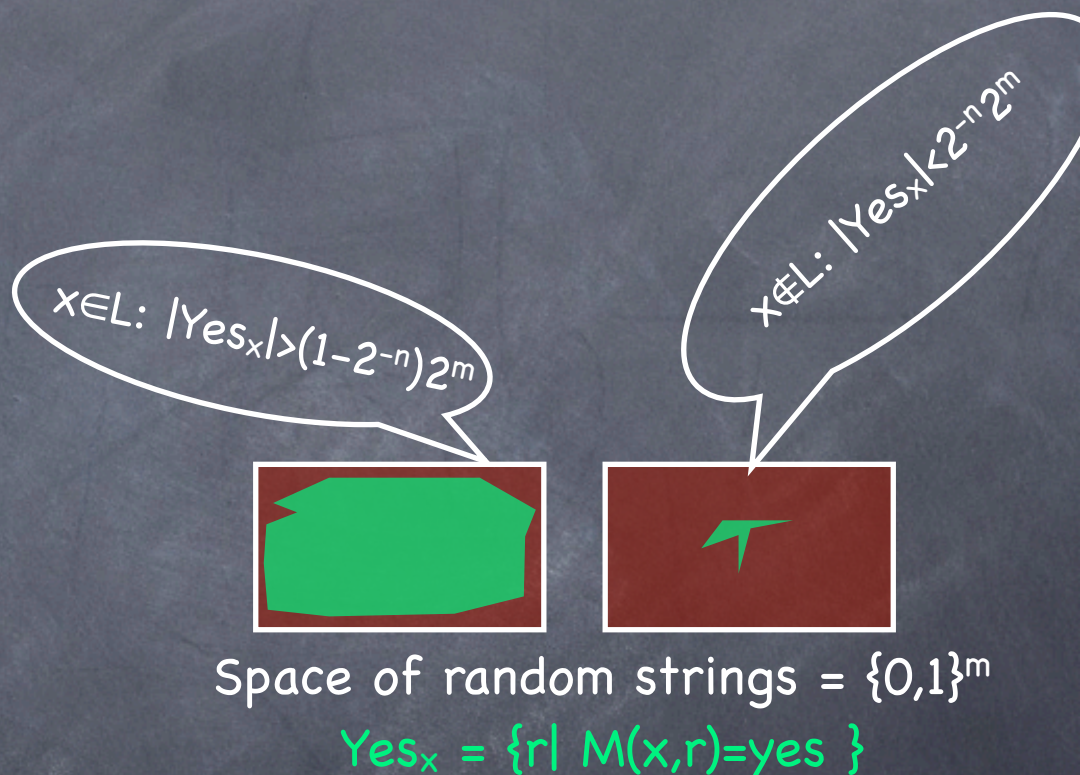

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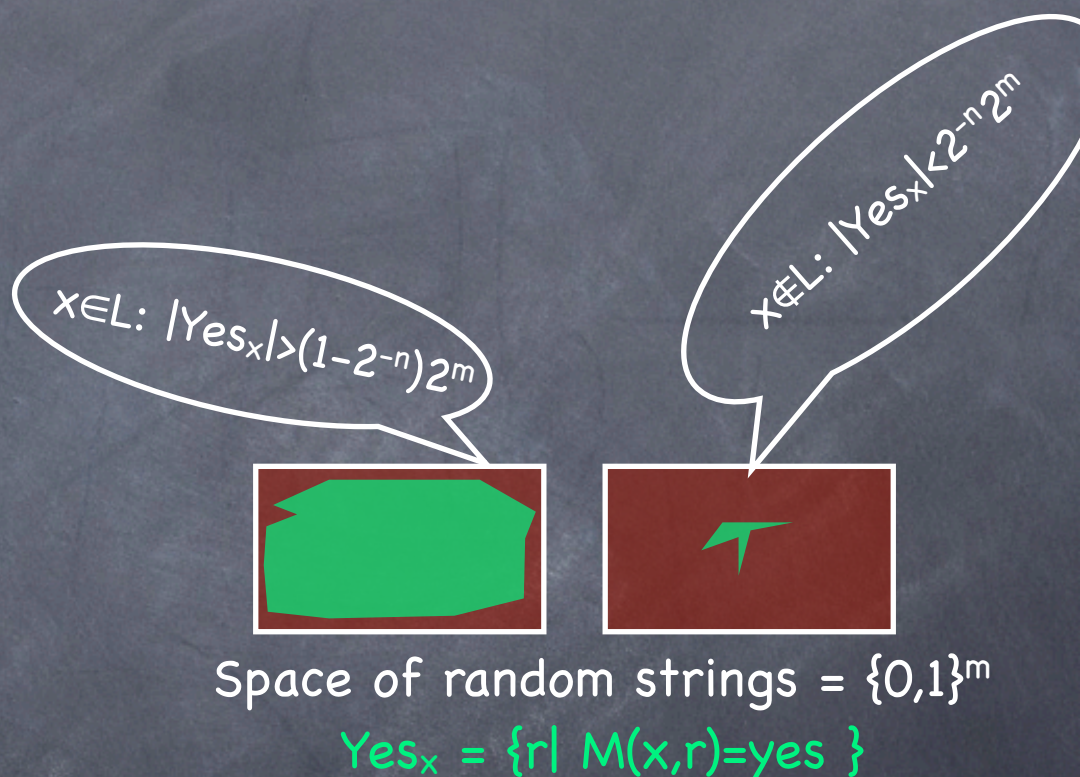

