Interactive Proofs

Lecture 17

IP = PSPACE

So far

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- IP

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- AM, MA

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- GNI $\in$ IP

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  - Using AM protocol for set lower-bound
  - In fact, IP[k] in AM[k+2]

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- PSPACE $\subseteq$ IP
  - Prover can convince verifier of high complexity statements

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  - Could then use the "fact" that IP[poly]=AM[poly]
    - Or modify the proof (as we'll do)

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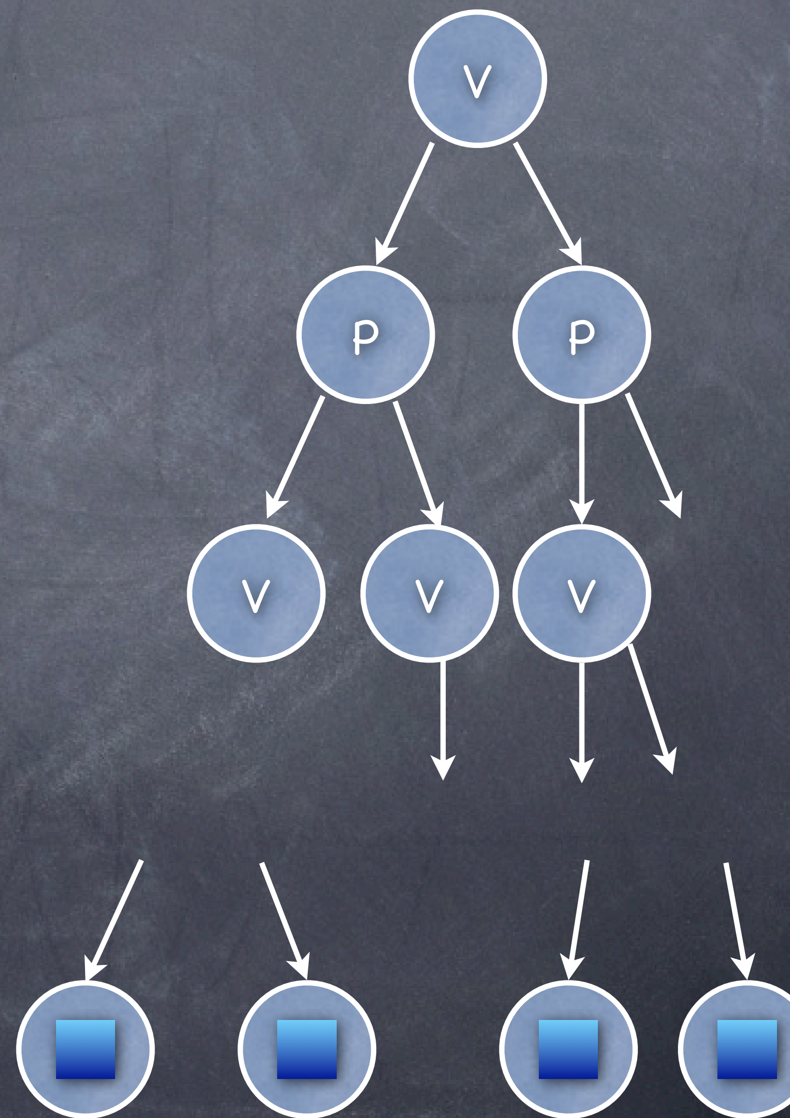

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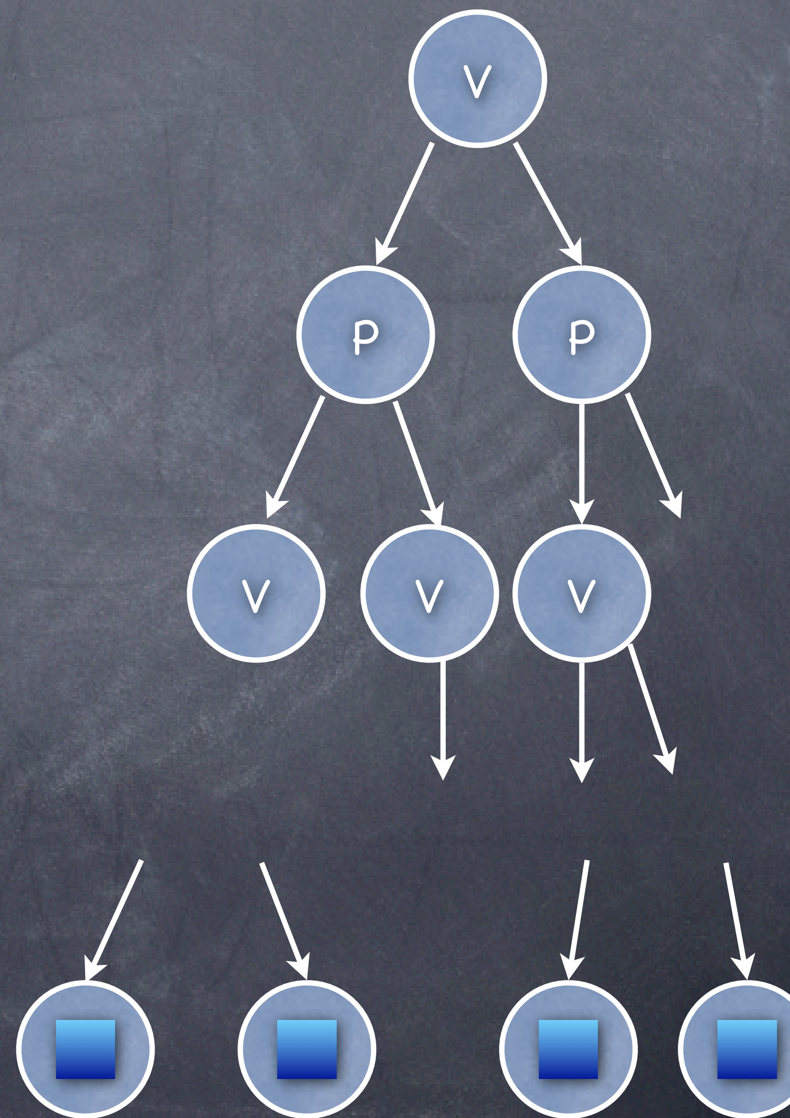