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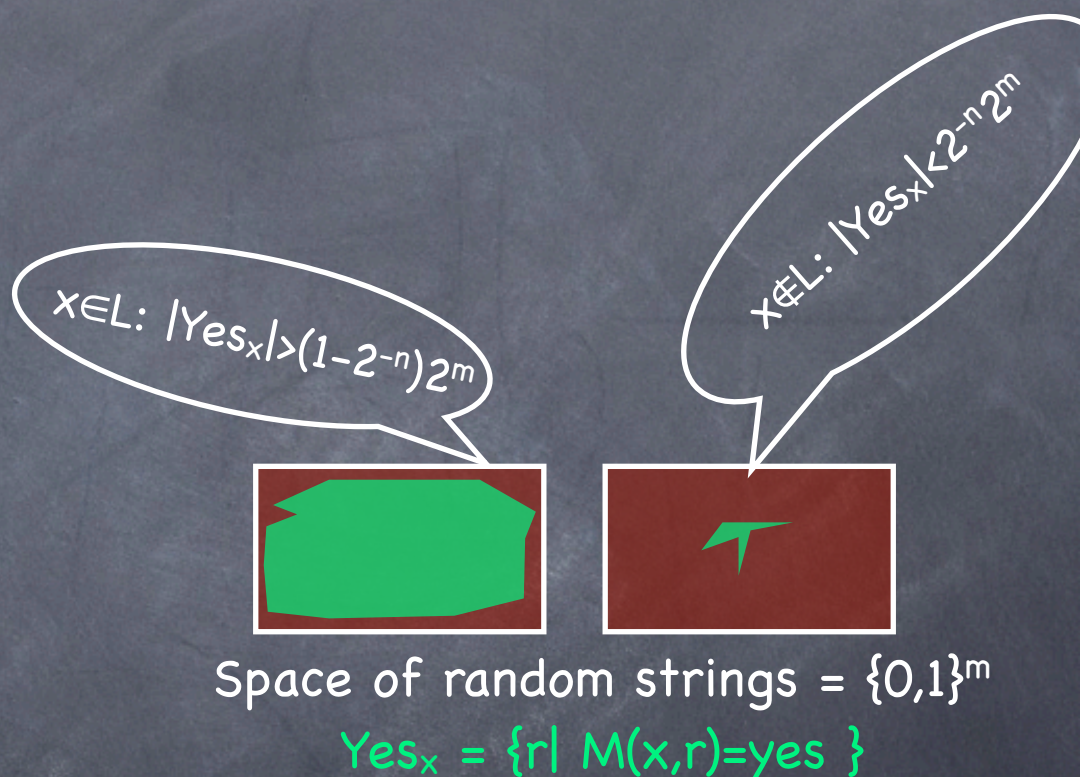