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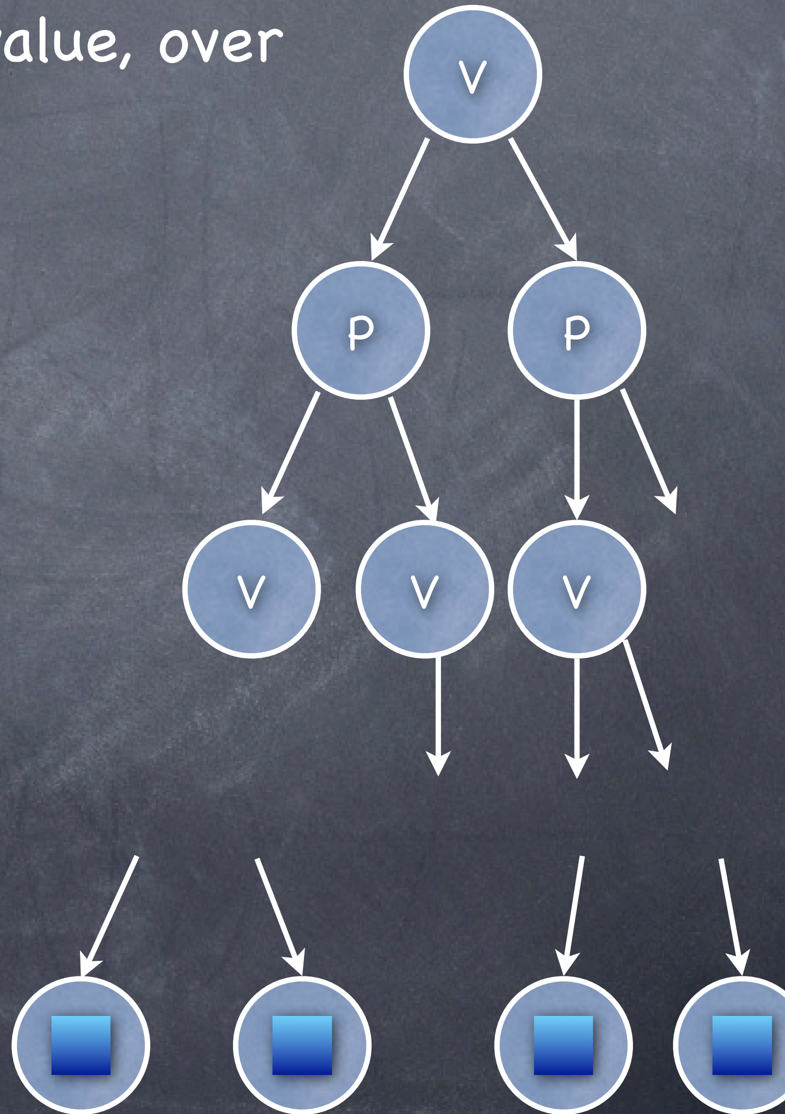

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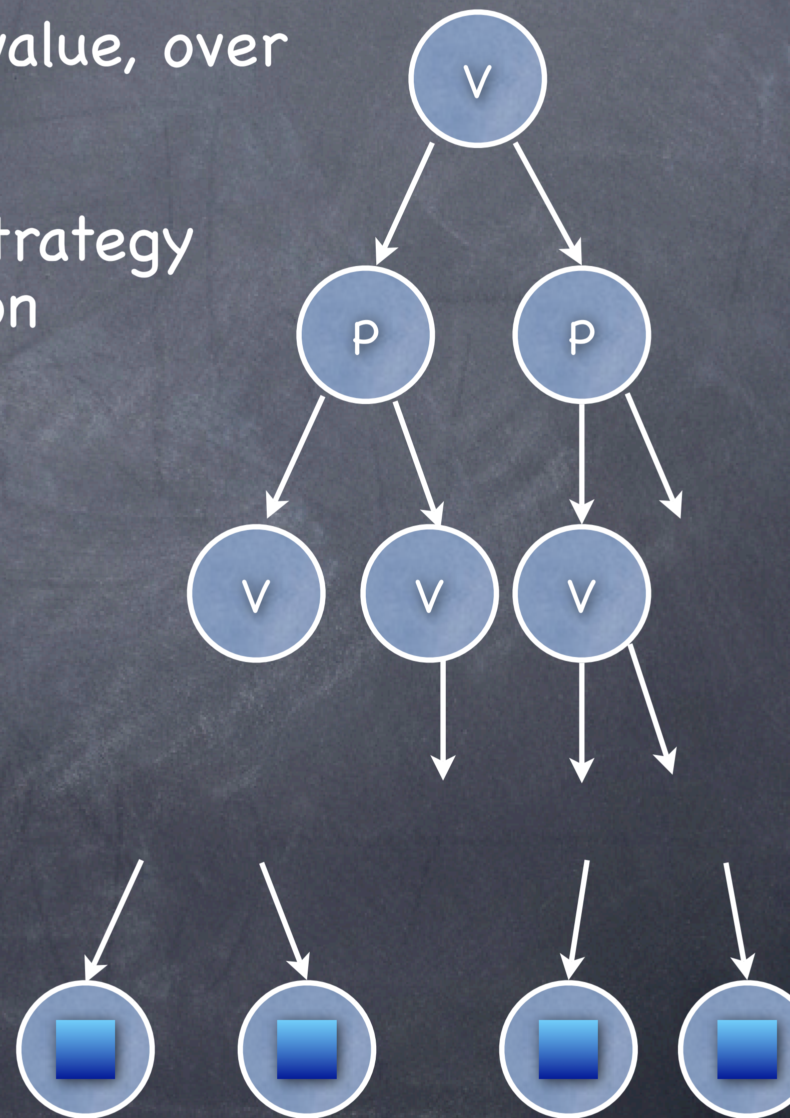

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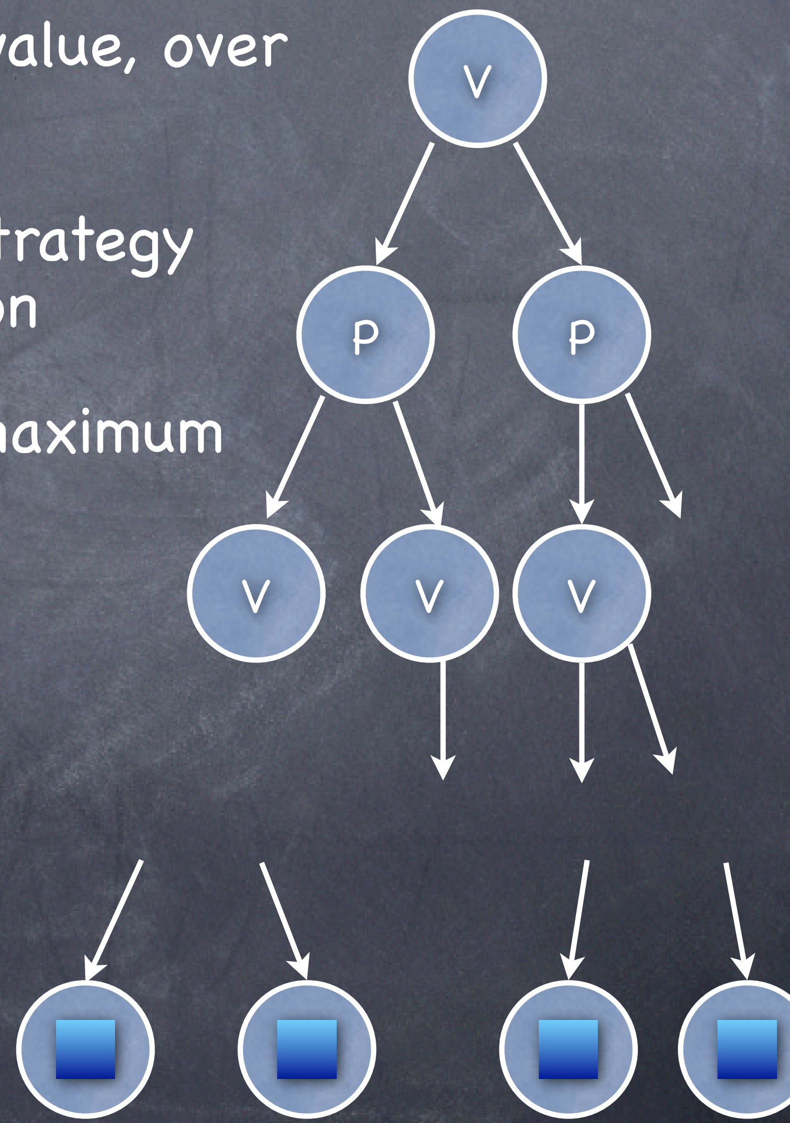

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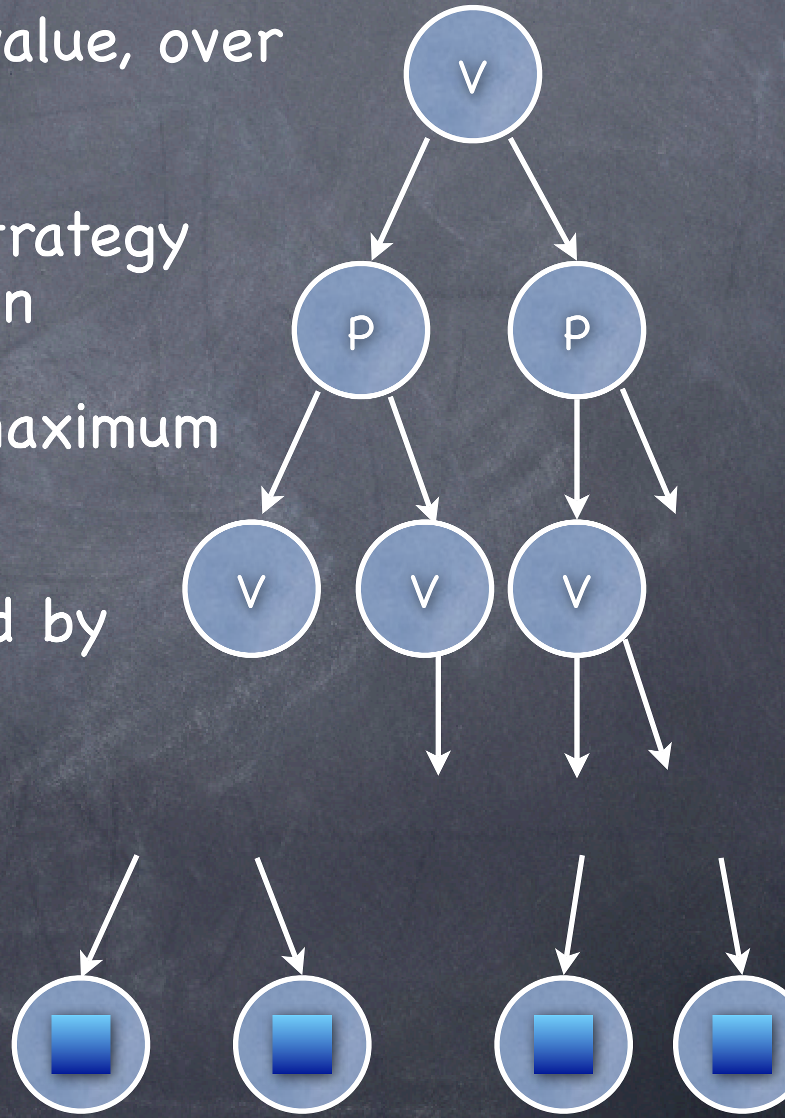