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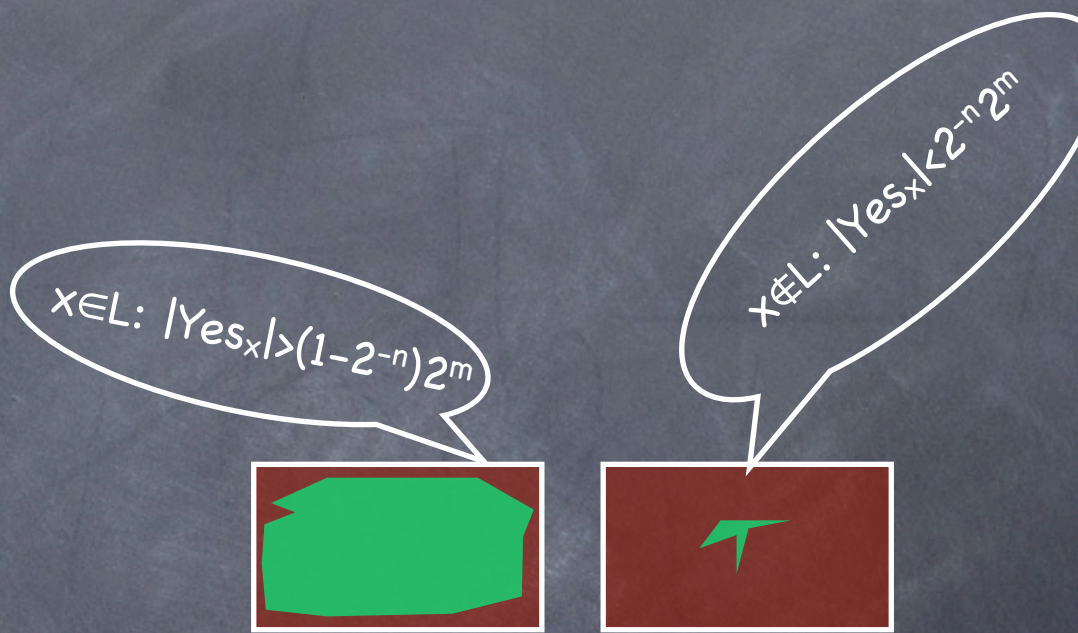

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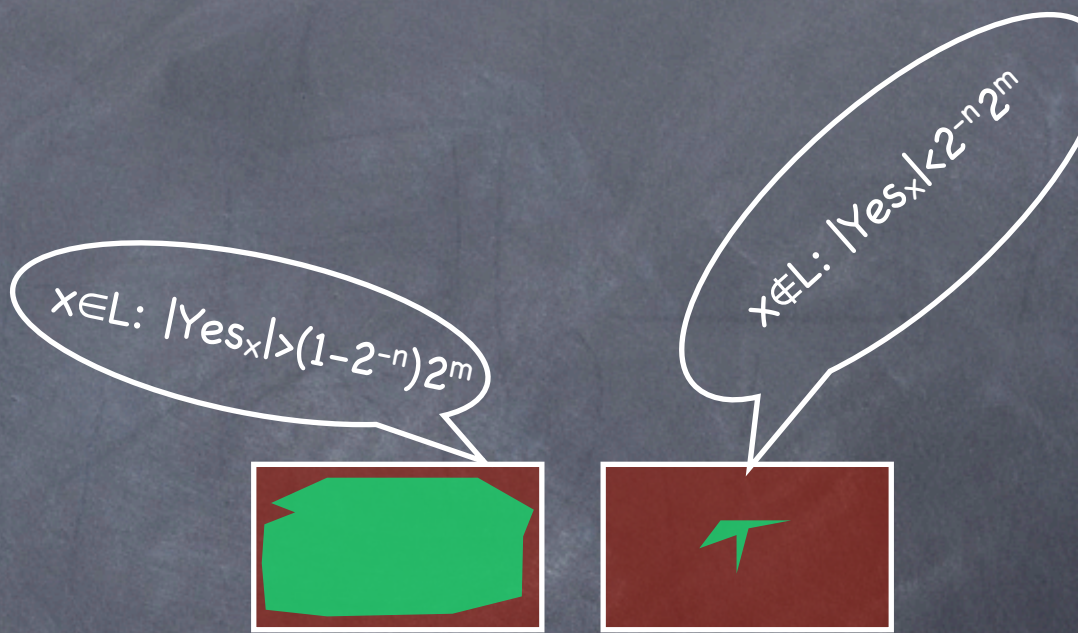