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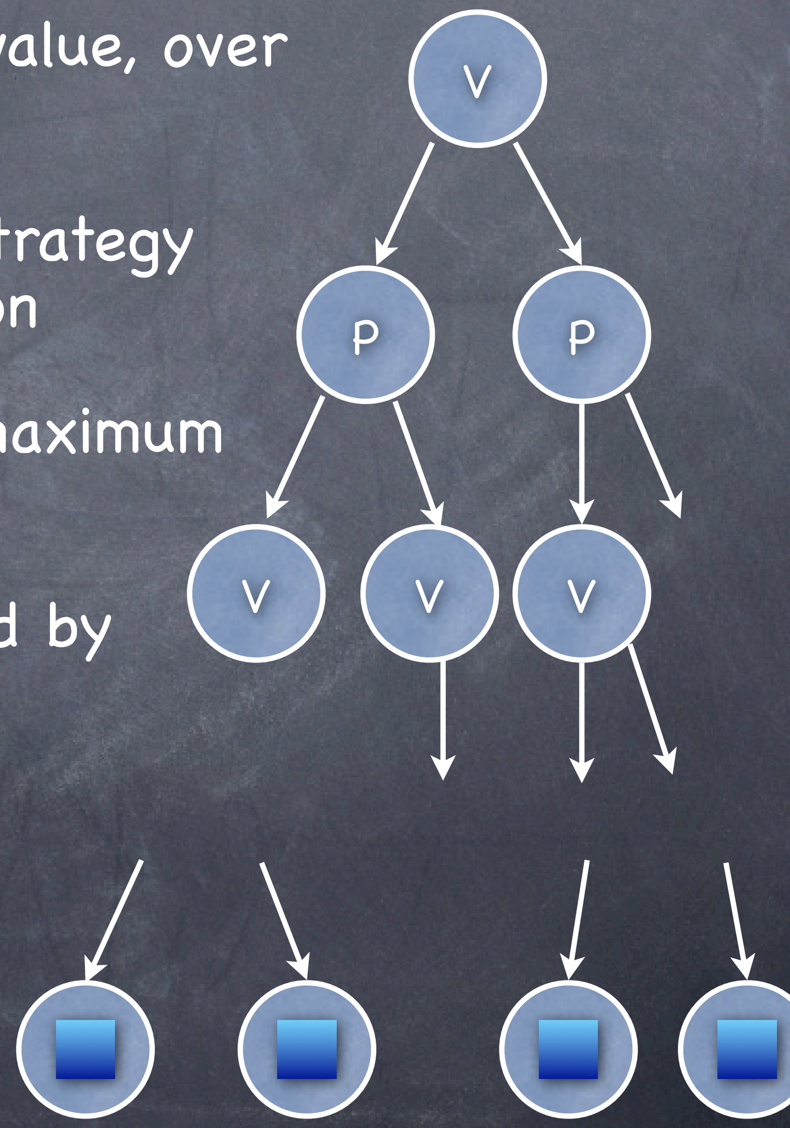

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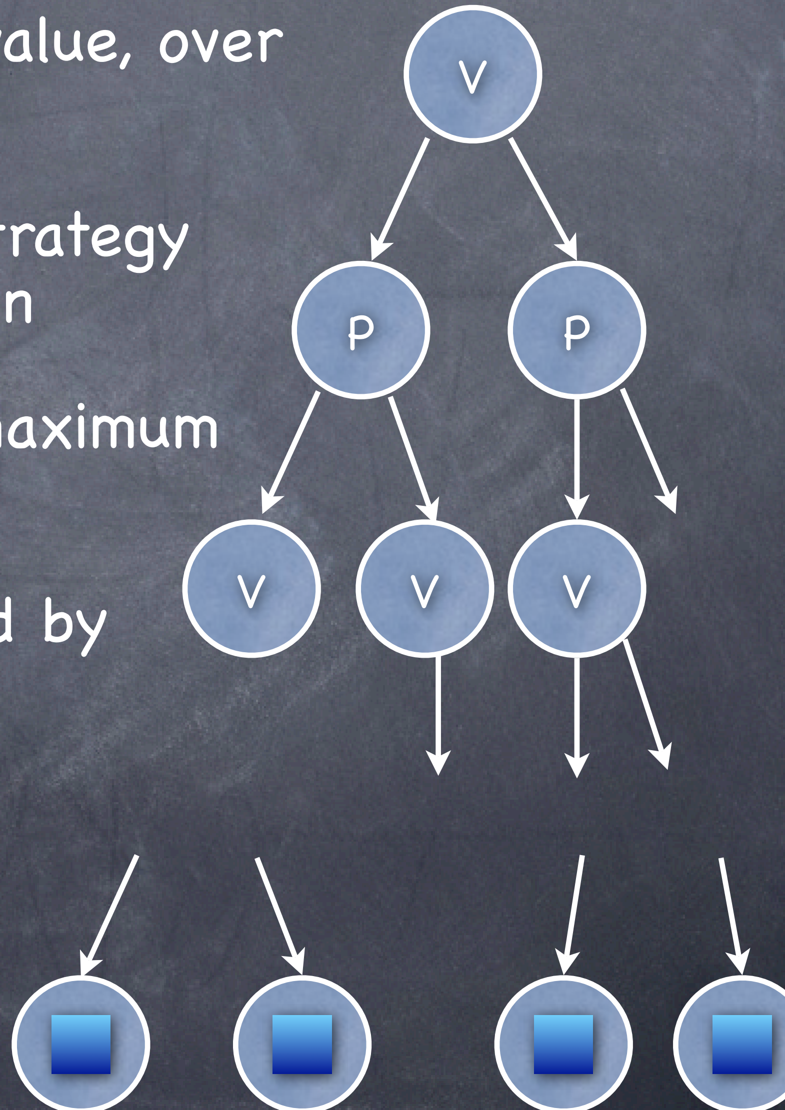

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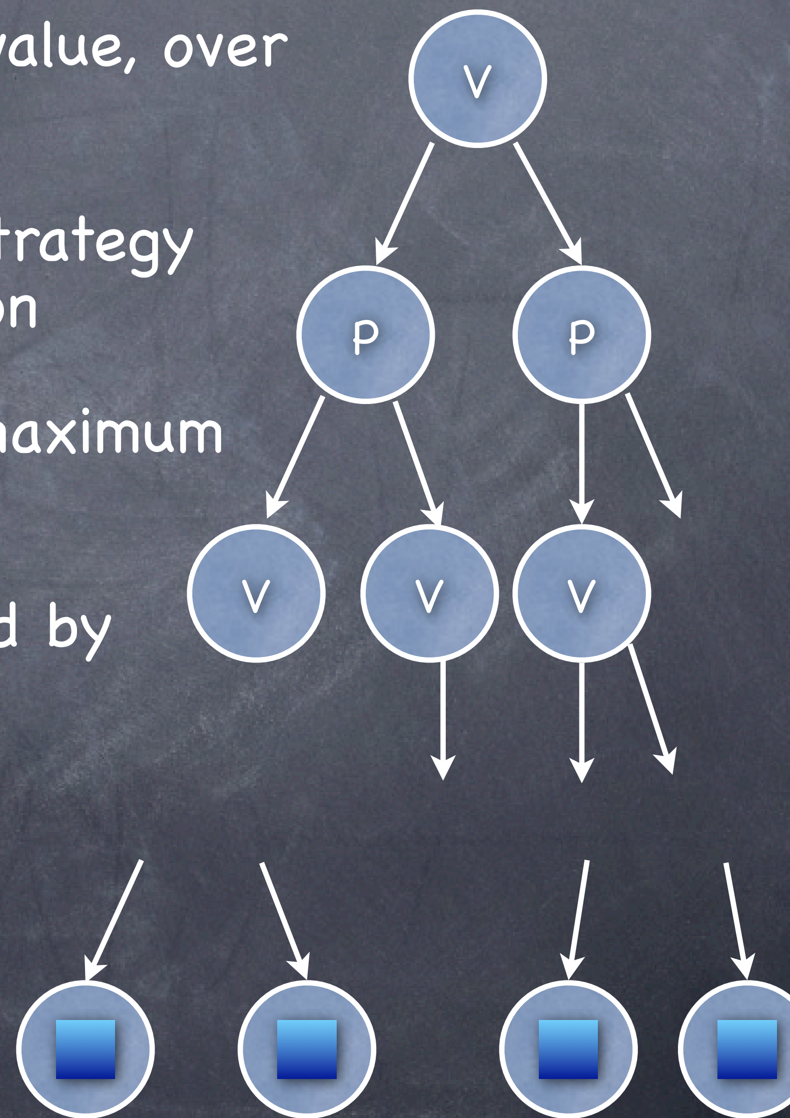
    - In PSPACE: depth polynomial



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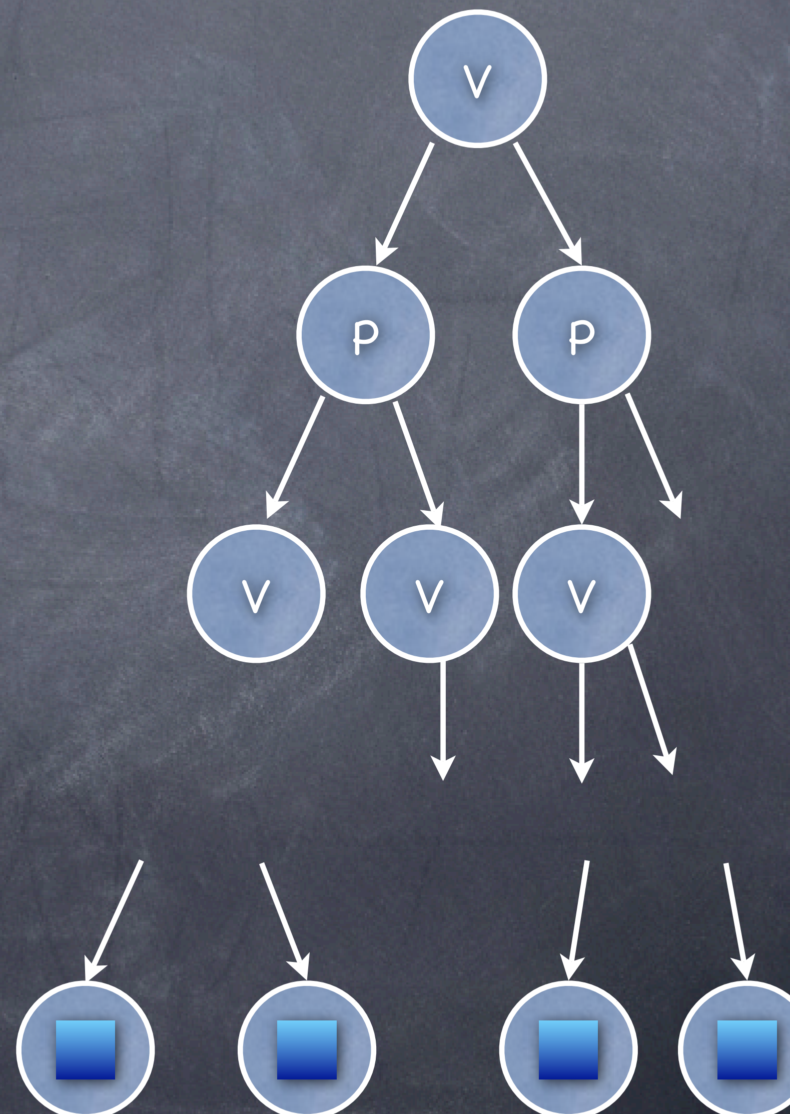