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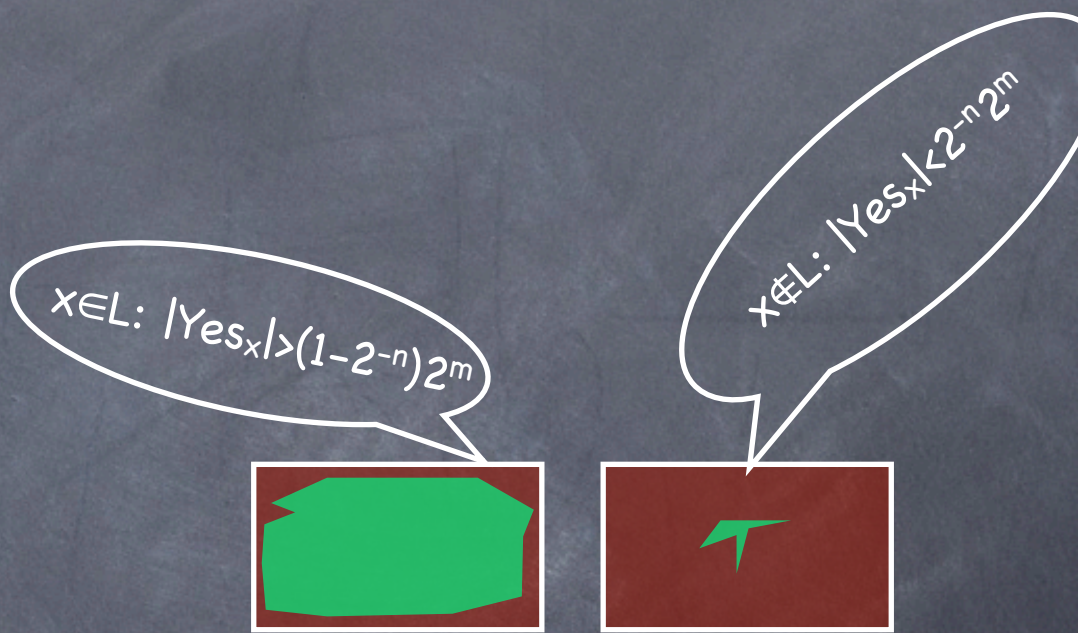

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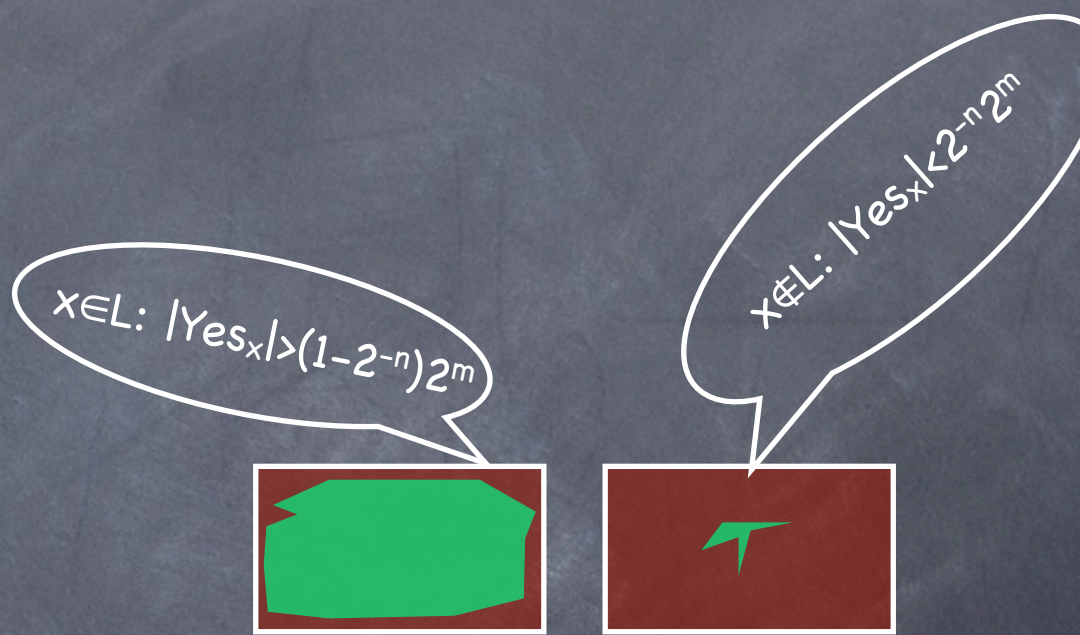


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