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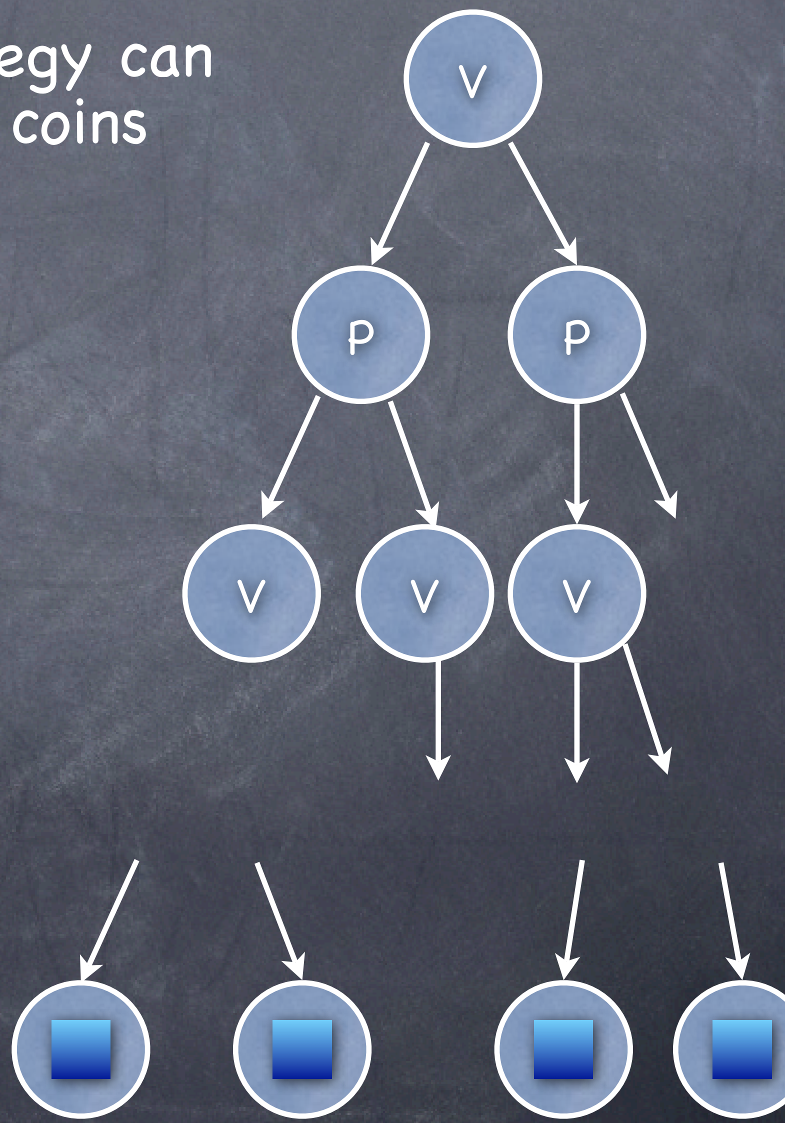

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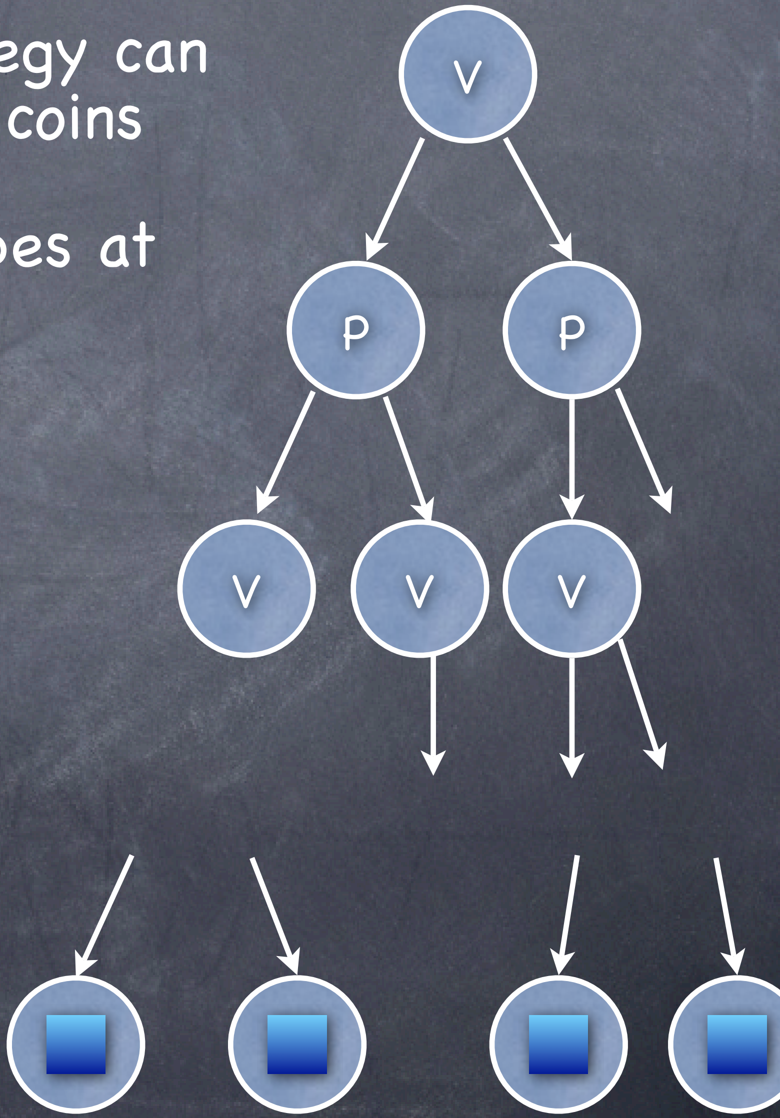

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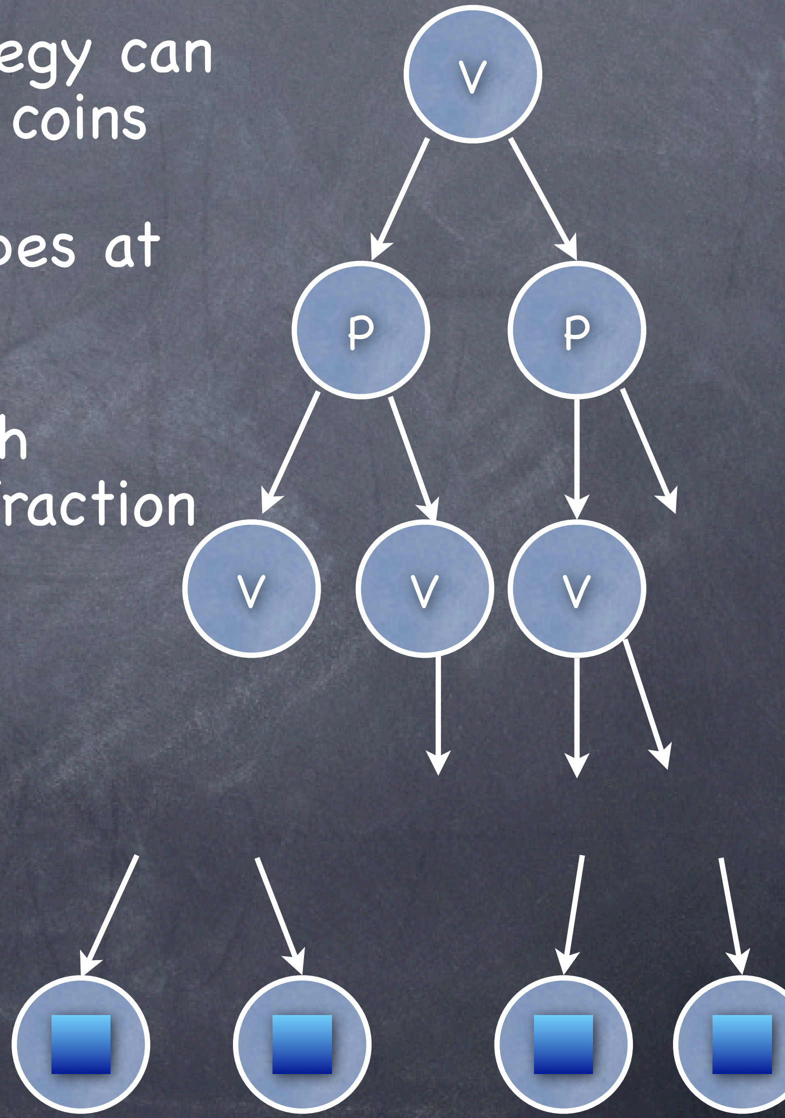

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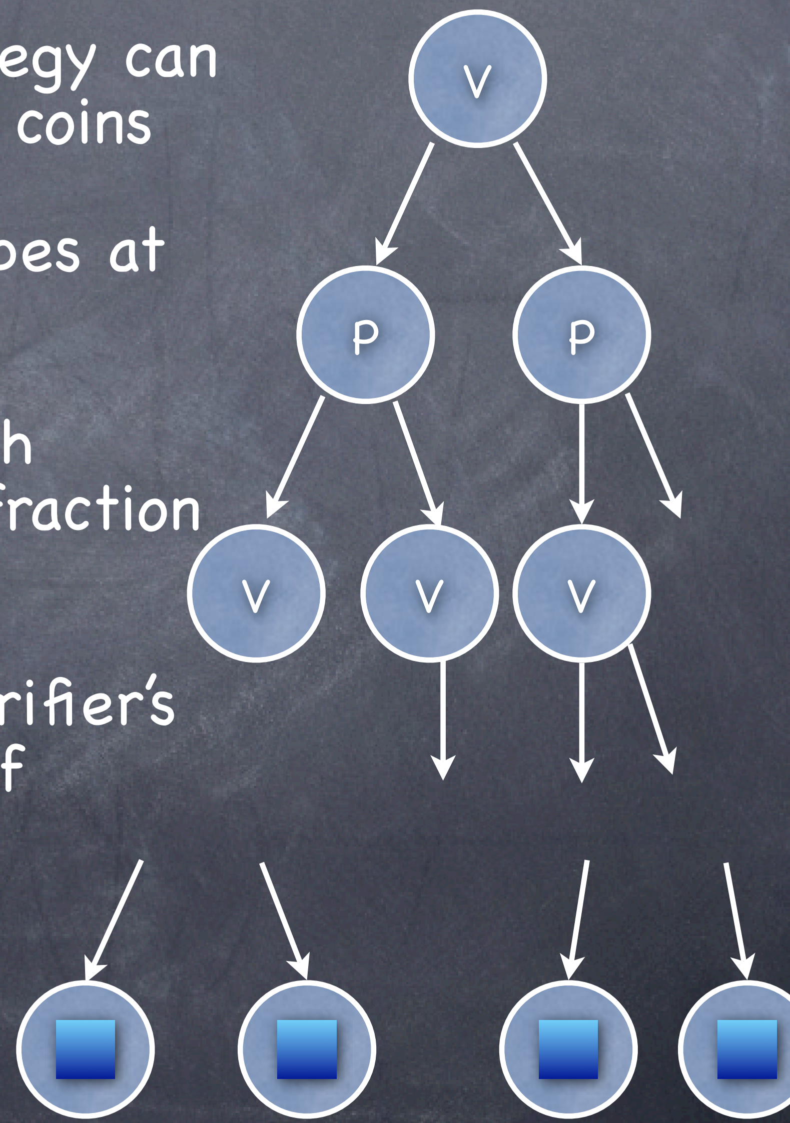

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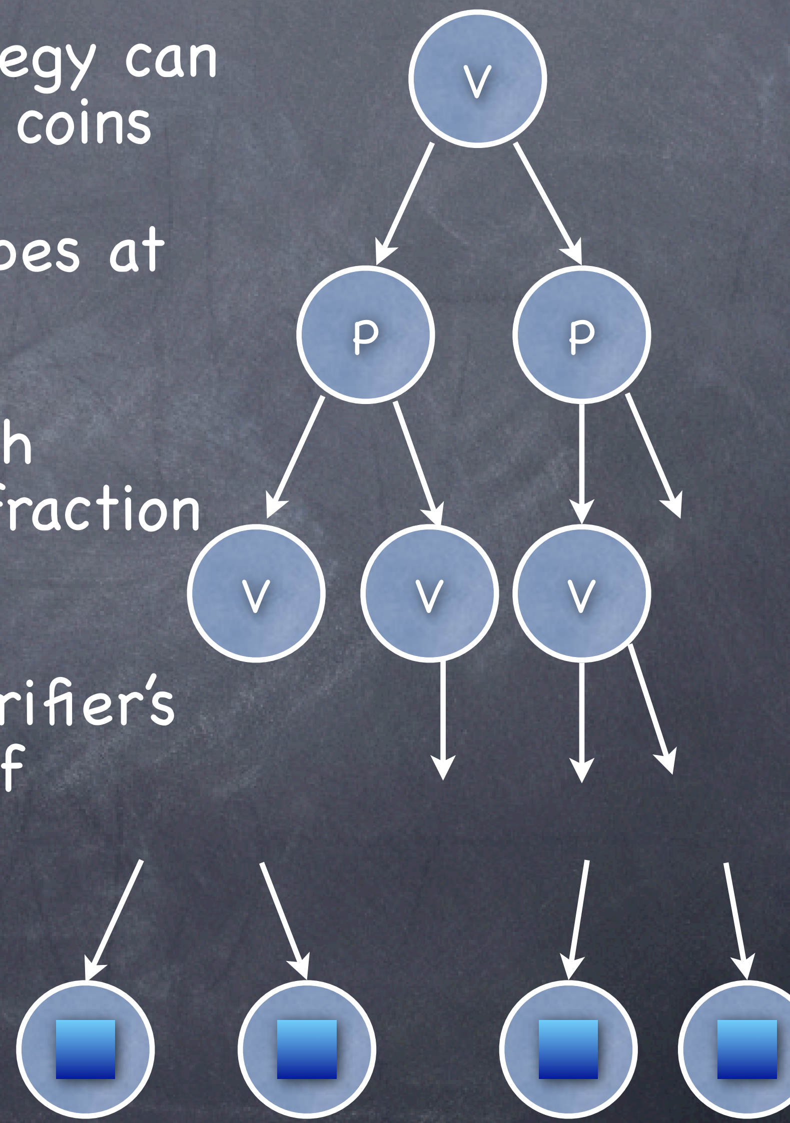

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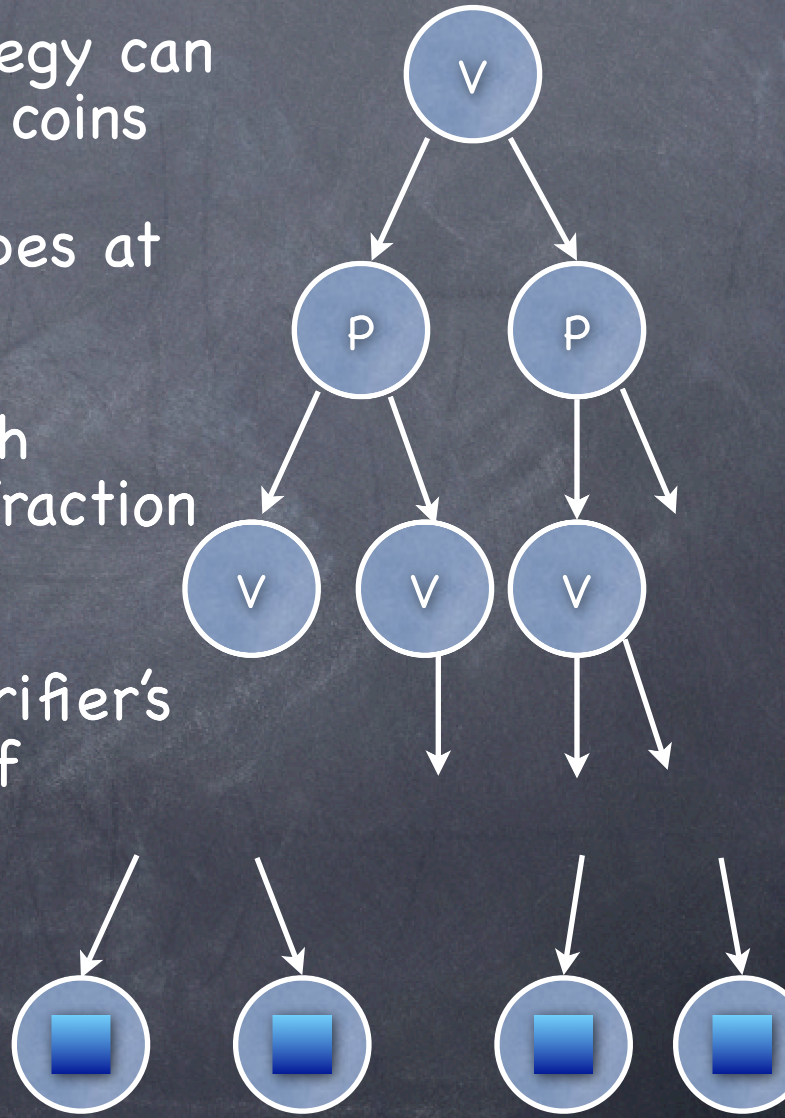

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  - Decide whether a QBF is true or not

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- Recall TQBF
  - Decide whether a QBF is true or not
  - QBF: $Q_1x_1\ Q_2x_2\ \ldots\ Q_nx_n\ F(x_1,\ldots,x_n)$ for quantifiers Q_i and a formula F on boolean variables

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      - Can always use a polynomial linear in each variable since $x^n=x$ for $x=0$ and $x=1$

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  - Proving > 0 is trivial
  - Consider proving = K (will be useful in the general case)

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Needs degree to be small

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  - Verifier checks K = T(0) + T(1). Still needs to check T=R

Only Σ, no Π

Needs degree to be small

Sum-check protocol

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    - Note: P_1 has degree at most d; verifier has oracle access to P_1 (as it knows a, and has oracle access to P)

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        - At most nd/p if n variables. Can take p exponential.

IP Protocol for TQBF

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  - Verifier picks random a, and asks prover to show $R'(a) = L(a)$
  - Verifier checks (as appropriate) $L(1).L(0) = K$ or $L(1)+L(0) = K$

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